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KILLING VECTOR FIELDS ON TANGENT SPHERE BUNDLES

Dedicated to the late Professor Hitoshi Takagi

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Abstract

The tangent bundle TM of a Riemannian manifold (M,g) admits a Riemannian metric G called the Sasaki metric. The general forms of Killing vector fields on (TM,G) are determined by Tanno [4]. The total space of the tangent sphere bundle $T^{\lambda}M$ is the set of all tangent vectors of (M,g) whose lengths are all equal to $\lambda(\neq 0)$, and it is a hypersurface of (TM,G). In the present paper we study Killing vector fields on $T^{\lambda}M$ which are fiber preserving. The main theorem of this paper shows that any fiber preserving Killing vector field on $(T^{\lambda}M, G^{\lambda})$ is extended to a Killing vector field on (TM,G). Moreover, we will find a Riemannian manifold (M,g) such that any Killing vector fields on T^1M is fiber preserving.

§ 1. Introduction

Let (M,g) be a Riemannian manifold of dimension n and ∇ its Levi-Civita connection. Let $\pi : TM \to M$ denote the bundle projection. For each $u \in TM$, we denote by V_u the kernel of $\pi_*|_{T_uTM}$. We call it the vertical subspace of T_uTM. The connection map $K : TTM \to TM$ corresponding to ∇ is defined by

$$K(A) = \lim_{t\to 0} \frac{\tau_0^t(u(t)) - u}{t} \quad \text{for } A \in T_uTM,$$

where $u(t)$, $-\varepsilon < t < \varepsilon$, is a differentiable curve on TM satisfying $u(0) = u$, $\dot{u}(0) = A$, and $\tau_0^t(u(t))$ denotes the parallel displacement of $u(t)$ from $\pi(u(t))$ to $\pi(u)$ along the geodesic are joining $\pi(u(t))$ and $\pi(u)$ in a normal neighborhood of $\pi(u)$. For each $u \in TM$, we denote by H_u the kernel of $K|_{T_uTM}$. We call it the horizontal subspace of T_uTM. At each point $u \in TM$, the tangent space T_uTM is decomposed as a direct sum $V_u \oplus H_u$. Then the Sasaki metric G on TM is defined by

$$G(Z,W) = g(\pi_*(Z), \pi_*(W)) + g(K(Z), K(W)) \quad \text{for } Z, W \in TTM.$$

We need some notation to explain the main result of this paper. $\mathscr{F}(M)$ denotes the ring of all C^{∞}-functions on M, $\mathscr{X}(M)$ the $\mathscr{F}(M)$-module of all C^{∞}-vector

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fields on M. For $X \in \mathscr{X}(M)$, there is uniquely $X^H \in \mathscr{X}(TM)$ and uniquely $X^V \in \mathscr{X}(TM)$ such that

$$\pi_*(X^H) = X, \quad K(X^H) = 0, \quad \pi_*(X^V) = 0, \quad K(X^V) = X.$$

We call X^H the horizontal lift of X and X^V the vertical lift of X. For $F \in \mathscr{F}(M)$, we define $F^c \in \mathscr{F}(TM)$ by $F^c(u) = uF$ for $u \in TM$. $X \in \mathscr{X}(M)$ has a unique lift X^c to TM such that $(X^c)(F^c) = (XF)^c$ for any $F \in \mathscr{F}(M)$. We call it the complete lift of X. For each $u \in TM$, there is a unique isomorphism $I_u : V_u \to T_{\pi(u)}M$ such that $(I_u(Z))(F) = Z(F^c)$ for any $F \in \mathscr{F}(M)$ and $Z \in V_u$. Let C be a tensor field of (1, 1) type on M. Then we define $\iota C \in \mathscr{X}(TM)$ and ${}^*C \in \mathscr{X}(TM)$ by

$$(\iota C)_u = (I_u^{-1} \circ C)(u) \ (u \in TM), \quad {}^*C = (I(\iota C))^H.$$

For each $\lambda > 0$, the set $T^\lambda M \stackrel{\text{def}}{=} \{u \in TM \mid g(u,u) = \lambda^2\}$ is considered as a hypersurface of TM and $j^\lambda : T^\lambda M \to TM$ denotes the immersion. Especially we call $T^1 M$ the unit tangent bundle of M. We denote by G^λ the induced metric on $T^\lambda M$. We define a diffeomorphism $f^\lambda : T^1 M \to T^\lambda M$ by $f^\lambda(u) = \lambda \cdot u$, $u \in T^1 M$. Put $\sigma_0 = \{u \in TM \mid g(u,u) = 0\}$. For $Z^1 \in \mathscr{X}(T^1 M)$, we define $\overline{Z^1} \in \mathscr{X}(TM \backslash \sigma_0)$ by

$$\overline{Z^1}_u = (j^\lambda \circ f^\lambda)_*((Z^1)|_{(j^\lambda \circ f^\lambda)^{-1}(u)}) \quad \text{for } u \in TM\backslash\sigma_0 \quad \text{and} \quad \lambda = \sqrt{g(u,u)}.$$

For $Z^\lambda \in \mathscr{X}(T^\lambda M)$, we define $\overline{Z^\lambda} \in \mathscr{X}(TM\backslash\sigma_0)$ by $\overline{Z^\lambda} = \overline{(f^\lambda)_*^{-1}(Z^\lambda)}$. $\overline{Z^\lambda}$ is tangent to $T^\lambda M$ and $\overline{Z^\lambda}|_{T^\lambda M} = j^\lambda_*(Z^\lambda)$. We often consider Z^λ to be a vector field on $TM\backslash\sigma_0$ by the correspondence $Z^\lambda \to \overline{Z^\lambda}$.

We call $X \in \mathscr{X}(TM)$ a vertical vector field on TM, if $X_u \in V_u$ for any $u \in TM$. If $\overline{X^\lambda}_u \in V_u$ for any $u \in TM\backslash\sigma_0$, $X^\lambda \in \mathscr{X}(T^\lambda M)$ is called a vertical vector field on $T^\lambda M$. We call $Z \in \mathscr{X}(TM)$ a fiber preserving vector field on TM, if the commutator product $[Z, X]$ is a vertical vector field on TM for any vertical $X \in \mathscr{X}(TM)$. If the commutator product $[Z^\lambda, X^\lambda]$ is a vertical vector field on $T^\lambda M$ for any vertical $X^\lambda \in \mathscr{X}(T^\lambda M)$, $Z^\lambda \in \mathscr{X}(T^\lambda M)$ is called a fiber preserving vector field on $T^\lambda M$.

The main purpose of this paper is to prove that any fiber preserving Killing vector field on $(T^\lambda M, G^\lambda)$ is extended to a Killing vector field on (TM, G). Namely we show the following.

THEOREM. *Let Z^λ be a Killing vector field on $(T^\lambda M, G^\lambda)$ which preserves the fiberes. Then there exists a Killing vector field Z on (TM, G) such that Z is tangent to $T^\lambda M$ and $Z|_{T^\lambda M} = j^\lambda_*(Z^\lambda)$.*

Conversely, let Z be a Killing vector field on (TM, G) which is tangent to $T^\lambda M$. Then there exists a fiber preserving Killing vector field Z^λ on $(T^\lambda M, G^\lambda)$ such that $j^\lambda_*(Z^\lambda) = Z|_{T^\lambda M}$.

Remark. If a Killing vector field Z on (TM, G) is tangent to $T^\lambda M$, then Z

is automatically a fiber preserving vector field on (TM, G). We will see it in the proof of Theorem.

This theorem and the result of Tanno [4] imply the following.

COROLLARY. *Let*
(i) *X be a Killing vector field on (M, g),*
(ii) *P be a $(1,1)$-tensor field on M, which satisfies*
(i-1) $\nabla P = 0$, *and*
(ii-2) $g(PU, V) + g(U, PV) = 0$ *for any* $U, V \in \mathscr{X}(M)$.
Then $(X^c + \iota P)|_{T^\lambda M}$ is considered as a fiber preserving Killing vector field on $(T^\lambda M, G^\lambda)$.

Conversely every fiber preserving Killing vector field on $(T^\lambda M, G^\lambda)$ is of this form.

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§2. Fiber preserving vector fields

Let (M, g) be a Riemannian manifold. For a chart (U, φ) of M, a chart $(\pi^{-1}(U), \tilde{\varphi})$ of the tangent bundle TM is naturally defined by

$$\tilde{\varphi}\left(y^i\left(\frac{\partial}{\partial x^i}\right)_p\right) = (x^1(p), \dots, x^n(p), y^1, \dots, y^n), \quad (y^i) \in \boldsymbol{R}^n,$$

where $\varphi(p) = (x^1(p), \dots, x^n(p))$ for $p \in U$. Using these charts, the horizontal subspace H_u and the vertical subspace V_u, $u \in TM$, of T_uTM are expressed by

$$H_u = \left\{ a^k\left(\frac{\partial}{\partial x^k}\right)_u - \Gamma^k_{ij} y^i a^j \left(\frac{\partial}{\partial y^k}\right) \middle| (a^j) \in \boldsymbol{R}^n \right\}, \quad V_u = \left\{ a^k\left(\frac{\partial}{\partial y^k}\right)_u \middle| (a^k) \in \boldsymbol{R}^n \right\}$$

and the components of the Sasaki metric G given by

$$G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} + g_{ab}\Gamma^a_{is}\Gamma^b_{jt} y^s y^t, \quad G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = g_{jb}\Gamma^b_{is} y^s, \quad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij},$$

where Γ^k_{ij}, $i, j, k = 1, \dots, n$, denote the Christoffel's symbols of the Riemannian metric g and there the Einstein convention for the summing is used. Let X be a vector field on M. Putting $X = X^k(\partial/\partial x^k)$ on U, we get

$$X^H = X^k \frac{\partial}{\partial x^k} - \Gamma^k_{ij} y^i X^j \frac{\partial}{\partial y^k}, \quad X^V = X^k \frac{\partial}{\partial y^k},$$

$$X^c = X^k \frac{\partial}{\partial x^k} + y^l \frac{\partial X^k}{\partial x^l} \frac{\partial}{\partial y^k}$$

on $\pi^{-1}(U)$.

Let Z be a fiber preserving vector field on TM and put $Z = Z^k(\partial/\partial x^k) + Z^{k+n}(\partial/\partial y^k)$ on $\pi^{-1}(U)$. From the definition of fiber preserving vector field, we can see that conditions

$$\frac{\partial}{\partial y^l} Z^k = 0 \quad \text{for } k,\ l = 1, \dots, n$$

are a necessary and sufficient condition for Z to be a fiber preserving vector field. For example, the complete lift X^c of a vector field X on M, and ιC for a tensor field of (1, 1) type on M are fiber preserving vector field. For a fiber preserving vector field Z on TM, we define a vector field $\underline{Z}$ on M by $(\underline{Z})_{\pi(u)} = \pi_*(Z_u)$, $u \in TM$. Let $\pi^\lambda : T^\lambda M \to M$ denotes the projection. For a fiber preserving vector field Z^λ on $T^\lambda M$, we define also a vector field on M by $(\underline{Z^\lambda})_{\pi(u)} = (\pi^\lambda)_*((Z^\lambda)_u)$, $u \in T^\lambda M$.

PROPOSITION 1. *If Z is a fiber preserving Killing vector field on (TM, G), then $\underline{Z}$ is a Killing vector field on (M, g).*

Proof. In a neighborhood of an arbitrary point $u_0 \in TM$, we use the coordinates such that $\Gamma^k_{ij}(\pi(u_0)) = 0$. Let $L_Z G$ denote the Lie derivative of G with respect to Z. From the conditions that $(L_Z G)(\partial/\partial x^i, \partial/\partial x^j) = 0$, we have that

$$\begin{aligned} &Z^k \frac{\partial}{\partial x^k}(g_{ij} + g_{ab}\Gamma^a_{is}\Gamma^b_{jt} y^s y^t) + Z^{k+n}(g_{ab}\Gamma^a_{is}\Gamma^b_{jk} + g_{ab}\Gamma^a_{ik}\Gamma^b_{js}) y^s \\ &+ \frac{\partial}{\partial x^i} Z^k \cdot (g_{kj} + g_{ab}\Gamma^a_{ks}\Gamma^b_{jt} y^s y^t) + \frac{\partial}{\partial x^i} Z^{k+n} \cdot g_{kb}\Gamma^b_{js} y^s \\ &+ \frac{\partial}{\partial x^j} Z^k \cdot (g_{ik} + g_{ab}\Gamma^a_{is}\Gamma^b_{kt} y^s y^t) + \frac{\partial}{\partial x^j} Z^{k+n} \cdot g_{kb}\Gamma^b_{is} y^s = 0, \end{aligned}$$

and hence, we see immediately that $(L_{\underline{Z}} g)(\partial/\partial x^i, \partial/\partial x^j)_{\pi(u_0)} = 0$. □

This Proposition is not used in this paper, but the method of the proof is applied to prove the following Lemma 1, (i).

Now we study fiber preserving Killing vector fields on $(T^\lambda M, G^\lambda)$. We denote by $i(M, g)$ the Lie algebra of Killing vector fields on (M, g).

LEMMA 1. *If Z^λ is a fiber preserving Killing vector field on $(T^\lambda M, G^\lambda)$, then we have that*

(i) $\underline{Z^\lambda} \in i(M, g)$,

(ii) $\overline{Z^\lambda}|_{T^\mu M} \in i(T^\mu M, G^\mu)$ *for any* $\mu > 0$,

(iii) $\overline{Z^\lambda} \in i(TM \backslash \sigma_0, G')$, *where we put* $G' = G|_{TM\backslash\sigma_0}$,

(iv) *There exists* $\overline{\overline{Z^\lambda}} \in i(TM, G)$ *such that* $\overline{\overline{Z^\lambda}}|_{TM\backslash\sigma_0} = \overline{Z^\lambda}$.

Proof. (i). In a neighborhood $\pi^{-1}(U)$ of an arbitrary point $u_0 \in T^\lambda M$, we use the coordinates such that $\Gamma_{ij}^k(\pi_\lambda(u_0)) = 0$. The horizontal lifts $(\partial/\partial x^i)^H$ and $(\partial/\partial x^j)^H$ of vector fields $\partial/\partial x^i$ and $\partial/\partial x^j$ on U are tangent to $T^\lambda M$ at any point of $\pi^{-1}(U) \cap T^\lambda M$, hence they can be considered as vector fields on $(\pi^\lambda)^{-1}(U)$. Then we see that

$$(L_{Z^\lambda} G^\lambda)\left(\left(\frac{\partial}{\partial x^i}\right)^H\Big|_{T^\lambda M}, \left(\frac{\partial}{\partial x^j}\right)^H\Big|_{T^\lambda M}\right)_{|j(u_0)} = (L_{\underline{Z^\lambda}} g)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_{|\pi(u_0)}.$$

This implies that $L_{\underline{Z^\lambda}} g = 0$ at each point $\pi^\lambda(u_0)$ of M. We proved the first statement of Lemma 1.

(ii). Let $W = \overline{Z^\lambda}|_{T^\mu M}$. Suppose that A^λ and B^λ are arbitrary fiber preserving vector fields on $T^\lambda M$. At any $u \in T^\mu M$, we have that

$$\begin{aligned}
&(L_W G^\mu)(\overline{A^\lambda}|_{T^\mu M}, \overline{B^\lambda}|_{T^\mu M}) \\
&= W G^\mu(\overline{A^\lambda}|_{T^\mu M}, \overline{B^\lambda}|_{T^\mu M}) - G^\mu([W, \overline{A^\lambda}|_{T^\mu M}], \overline{B^\lambda}|_{T^\mu M}) \\
&\quad - G^\mu(\overline{A^\lambda}|_{T^\mu M}, [W, \overline{B^\lambda}|_{T^\mu M}]) \\
&= \overline{Z^\lambda} G(\overline{A^\lambda}, \overline{B^\lambda}) - G^\mu([(f^\mu \circ (f^\lambda)^{-1})_* Z^\lambda, (f^\mu \circ (f^\lambda)^{-1})_* A^\lambda], \overline{B^\lambda}|_{T^\mu M}) \\
&\quad - G^\mu(\overline{A^\lambda}|_{T^\mu M}, [(f^\mu \circ (f^\lambda)^{-1})_* Z^\lambda, (f^\mu \circ (f^\lambda)^{-1})_* B^\lambda]) \\
&= \overline{Z^\lambda} G(\overline{A^\lambda}, \overline{B^\lambda}) - G^\mu(\overline{[Z^\lambda, A^\lambda]}|_{T^\mu M}, \overline{B^\lambda}|_{T^\mu M}) - G^\mu(\overline{A^\lambda}|_{T^\mu M}, \overline{[Z^\lambda, B^\lambda]}|_{T^\mu M}). \\
&= \overline{Z^\lambda} G(\overline{A^\lambda}, \overline{B^\lambda}) - G(\overline{[Z^\lambda, A^\lambda]}, \overline{B^\lambda}) - G(\overline{A^\lambda}, \overline{[Z^\lambda, B^\lambda]}).
\end{aligned}$$

Since Z^λ, A^λ and B^λ are fiber preserving vector fields on $T^\lambda M$, there exist Z^i, A^j, $B^k \in \mathscr{F}(\boldsymbol{R}^n)$ and Z^{i+n}, A^{j+n}, $B^{k+n} \in \mathscr{F}(\boldsymbol{R}^{2n})$, $i, j, k = 1, \ldots, n$ such that

$$j_*^\lambda(Z^\lambda) = Z^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i} + Z^{i+n}(x^1, \ldots, x^n, y^1, \ldots, y^n)\frac{\partial}{\partial y^i},$$

$$j_*^\lambda(A^\lambda) = A^j(x^1, \ldots, x^n)\frac{\partial}{\partial x^j} + A^{j+n}(x^1, \ldots, x^n, y^1, \ldots, y^n)\frac{\partial}{\partial y^j},$$

$$j_*^\lambda(B^\lambda) = B^k(x^1, \ldots, x^n)\frac{\partial}{\partial x^k} + B^{k+n}(x^1, \ldots, x^n, y^1, \ldots, y^n)\frac{\partial}{\partial y^k}.$$

Here, we define a function r on TM by $r(u)=\sqrt{g(u,u)}$ $(u \in TM)$. From the definition of the extended vector fields $\overline{Z^\lambda}$, $\overline{A^\lambda}$ and $\overline{B^\lambda}$, the vertical parts of $\overline{Z^\lambda}$, $\overline{A^\lambda}$ and $\overline{B^\lambda}$ are in proportion to the value of the function r, and hence we see that

$$\overline{Z^\lambda}=Z^i(x^1,\dots,x^n)\frac{\partial}{\partial x^i}+\frac{r}{\lambda}\cdot Z^{i+n}\left(x^1,\dots,x^n,\frac{\lambda y^1}{r},\dots,\frac{\lambda y^n}{r}\right)\frac{\partial}{\partial y^i},$$

$$\overline{A^\lambda}=A^i(x^1,\dots,x^n)\frac{\partial}{\partial x^j}+\frac{r}{\lambda}\cdot A^{j+n}\left(x^1,\dots,x^n,\frac{\lambda y^1}{r},\dots,\frac{\lambda y^n}{r}\right)\frac{\partial}{\partial y^j},$$

$$\overline{B^\lambda}=B^k(x^1,\dots,x^n)\frac{\partial}{\partial x^k}+\frac{r}{\lambda}\cdot B^{k+n}\left(x^1,\dots,x^n,\frac{\lambda y^1}{r},\dots,\frac{\lambda y^n}{r}\right)\frac{\partial}{\partial y^k}.$$

Then, for any $u \in T^\mu M$, we have

$$\begin{aligned}
G(\overline{A^\lambda},\overline{B^\lambda})|_u &= \Bigg(A^jB^k(g_{jk}+g_{ab}\Gamma^a_{js}\Gamma^b_{kt}y^sy^t)+\frac{r}{\lambda}A^jB^{k+n}g_{kb}\Gamma^b_{js}y^s \\
&\quad +\frac{r}{\lambda}A^{j+n}B^kg_{jb}\Gamma^b_{ks}y^s+\left(\frac{r}{\lambda}\right)^2A^{j+n}B^{k+n}g_{jk}\Bigg)\Bigg|_u \\
&= \Bigg(\left(1-\left(\frac{r}{\lambda}\right)^2\right)A^jB^kg_{jk}+\left(\frac{r}{\lambda}\right)^2\Bigg\{A^jB^k(g_{jk}+g_{ab}\Gamma^a_{js}\Gamma^b_{kt}\frac{\lambda y^s}{r}\frac{\lambda y^t}{r}) \\
&\quad +A^jB^{k+n}g_{kb}\Gamma^b_{js}\frac{\lambda y^s}{r}+A^{j+n}B^kg_{jb}\Gamma^b_{ks}\frac{\lambda y^s}{r}+A^{j+n}B^{k+n}g_{jk}\Bigg\}\Bigg)\Bigg|_u \\
&= \left(1-\left(\frac{\mu}{\lambda}\right)^2\right)g(\underline{A^\lambda},\underline{B^\lambda})|_{\pi^\mu(u)}+\left(\frac{\mu}{\lambda}\right)^2G^\lambda(A^\lambda,B^\lambda)|_{(f^\lambda\circ(f^\mu)^{-1})(u)},
\end{aligned}$$

and hence

$$\begin{aligned}
\overline{Z^\lambda}G(\overline{A^\lambda},\overline{B^\lambda})|_u &= \left(1-\left(\frac{\mu}{\lambda}\right)^2\right)\underline{Z^\lambda}g(\underline{A^\lambda},\underline{B^\lambda})|_{\pi^\mu(u)} \\
&\quad +\left(\frac{\mu}{\lambda}\right)^2Z^\lambda G^\lambda(A^\lambda,B^\lambda)|_{(f^\lambda\circ(f^\mu)^{-1})(u)}, \\
G(\overline{[Z^\lambda,A^\lambda]},\overline{B^\lambda})|_u &= \left(1-\left(\frac{\mu}{\lambda}\right)^2\right)g([\underline{Z^\lambda},\underline{A^\lambda}],\underline{B^\lambda})|_{\pi^\mu(u)} \\
&\quad +\left(\frac{\mu}{\lambda}\right)^2G^\lambda([Z^\lambda,A^\lambda],B^\lambda)|_{(f^\lambda\circ(f^\mu)^{-1})(u)}, \\
G(\overline{A^\lambda},\overline{[Z^\lambda,B^\lambda]})|_u &= \left(1-\left(\frac{\mu}{\lambda}\right)^2\right)g(\underline{A^\lambda},[\underline{Z^\lambda},\underline{B^\lambda}])|_{\pi^\mu(u)} \\
&\quad +\left(\frac{\mu}{\lambda}\right)^2G^\lambda(A^\lambda,[Z^\lambda,B^\lambda])|_{(f^\lambda\circ(f^\mu)^{-1})(u)}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&(L_W G^\mu)(\overline{A^\lambda}|_{T^\mu M}, \overline{B^\lambda}|_{T^\mu M}) \\
&= \left(1 - \left(\frac{\mu}{\lambda}\right)^2\right)(L_{\underline{Z^\lambda}} g)(\underline{A^\lambda}, \underline{B^\lambda})|_{\pi^\mu(u)} + \left(\frac{\mu}{\lambda}\right)^2 (L_{Z^\lambda} G^\lambda)(A^\lambda, B^\lambda)|_{(f^\lambda \circ (f^\mu)^{-1})(u)} \\
&= 0.
\end{aligned}$$

The last equality in the above follows from $Z^\lambda \in i(T^\lambda M, G^\lambda)$ and (i) of this lemma. We proved the second statement of Lemma 1.

(iii). Let $\Phi(u) = g(u,u)$ for $u \in TM\backslash\sigma_0$. Then the gradient vector field of Φ, $\operatorname{grad}\Phi = 2y^i(\partial/\partial y^i)$, is orthogonal to $T^\lambda M$ at any point of $T^\lambda M$. Since we know $L_{\overline{Z^\lambda}} \operatorname{grad}\Phi = 0$, we have that

$$(L_{\overline{Z^\lambda}} G')(\overline{A^\lambda}, \operatorname{grad}\Phi) = 0, \quad (L_{\overline{Z^\lambda}} G')(\operatorname{grad}\Phi, \operatorname{grad}\Phi) = 0$$

for any $A^\lambda \in \mathscr{X}(T^\lambda M)$. The statement (iii) follows from this fact and (ii) of this lemma.

(iv). Let $\overline{Z^\lambda} = \overline{Z^\lambda}^k(\partial/\partial x^k) + \overline{Z^\lambda}^{k+n}(\partial/\partial y^k)$. From (iii) of this lemma we know that

$$(L_{\overline{Z^\lambda}} G')\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0$$

on $TM\backslash\sigma_0$, which implies that

$$\overline{Z^\lambda}^k\left(\frac{\partial}{\partial x^k} g_{ij}\right) + \left(\frac{\partial}{\partial y^i} \overline{Z^\lambda}^{k+n}\right) \cdot g_{kj} + \left(\frac{\partial}{\partial y^j} \overline{Z^\lambda}^{k+n}\right) \cdot g_{ik} = 0.$$

Since $\overline{Z^\lambda}$ is a fiber preserving vector field on $T^\lambda M$, we may suppose that $(\partial/\partial y^l)\overline{Z^\lambda}^k = 0$. Differentiating the left hand side of the formula above with respect to y^l, we get

$$(\dagger) \qquad \left(\frac{\partial^2}{\partial y^l \partial y^i} \overline{Z^\lambda}^{k+n}\right) \cdot g_{kj} + \left(\frac{\partial^2}{\partial y^l \partial y^j} \overline{Z^\lambda}^{k+n}\right) \cdot g_{ik} = 0.$$

Putting $i = j$ in the formula ($\dagger$), we have that

$$\left(\frac{\partial^2}{\partial y^l \partial y^i} \overline{Z^\lambda}^{k+n}\right) \cdot g_{ki} = 0 \quad \left(\text{or} \left(\frac{\partial^2}{\partial y^i \partial y^j} \overline{Z^\lambda}^{k+n}\right) \cdot g_{ik} = 0\right).$$

Therefore, putting $l = i$ in the formula ($\dagger$), we have that

$$\left(\frac{\partial^2}{\partial y^i \partial y^i} \overline{Z^\lambda}^{k+n}\right) \cdot g_{kj} = 0.$$

Hence we get $(\partial^2/\partial y^i \partial y^i)\overline{Z^\lambda}^m = 0$ for $n < m \leq 2n.$ From the definition, $\overline{Z^\lambda}^m$ are in proportion to the value of the function r, $\overline{Z^\lambda}^m$ is of the form

$$\overline{Z^\lambda}^m = A_k^m(x^1, \dots, x^n) \cdot y^k,$$

where A_k^m are some functions on $\boldsymbol{R}^n$. Since $\overline{Z^\lambda}^m$ is a smooth function on $TM \backslash \sigma_0$, we see that

$$A_k^m(x^1, \dots, x^n) = \overline{Z^\lambda}^m(x^1, \dots, x^n, 0, \dots, \overset{k}{\check{1}}, \dots, 0) \in \mathscr{F}(\boldsymbol{R}^n),$$

for each m with $n < m \leq 2n$. Therefore, $\overline{Z^\lambda}^m \in \mathscr{F}(TM \backslash \sigma_0)$ is extended to a differentiable function on TM, hence $\overline{Z^\lambda}$ can be extended to a vector field $\overline{\overline{Z^\lambda}}$ on TM such that

$$\overline{\overline{Z^\lambda}} = \begin{cases} \overline{Z^\lambda}, & \text{on } TM \backslash \sigma_0 \\ j_*^0(\underline{Z^\lambda}), & \text{on } \sigma_0 \end{cases}$$

where $j^0 : M \to TM$ denotes the natural immersion. This satisfies the equations $L_{\overline{\overline{Z^\lambda}}} G = 0$ on σ_0. We proved all statements of Lemma 1. □

Before we prove Theorem, we review the results of Tanno ([4], Theorem A.).

For $X \in \mathscr{X}(M)$, we define $X^\# \in \mathscr{X}(TM)$ by $X^\# = X^V +^* (T_X)$, where T_X is a tensor filed of (1, 1) type on M such that

$$g(T_X U, V) + g(U, \nabla_V X) = 0 \quad \text{for any } U, V \in \mathscr{X}(M).$$

The general forms of Killing vector fields on (TM, G) are given by

THEOREM A. (Tanno) *Let (TM, G) be the tangent bundle with the Sasaki metric of a Riemannian manifold (M, g). Let*

(i) *X be a Killing vector field on (M, g),*

(ii) *P be a (1, 1)-tensor field on M, which satisfies*

(ii-1) *$\nabla P = 0$, and*

(ii-2) *$g(PU, V) + g(U, PV) = 0$ for any $U, V \in \mathscr{X}(M)$,*

(iii) *Y be a vector field on (M, g), which satisfies*

(iii-1) *$(\nabla^2 Y)(U, V) + (\nabla^2 Y)(V, U) = 0$ for any $U, V \in \mathscr{X}(M)$, and*

(iii-2) *$R(W, T_Y(U))V + R(W, T_Y(V))U = 0$ for any $U, V, W \in \mathscr{X}(M)$.*

Then the vector field Z on TM defined by $Z = X^c + \iota P + Y^\#$ is a Killing vector field on (TM, G). Conversely every Killing vector field on (TM, G) is of this form.

Now, we will prove Theorem.

Proof of Theorem. By (iv) of Lemma 1, we have the necessary condition. By the results of Tanno, we will prove the converse part of Theorem. Suppose that there exists a Killing vector field Z on (TM, G) such that it is tangent to $T^\lambda M$ at any point of $T^\lambda M$. By Theorem A, there exists a Killing vector field X on (M, g), a tensor field P of (1, 1) type on M and a vector field Y on M such that Z is decomposed as $Z = X^c + \imath P + Y^{\#}$. It is easy to see that X^c and $\imath P$ are tangent to $T^{\lambda}M$ at any point of $T^{\lambda}M$. Therefore $Y^{\#} = Z - X^c - \imath P$ is tangent to $T^{\lambda}M$. Hence, $G(Y^{\#}, \operatorname{grad}\Phi)|_u = 0$ for any $u \in T^{\lambda}M$. Put $Y = Y^{\imath}(\partial/\partial x^i)$. We have

$$G(Y^{\#}, \operatorname{grad}\Phi)|_u = g_{ij}Y^iy^j \quad \text{for any } (y)^i \in \boldsymbol{R}^n, g_{kl}y^ky^l = \lambda^2.$$

We see that Y is identically zero on M, which implies $Z = X^c + \imath P$. Since X^c and $\imath P$ preserve fibers of TM, Z preserves fibers of TM. By Sasaki ([3], II, Lemma 1), $Z|_{T^{\lambda}M}$ is considered as a Killing vector field on $(T^{\lambda}M, G^{\lambda})$. In consequence

$$Z|_{T^{\lambda}M} = (X^c + \imath P)|_{T^{\lambda}M},$$

is considered as a fiber preserving Killing vector field on $(T^{\lambda}M, G^{\lambda})$. □

§3. An example

When M is the sphere of radius λ in the Euclidean space, there is a Killing vector field on $(T^{\lambda}M, G^{\lambda})$, which is not fiber preserving [4]. In this section, we will find a Riemannian manifold M, on which any Killing vector fields on the unit tangent bundle are fiber preserving.

PROPOSITION 2. *Let (M, g) be a space of constant curvature c, where the dimension of M is greater than two and the curvature c satisfies $-0.30 < c < 0.32$. Then every Killing vector fields on the unit tangent bundle are fiber preserving.*

Proof. We will identify $j_*^{\lambda}(TT^{\lambda}M)$ with $TTM|_{T^{\lambda}M}$. For each $u \in T^{\lambda}M$, set

$$H_u^{\lambda} = H_u \cap T_uT^{\lambda}M, \quad V_u^{\lambda} = V_u \cap T_uT^{\lambda}M.$$

Then we have $T_uT^{\lambda}M = H_u^{\lambda} \oplus V_u^{\lambda}$ for each $u \in T^{\lambda}M$. It is easy to see that a necessary and sufficient condition that $Z \in \mathscr{X}(T^{\lambda}M)$ is fiber preserving is

$$(\psi_s)_*(X) \in V_{\psi_s(u)}^{\lambda} \text{ for any } X \in V_u^{\lambda}, u \in T^{\lambda}M,$$

where $\{\psi_s | 0 < s < \varepsilon\}$ denotes a local 1-parameter group of local transformations of Z. We need Lemma to prove Proposition.

LEMMA 2. *If a vector field Z on $T^{\lambda}M$ is not fiber preserving, then there exist $u_0 \in T^{\lambda}M$, $Y_0 \in V_{u_0}^{\lambda}$ $(Y_0 \neq 0)$ and, ε_0 with $0 < \varepsilon_0 < \varepsilon$, such that the horizontal part of $(\psi_s)_*(Y_0)$ is not zero on $0 < s < \varepsilon_0$.*

Proof. From the assumption, there exist $u \in T^{\lambda}M$, $Y \in V_u^{\lambda}$ $(Y \neq 0)$ and $t > 0$ such that $(\psi_t)_*(Y) \notin V_{\psi_t(u)}^{\lambda}$. Set

$$t_0 = \sup\{s | -\varepsilon < s \leq t, (\psi_s)_*(Y) \in V_{\psi_s(u)}^{\lambda}\}.$$

Since ψ_s is continuous, we have $t_0 < t$. Put

$$u_0 = \psi_{t_0}(u), \quad Y_0 = (\psi_{t_0})_*(Y), \quad \varepsilon_0 = t - t_0.$$

They satisfy the conditions stated in Lemma 2. □

The shape operator A^λ of $T^\lambda M$ in TM is computed by Blair [1]:

$$A^\lambda(X) = \begin{cases} 0, & \text{for } X \in H_u^\lambda, \\ -X/\lambda, & \text{for } X \in V_u^\lambda. \end{cases}$$

Let R, $\bar{R}$ and $\bar{R}^\lambda$ denote the curvature tensor of (M, g), (TM, G) and $(T^\lambda M, G^\lambda)$ respectively. And let α^λ denotes the second fundamental form of $T^\lambda M$ in TM. Then the equation of Gauss of $T^\lambda M$ in TM is

$$\bar{R}^\lambda(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + \alpha^\lambda(Y, Z)\alpha^\lambda(X, W) - \alpha^\lambda(X, Z)\alpha^\lambda(Y, W),$$

for $X, Y, Z, W \in TT^\lambda M$. The curvature tensor $\bar{R}$ of (TM, G) are calculated by Kowalski [2].

Now we are in a position to prove the Proposition. Suppose that there exists a Killing vector fields Z on $(T^\lambda M, G^\lambda)$ which is not fiber preserving. Then, by Lemma 2, there are $u_0 \in T^\lambda M$, $Y_0 \in V_{u_0}^\lambda (G_\lambda(Y_0, Y_0) = 1)$ and $\varepsilon > 0$ such that the horizontal part of $(\psi_s)_*(Y_0)$ is not zero on $0 < s < \varepsilon_0$. We define a vector field $P(s)$, along the curve $\psi_s(u_0)$ of $T_\lambda M$ by $P(s) = (\psi_s)_*(Y_0)$, $0 \le s < \varepsilon_0$. Let

$$P(s) = \boldsymbol{X}(s) + \boldsymbol{Y}(s), \quad \boldsymbol{X}(s) \in H_{\psi_s(u_0)}^\lambda, \quad \boldsymbol{Y}(s) \in V_{\psi_s(u_0)}^\lambda$$

be the orthogonal decomposition of $P(s)$. By taking ε_0 sufficiently small, if necessary, we may suppose $\boldsymbol{Y}(s) \neq 0$ for $0 \le s < \varepsilon_0$, because $\boldsymbol{Y}(0) = Y_0 \neq 0$. Put

$$\begin{aligned}
a(s) &= \sqrt{G^\lambda(\boldsymbol{X}(s), \boldsymbol{X}(s))}, \quad \text{for } 0 \le s < \varepsilon_0, \\
b(s) &= \sqrt{G^\lambda(\boldsymbol{Y}(s), \boldsymbol{Y}(s))}, \quad \text{for } 0 \le s < \varepsilon_0, \\
X(s) &= \begin{cases} 0, & \text{for } s = 0, \\ \boldsymbol{X}(s)/a(s), & \text{for } 0 < s < \varepsilon_0, \end{cases} \\
Y(s) &= \boldsymbol{Y}(s)/b(s), \quad \text{for } 0 \le s < \varepsilon_0.
\end{aligned}$$

Then we have $P(s) = a(s)X(s) + b(s)Y(s)$ for $0 \le s < \varepsilon_0$. Remark that $a(s)$ is a continuous function satisfying $a(s) \neq 0$ for $0 < s < \varepsilon_0$. We take a vector $\bar{Y}_0$ in $V_{u_0}^\lambda$ such that $G^\lambda(Y_0, \bar{Y}_0) = 0$ and $G^\lambda(\bar{Y}_0, \bar{Y}_0) = 1$. Put $\bar{P}(s) = (\psi_s)_*(\bar{Y}_0)$, $0 \le s < \varepsilon_0$, and let

$$\bar{P}(s) = \bar{\boldsymbol{X}}(s) + \bar{\boldsymbol{Y}}(s), \quad \bar{\boldsymbol{X}}(s) \in H_{\psi_s(u_0)}^\lambda, \quad \bar{\boldsymbol{Y}}(s) \in V_{\psi_s(u_0)}^\lambda$$

be the orthogonal decomposition of $\bar{P}(s)$. If necessary, we take ε_0 sufficiently small, and put

$$\bar{a}(s) = \sqrt{G^\lambda(\bar{\boldsymbol{X}}(s), \bar{\boldsymbol{X}}(s))}, \quad \text{for } 0 \le s < \varepsilon_0,$$

$$\bar{b}(s) = \sqrt{G^\lambda(\bar{\boldsymbol{Y}}(s), \bar{\boldsymbol{Y}}(s))}, \quad \text{for } 0 \le s < \varepsilon_0,$$

$$\bar{X}(s) = \begin{cases} 0, & \text{when } \bar{a}(s) = 0, \\ \bar{\boldsymbol{X}}(s)/\bar{a}(s), & \text{when } \bar{a}(s) \neq 0, \end{cases}$$

$$\bar{Y}(s) = \bar{\boldsymbol{Y}}(s)/\bar{b}(s), \quad \text{for } 0 \le s < \varepsilon_0.$$

Then we have $\bar{P}(s) = \bar{a}(s)\bar{X}(s) + \bar{b}(s)\bar{Y}(s)$ for $0 \le s < \varepsilon_0$. Since ψ_s is an isometric mapping, we see that

$$1 = G^\lambda(Y_0, Y_0) = G^\lambda(P(s), P(s)) = a(s)^2 + b(s)^2,$$

$$1 = G^\lambda(\bar{Y}_0, \bar{Y}_0) = G^\lambda(\bar{P}(s), \bar{P}(s)) = \bar{a}(s)^2 + \bar{b}(s)^2,$$

$$0 = G^\lambda(Y_0, \bar{Y}_0) = a(s)\bar{a}(s)G^\lambda(X(s), \bar{X}(s)) + b(s)\bar{b}(s)G^\lambda(Y(s), \bar{Y}(s)).$$

On the other hand, from the definitions of $X(s)$, $\bar{X}(s)$, $Y(s)$ and $\bar{Y}(s)$, we have that

$$G^\lambda(X(s), X(s)) = \begin{cases} 0, & \text{for } s = 0, \\ 1, & \text{for } 0 < s < \varepsilon_0, \end{cases}$$

$$G^\lambda(Y(s), Y(s)) = 1, \quad \text{for } 0 \le s < \varepsilon_0,$$

$$G^\lambda(\bar{X}(s), \bar{X}(s)) \le 1, \quad \text{for } 0 \le s < \varepsilon_0,$$

$$G^\lambda(\bar{Y}(s), \bar{Y}(s)) = 1, \quad \text{for } 0 \le s < \varepsilon_0.$$

For each s with $0 \le s < \varepsilon_0$, put $k(\lambda, s) = G^\lambda(\bar{R}^\lambda(P(s), \bar{P}(s))\bar{P}(s), P(s))$. When M is of constant sectional curvatures c, it is known that $R(U, V)W$ is of the form

$$R(U, V)W = c\{g(V, W)U - g(U, W)V\}, \quad \text{for } U, V, W \in TM,$$

and we have that

$$\begin{aligned} k(\lambda, s) = {} & (a\bar{a})^2\{c(1 - G^\lambda(X, \bar{X})^2) \\ & + \frac{3}{4}c^2(2g(\pi_*^\lambda(X), \psi_s(u_0)) \cdot g(\pi_*^\lambda(\bar{X}), \psi_s(u_0)) \cdot G^\lambda(X, \bar{X}) \\ & - g(\pi_*^\lambda(X), \psi_s(u_0))^2 - g(\pi_*^\lambda(\bar{X}), \psi_s(u_0))^2)\} \\ & + (a\bar{b})^2\left\{\frac{1}{4}c^2(g(\pi_*^\lambda(X), K(\bar{Y}))^2 + g(\pi_*^\lambda(X), \psi_s(u_0))^2)\right\} \\ & + (\bar{a}b)^2\left\{\frac{1}{4}c^2(g(\pi_*^\lambda(\bar{X}), K(Y))^2 + g(\pi_*^\lambda(\bar{X}), \psi_s(u_0))^2)\right\} \\ & + a\bar{a}b\bar{b}\{3c(g(\pi_*^\lambda(X), K(Y)) \cdot g(\pi_*^\lambda(\bar{X}), K(\bar{Y})) \end{aligned}$$

$$
\begin{aligned}
&- g(\pi_*^\lambda(X), K(\bar{Y})) \cdot g(\pi_*^\lambda(\bar{X}), K(Y))) \\
&- \frac{1}{2} c^2 (2g(\pi_*^\lambda(X), K(Y)) \cdot g(\pi_*^\lambda(\bar{X}), K(\bar{Y})) \\
&- g(\pi_*^\lambda(X), K(\bar{Y})) \cdot g(\pi_*^\lambda(\bar{X}), K(Y)) \\
&+ g(\pi_*^\lambda(X), \psi_s(u_0)) \cdot g(\pi_*^\lambda(\bar{X}), \psi_s(u_0)) \cdot G^\lambda(Y, \bar{Y}))\} \\
&+ (b\bar{b}/\lambda)^2 (1 - G^\lambda(Y, \bar{Y})^2).
\end{aligned}
$$

Especially, we know $k(\lambda, 0) = 1/\lambda^2$. In the following we shall suppose that $\lambda = 1$. Then we get

$$
\begin{aligned}
k(1, s) \leq 1 - a^2 \Bigg\{ &\left(1 - \frac{1}{2}c^2 - (|c| + 1)a^2\right) \cdot \left|\frac{\bar{a}}{a}\right|^2 \\
&- \left(2 \cdot |3c - \frac{1}{2}c^2| + \frac{1}{2}c^2\right) \cdot \left|\frac{\bar{a}}{a}\right| + 1 - \frac{1}{2}c^2 \Bigg\}.
\end{aligned}
$$

From this inequality there exists a positive number $\varepsilon_0' > 0$ $(\varepsilon_0' < \varepsilon_0)$ such that, if $(6 - 2\sqrt{14})/5 < c < -6 + 2\sqrt{10}$, then $k(1, s) < 1$ for $0 < s < \varepsilon_0'$. But since ψ_s is the isometric mapping, we know that $k(1, s) = k(1, 0) = 1$. This gives a contradiction. Since $(6 - 2\sqrt{14})/5 < -0.3$ and $0.32 < -6 + 2\sqrt{10}$, we proved Proposition. □

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