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ON d-SPANNEDNESS OF THE ADJOINT BUNDLES ON POLARIZED MANIFOLDS

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1. Introduction

Let X be a projective variety and L be an ample Cartier divisor. Then the pair (X, L) is called a polarized variety.

Let K be an algebraically closed field of characteristic $p \geq 0$. Let X be an n-dimensional smooth variety over K. Denote by $X^{[r]}$ the Hilbert scheme of all zero dimensional subschemes $(Z, \mathcal{O}_Z)$ of X with $\text{length}(\mathcal{O}_Z) = r$. An element of $X^{[r]}$ is sometimes called a zero cycle.

DENITION ([BB (0.0)]). An element $(Z, \mathcal{O}_Z)$ of $X^{[r]}$ is called curvilinear if $\dim T_v Z \leq 1$ for every $v \in Z_{\text{red}}$.

DENITION ([BB (0.0)]). Let L be a line bundle on X and let $d \geq 0$. We say that L is d-spanned if $\Gamma(L) \to \Gamma(\mathcal{O}_Z(L))$ is surjective for any curvilinear zero cycle $Z \in X^{[d+1]}$.

Note that L is 0-spanned if and only if L is generated by global sections. Note also that L is 1-spanned if and only if L is very ample.

Note that if L is d-spanned and $(Z, \mathcal{O}_Z)$ is a curvilinear zero cycle with $\text{length}(\mathcal{O}_Z) = m \leq d + 1$, then the linear span $\langle Z \rangle$ by $H^0(X, L)$ is isomorphic to $\boldsymbol{P}^{m-1}$.

On the d-spannedness, the following Fujita's conjecture is well known.

CONJECTURE ([F1]). *Let (X, L) an smooth polarized n-dimensional variety over $\boldsymbol{C}$. If $L^n > 1$ then*
(1) *$K_X + tL$ is spanned for $t \geq n$.*
(2) *$K_X + tL$ is very ample for $t \geq n + 1$.*

This conjecture is true if $n \leq 2$ by [R]. The case (1) of above conjecture is also true if $n \leq 4$ (by [EL] and [F2] if $n = 3$ and by [K] if $n = 4$). If $n = 2$, the following theorem holds.

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THEOREM ([BFS Theorem (2.2)]). *Let (S, L) be a polarized surface and let $d \geq 0$. If $L^2 > 1$, then $K_S + tL$ is d-spanned for any $t \geq d+2$.*

When $L^n = 1$, there are some examples where $K_X + nL$ is not spanned, such as $(\boldsymbol{P}^n, \mathcal{O}(1))$, a hypersurface M of degree 6 in the weighted projective space $\boldsymbol{P}(3, 2, 1, \ldots, 1)$ with $\mathcal{O}_M(1)$, and so on. We construct new series of such examples for d-spannedness.

The main result is the following:

MAIN RESULT. *For any integers $d \geq 0$, $n \geq 2$, and for any algebraically closed field K, there exists an n-dimensional polarized manifold (X, L) of general type with $L^n = 1$ over K such that $K_X + (n+d)L$ is not d-spanned.*

2. Preliminaries

Let $q \geq 5$ be a prime number such that $q \neq p$. Let $G \cong \boldsymbol{Z}/q\boldsymbol{Z}$ be a cyclic group of order q generated by the primitive q-th root of unity. Given a non-negative integer n and a sequence $w_0, \ldots, w_{n+1}$ with $0 \leq w_0 < \cdots < w_{n+1} \leq q-1$, we define an action of G on $\boldsymbol{P}_K^{n+1}$ by

$$g \cdot (z_0 : \cdots : z_{n+1}) = (g^{w_0} z_0 : \cdots : g^{w_{n+1}} z_{n+1})$$

for $g \in G$.

Denote by S_d^j the set of monomials $\{z_{i_1}, \ldots, z_{i_d}\}$ of degree d such that $w_{i_1} + \cdots + w_{i_d} \equiv j$. Here and throughout this paper, $\equiv$ means conjugate modulo q, i.e. $=$ in $\boldsymbol{Z}/q\boldsymbol{Z} = \boldsymbol{F}_q$. Let X be a smooth hypersurface in $\boldsymbol{P}_K^{n+1}$ defined by a G-invariant homogeneous polynomial F of degree q. Let

$$F = \sum_{f_i \in S_q^0} \alpha_i f_i = \alpha_0 z_0^q + \cdots + \alpha_{n+1} z_{n+1}^q + \cdots = 0 \quad \text{for } \alpha_i \in K,$$

and we assume $\alpha_j \neq 0$ for all $j = 0, \ldots, n+1$.

Note that F does not contain monomials of the form $z_a^i z_b^{q-i}$, $a \neq b$ and $i \not\equiv 0$. Indeed, otherwise, $(w_a - w_b)i \equiv 0$. Since $w_a \not\equiv w_b$, this implies $i \equiv 0$.

LEMMA 1. *This action of G on X is fixed point free. Therefore the quotient space $M = X/G$ is smooth.*

Proof. Assume that $gx = x$ for $x = (x_0 : \cdots : x_{n+1}) \in X$ and for $g \neq 1$. Take j such that $x_j \neq 0$. Since $gx = (g^{w_0} x_0 : \cdots : g^{w_{n+1}} x_{q-1}) = x$, we have $(g^{w_i - w_j} - 1)x_i = 0$ for any $i \neq j$. Since the order of g is q, $g^{w_i - w_j} \neq 1$ for $i \neq j$. Hence we have $x_i = 0$ for $i \neq j$. Therefore $x = (0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ but this point is not on X since $\alpha_j \neq 0$. □

Let $\pi : X \to M$ be the natural morphism which is a covering of degree q.

For $a \neq b$, let $x_{a,b}$ be a point on X defined by $z_j = 0$ for all $j \neq a, b$. Such a point satisfies the equation $\alpha_a z_a^q + \alpha_b z_b^q = 0$, hence is unique up to the G-action, and defines a unique point on M, which will be denoted by $y_{a,b}$.

3. The fundamental case

In this section we assume $n = q - 2$. Let X and $F = \alpha_0 z_0^q + \cdots + \alpha_{n+1} z_{n+1}^q + \cdots$ be as before. We assume that $\alpha_0, \dots, \alpha_{n+1} \neq 0$ and X is smooth. Set $F_i = (\partial/\partial z_i)F$. Since X is smooth, $F_0(x) = \cdots = F_{q-1}(x) = 0$ does not occur at any $x \in X$.

PROPOSITION 2. *The canonical sheaf of $M = X/G$ is trivial.*

Proof. The natural surjective morphism $\pi : X \to M$ is the étale Galois covering. Since $H^i(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}^{n+1}}(-q)) = 0$ for $1 \leq i \leq n$, we have

$$H^i(X, \mathcal{O}_X) = 0 \qquad \text{for } 1 \leq i \leq n-1. \tag{1}$$

Since π is finite morphism, we have $H^i(M, \pi_* \mathcal{O}_X) = H^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq n-1$. Hence we have

$$H^i(M, \mathcal{O}_M) = 0 \qquad \text{for } 1 \leq i \leq n-1, \tag{2}$$

because of $\mathcal{O}_M$ is a component of direct sum of $\pi_* \mathcal{O}_X$. By the adjunction formula we have $\omega_X \cong \mathcal{O}_X$. Hence we have

$$H^n(X, \mathcal{O}_X) \cong H^0(X, \omega_X) = H^0(X, \mathcal{O}_X) = K, \tag{3}$$

by the Serre duality. Since $n = q - 2$ is an odd number, we have $\chi(\mathcal{O}_X) = 0$ by (1) and (3). Hence we have

$$\chi(\mathcal{O}_M) = (1/\deg \pi)\chi(\mathcal{O}_X) = 0. \tag{4}$$

Since (1) and (4), we have

$$h^0(M, \omega_M) = h^n(M, \mathcal{O}_M) = h^0(M, \mathcal{O}_M) = 1. \tag{5}$$

Since $\omega_X \cong \mathcal{O}_X$, hence ω_M is numerically trivial as an element of $\mathrm{Pic}(M)$. Since (5), we have $\omega_M \cong \mathcal{O}_M$. $\square$

Let H_j be the divisor on X defined by $(z_j = 0)$. Then G acts on H_j freely. Hence $D_j = H_j/G$ is an ample divisor on M. Since $H_j^n = q$ and $\pi^* D_j = H_j$, we have $D_j^n = 1$.

Let $N = D_1 - D_0$ and $N' = D_j - D_{j-1}$. Since

$$\mathrm{div}\left(\frac{z_{j-1} z_1}{z_j z_0}\right) = H_j - H_{j-1} - H_1 + H_0$$

and $f = z_{j-1}z_1/(z_jz_0)$ is G-invariant, thus $f \in K(M)$ and $\mathrm{div}(f) = N - N'$. Therefore N' and N are linearly equivalent. Hence we have $D_i - D_j = (i-j)N$ in $\mathrm{Pic}(M)$. Moreover $\mathcal{O}_M(N)$ is a q-torsion element in $\mathrm{Pic}(M)$.

Let $E = \sum_{i=0}^{q-1} a_iD_i + jN$ be a divisor on M where each $a_i \geq 0$ and $j \geq 0$. Since $D_i - D_0 = iN$, we can write $E = tD_0 + j'N$ where $t = \sum_{i=0}^{q-1} a_i$ and $j' \geq 0$. We consider divisors of the form $tD + jN$, where $D = D_0$.

THEOREM 3. *Let M be as before.*

(1) *$|K_M + nD + jN|$ has finite base points. They are exactly the $y_{a,b}$'s such that $a + b + j \equiv 0$.*

(2) *$K_M + (n+d)D + jN$ is not d-spanned for $d \leq q-1$.*

Proof. For simplicity, denote H_0 by H. Recall that $K_M = 0$ in this case.

(1) Since

$$\pi^* : H^0(M, nD + jN) \to H^0(X, nH)$$

is injective, we identify $H^0(M, nD + jN)$ with its image of π^*. Then S_n^j can be viewed as a basis of $H^0(M, nD + jN)$.

Let $a \neq b$. Take a point $x_{a,b}$ in $\pi^{-1}(y_{a,b})$. Assume that $a + b \equiv -j$. We will show that $y_{a,b}$ is a base point of $|K_M + nD + jN|$. If not, there exists $f \in S_n^j$ such that $f(x_{a,b}) \neq 0$. Since $z_i = 0$ for $i \neq a, b$ at $x_{a,b}$, f must be of the form $f_{a,b}^{(i)} = z_a^i z_b^{n-i}$ for $i = 0, \dots, n$. Hence

$$|f_{a,b}^{(i)}| = ai + b(q-2-i) \equiv (a+b)i - 2b(i+1) \equiv -ji - 2b(i+1) \equiv j.$$

Hence we have $(2b + j)(i+1) \equiv 0$. Since $i \neq q-1$, we have $j \equiv -2b$. Since $a + b \equiv -j \equiv 2b$, we have $a = b$, which is a contradiction. Therefore $y_{a,b}$ is a base point of $|K_M + nD + jN|$.

Conversely, let $x = (x_0 : \cdots : x_{q-1})$ be a base point of $|K_M + nD + jN|$. Let $t \equiv -j/2$. Since $f = z_t^n \in S_n^j$, we have $x_t = 0$ at x. Suppose that $x_a \neq 0$ for $a \neq t$. Consider the form $f = z_a^i z_b^{n-i}$ where $b \neq a$ and $i = 0, \dots, n$. Since $n = q-2$, $f \in S_n^j$ if and only if $(a-b)i - 2b \equiv j$. For any a, b, j with $a \neq b$, this relation gives a unique $i \in \boldsymbol{Z}/q\boldsymbol{Z}$. Moreover $i \equiv q-1$ only if $a + b \equiv -j$. Hence, if $a + b + j \not\equiv 0$, we have $z_a^i z_b^{n-i} \in S_n^j$ for some $i \leq n$. Since x is a base point and $x_a \neq 0$, we have $x_b \neq 0$ for $b \not\equiv -a-j$. This shows that the base points of $|K_M + nD + jN|$ are exactly the points $y_{a,b}$ with $a + b + j \equiv 0$.

(2) We assume that $d \geq 1$ (the case of $d = 0$ is (1)). As in (1), we may regard S_{n+d}^j as the basis of $H^0((n+d)D + jN)$.

Take a, b and $c \equiv (a+b+j)/d$. If $c \equiv a$ or $c \equiv b$, we can replace (a,b) by $(a+1, b-1)$ or $(a-1, b+1)$ so that a, b, c are mutually distinct in $\boldsymbol{F}_q$, because $q \geq 5$.

Let $L_c = X \cap (\bigcap_{i \neq a,b,c} (z_i = 0))$ and let $Z = (d+1)y_{a,b}$ be a curvilinear zero cycle along $\pi(L_c)$. The image of the map $\rho : H^0(M, (n+d)D + jN) \to H^0(Z, (n+d)D + jN)$ is generated by monomials h in S_{n+d}^j such that $h(L_c) \neq 0$. They are of the form $h = z_c^k z_a^i z_b^{n+d-i-k}$ with $k \leq d$. In case

$k = d$, we have $i \le n = q - 2$ and $i + 1 \not\equiv 0$. But $j \equiv cd + ai - (2 + i)b$ yields $(a - b)(i + 1) \equiv 0$ since $c \equiv (a + b + j)/d$. This contradicts $a \ne b$, so we conclude $k < d$.

Next, recall that $F|_{L_c}$ is of the form $z_a^q + z_b^q + z_c^q +$ (*other terms*). Hence $z_a^q|_{L_c}$ can be expressed by terms of smaller degrees in z_a. Thus we may assume that g in of the form $z_c^k z_a^i z_b^{n+d-i-k}$ with $k < d$ and $0 \le i < q$. Since $j \equiv kc + ai + b(n + d - i - k) \equiv (a - b)i + k(c - b) + b(n + d)$, i is uniquely determined for each $k = 0, \dots, d - 1$. Therefore $\mathrm{Im}(\rho)$ is generated by d monomials, so $\dim \mathrm{Im}(\rho) \le d$, hence ρ is not surjective, and $K_M + (n + d)D + jN$ is not d-spanned. □

We get the following corollary by setting $j = 0$:

COROLLARY. *$|K_M + nD|$ has finitely many base points and $K_M + (n + d)D$ is not d-spanned for $0 \le d \le q - 1$.*

4. General case

By taking a divisor D_i in the preceding example, we will get examples of dimension $q - 3$. This process gives examples of any dimension.

Let X' be a smooth hypersurface in $\boldsymbol{P}^{n+1}$ defined by a G-invariant homogeneous polynomial of degree q as in section 2, and set $M' = X'/G$, where the G-action is given by $g(z'_0 : \cdots : z'_{n+1}) = (g^{w_0} z'_0 : \cdots : g^{w_{n+1}} z'_{n+1})$ for some $w_0, \dots, w_{n+1}$ with $0 \le w_0 < w_1 < \cdots < w_{n+1} \le q - 1$. Let $T = \{0, \cdots, q - 1\} \setminus \{w_0, \dots, w_{n+1}\}$ and $s = \# T = q - 2 - n$. Then there are a linear embedding $\iota : \boldsymbol{P}^{n+1} \subset \boldsymbol{P}^{q-1}$ and a smooth hypersurface X in $\boldsymbol{P}^{q-1} = \{(z_0 : \cdots : z_{q-1})\}$ defined by a G-invariant homogeneous polynomial of degree q such that $X' = X \cap \boldsymbol{P}^{n+1}$, where the G-action on $\boldsymbol{P}^{q-1}$ is given by $g : z_j \mapsto g^j z_j$ and $z'_j = \iota^* z_{w_j}$. The pair X and $M = X/G$ is a fundamental one as in section 3. Moreover M' is identified with the submanifold $\bigcap_{i \in T} D_i$ in M. Set $D' = D_{w_0}|_{M'}$ and $N' = N|_{M'} \in \mathrm{Pic}(M')$. Then

$$K_{M'} + tD' + jN' = \left(K_M + (s + t)D_{w_0} + \left(\sum_{i \in T} (i - w_0) + j \right) N \right) \Bigg|_{M'}$$
$$= \left(K_M + (s + t)D + \left(\sum_{i \in T} i + tw_0 + j \right) N \right) \Bigg|_{M'}$$

by the adjunction formula.

Since H is an ample divisor on X, we have

$$0 = H^1(X, \mathcal{O}_X(-sH)) \cong H^1(M, \pi_* \mathcal{O}_X(-sH)) = \oplus_j H^1(M, \mathcal{O}_M(-sD + jN)).$$

Hence we have $H^1(M, \mathcal{O}_M(-sD + jN)) = 0$ for any j. Therefore the restriction map $H^0(M, K_M + (s + t)D + (\sum i + tw_0 + j)N) \to H^0(M', K_{M'} + tD' + jN')$ is surjective for any t, j. Hence $|K_{M'}+tD'+jN'|$ has a base point if $|K_M+(s+t)D+j'N|$ has a base point on $\bigcap_{i\in T} D_i$ for $j'=\sum_{i\in T} i+tw_0+j$. Similarly, $K_{M'}+tD'+jN'$ is not d-spanned if $H^0(M,K_M+(s+t)D+j'N)$ is not d-spanned at a point in $\bigcap_{i\in T} D_i$ along curvilinear zero cycles in M'.

By this observation and Theorem 3, we get the following:

THEOREM 4. *Let $n\geq 2$ and let M' be as before. For $a\neq b$, let $y_{a,b}$ be the point on M' defined by $z_i'=0$ for all $i\neq a,b$. Let $j'=\sum_{i\in T} i+(n+d)w_0+j$. Then*

(1) *If there exist a, b such that $w_a+w_b+j'\equiv 0$ and that $a\neq b$ then $y_{a,b}$ is a base point of $|K_{M'}+nD'+jN'|$.*

(2) *If there exists w_c such that $w_c\equiv (w_a+w_b+j')/d$ and that c is different from a, b then $K_{M'}+(n+d)D'+jN'$ is not d-spanned for $d\leq q-1$.*

Proof. If $q=n+2$, this is just Theorem 3. Let $q>n+2$.

(1) By Theorem 3, $y_{a,b}$ is a base point of $|K_M+(q-2)D+j'N|$ if $w_a+w_b+j'\equiv 0$. Hence the above observation applies.

(2) By Theorem 3, $|K_M+(q-2+d)D+j'N|$ is not d-spanned along the curvilinear zero cycle Z over $y_{a,b}$ along L_c. Hence the above observation applies. □

If $n=q-2$ then the canonical sheaf of M is trivial and M is of Kodaira dimension 0. If $n<q-2$, the canonical sheaf of M is ample and M is of general type.

Now we give explicit examples by applying Theorem 4.

(1) Consider first the case where $n\geq 3$ and $d\geq 0$. Choose a prime number q such that $q\neq p$ and that $q\geq \max(n+3,d+1)$.

Assume that $q-2-n=2m$ is even and set

$$T=\left\{\frac{q-1}{2}-(m-1),\frac{q-1}{2}-(m-2),\dots,\frac{q-1}{2}+m\right\}.$$

Then we have $\sum_{i\in T} i=0$ and $\{0,1,2,q-2,q-1\}\subset\{w_0,\dots,w_{n+1}\}$. Let $D'=D_0|_{M'}$. In this case, we have $j'\equiv 0$. Let $\{a,b\}=\{1,q-1\}$. Then we have $a+b+j'\equiv 0$ and $c\equiv (a+b+j')/d\equiv 0$. Hence $K_{M'}+nD'$ has a base point $y_{1,q-1}$ and $K_{M'}+(n+d)D'$ is not d-spanned along L_0 at $y_{1,q-1}$.

Assume next that $q-2-n=2m+1$ is odd. Note that $(q-1)/2+m<q-3$. Indeed, $2(q-3)-(q-1)-2m=q-2m-5=n-2>0$. Let

$$T=\left\{\frac{q-1}{2}-(m-1),\frac{q-1}{2}-(m-2),\dots,\frac{q-1}{2}+m,q-3\right\}.$$

Then we have $\sum_{i\in T} i=-3$ and $\{0,1,2,q-2,q-1\}\subset\{w_0,\dots,w_{n+1}\}$. Let $D'=D_0|_{M'}$. In this case, we have $j'\equiv -3$. Let $\{a,b\}=\{1,2\}$. Then we have $a+b+j'\equiv 0$ and $c\equiv (a+b+j')/d\equiv 0$. Hence $K_{M'}+nD'$ has a base point $y_{1,2}$ and $K_{M'}+(n+d)D'$ is not d-spanned along L_0 at $y_{1,2}$.

(2) Consider next the case where $n = 2$ and $d \geq 0$. Choose a prime number q with $q \geq \max(n+3 = 5, d+3)$ and set $c \equiv -1/(d+1) \in \boldsymbol{Z}/q\boldsymbol{Z}$. Since $q \geq d+3$, we have $c \neq 0, 1$. Let $a = 0$ and b be different from $1, a, c$. Define the ambient space $\boldsymbol{P}^{n+1}$ of X to be $\bigcap_{i \neq 1,a,b,c} (x_i = 0)$ in $\boldsymbol{P}^{q-1}$. Then $(x_a : x_1 : x_b : x_c)$ gives $i \neq 1, a, b, c$ homogeneous coordinater of $\boldsymbol{P}^{n+1}$. Let $D' = D_0|_{M'}$. In this case, we have $j' \equiv -(1+a+b+c)$. If $d = 0$ then $a + b + j' \equiv -(c+1) \equiv 0$. Hence $K_{M'} + nD'$ has a base point $y_{a,b}$. If $d > 0$ then $(a+b+j')/d \equiv -(1+c)/d \equiv c$. Hence $K_{M'} + (n+d)D'$ is not d-spanned along L_c at $y_{a,b}$.

Thus we have the examples of $n \geq 2$ for any $d \geq 0$ and $p \geq 0$.

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