Z. H. HOU
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A PINCHING PROBLEM ON SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR FIELD IN A SPHERE

ZHONG HUA HOU

Abstract

Let M^n be a closed oriented submanifold with nonzero parallel mean curvature vector field immersed into a unit sphere S^{n+p}. Denote by S the square of the length of the second fundamental form. We consider a pinching problem on S. We give a pinching constant C on S which depends only on n and p. It is better than one given by Xu [12]. When $p = 1, 2$ or $n \geq 8$, we show that it is the best possible among this kind of pinching constants. We also characterize those M^n with $S = C$.

1. Introduction

Let M^n be a closed oriented submanifold of dimension n with parallel mean curvature vector field immersed into an $(n+p)$-dimensional unit sphere S^{n+p}. Denote by H the mean curvature and by S the square of the length of the second fundamental form. We propose to consider *the pinching problem on* S, that is, *finding a constant* C *such that, if* $S < C$ *on* M^n, *then* M^n *is totally umbilical.* The constant C so obtained is called *the pinching constant of* S. Moreover, for any $\varepsilon > 0$, if there exists an M^n in S^{n+p} such that M^n is not totally umbilical and $C \leq S < C + \varepsilon$, we say that C is *the best possible pinching constant.* It is known that, for a closed oriented submanifold of dimension n with parallel mean curvature vector field immersed into an $(n+p)$-dimensional unit sphere S^{n+p}, it is totally umbilical if and only if it is an n-sphere in S^{n+p}.

When M^n is minimal, Simons [11] obtained a pinching constant $n/(2 - 1/p)$ of S and showed that it can be attained. Chern-do Carmo-Kobayashi [3] and Lawson [6] classified those minimal submanifolds with $S = n/(2 - 1/p)$ in S^{n+p}. When $p \geq 2$, Li's [7] improved Simons' pinching constant to $2n/3$, and showed that it can be attained only by Veronese surface in a totally geodesic S^4 of S^{n+p}.

The pinching problem on S for submanifolds with parallel mean curvature vector field immersed into a sphere was firstly studied by Okumura [8, 9]. Up to now, there are many remarkable results obtained. The pinching constant depending on H was firstly obtained by Okumura and improved by Alencar-do Carmo [1] (for $p = 1$) and by Xu [13] (for $p \geq 1$). Since H is a geometric

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invariant depending on a specific immersion, it is meaningful to give a pinching constant independent of any specific immersions.

The pinching constant depending only on n and p was firstly obtained by Yau [14]. He proved that, for a closed submanifold M^n with parallel mean curvature vector field immersed into S^{n+p}, $p > 1$, if $S \leq n/\{3 + \sqrt{n} - 1/(p-1)\}$, then M^n lies in a totally geodesic S^{n+1} of S^{n+p}. This result was improved by Xu [12], who showed that, under the same assumptions as above, if $S \leq \min\{2n/(1+\sqrt{n}), n/\{2 - 1/(p-1)\}\}$, then M^n lies in a totally geodesic S^{n+1} of S^{n+p}. Furthermore, under additional assumptions, Xu [12] proved that M^n is totally umbilical.

Among all the possible pinching constants depending only on n and p, it is significant to find the best possible one. The author [5] showed that $2\sqrt{n-1}$ is the best possible pinching constant depending on n for $p = 1$. In this paper, we will give a pinching constant C of S depending on n and p. It is better than the one given by Xu [12]. When $p = 1, 2$ or $n \geq 8$, we assert that it is the best possible one among this kind of pinching constants. We also characterize those M^n with $S = C$.

Precisely, we propose to prove the following theorems. Denote by $S^n(r)$ the standard n-dimensional sphere of radius r and define C by

$$(*) \qquad C = \min\left\{2\sqrt{n-1}, \frac{n}{1 + (1/2)\mathrm{sgn}(p-2)}\right\}$$

where $\mathrm{sgn}(\cdot)$ is the standard sign function. Then we have:

THEOREM 1. *Let M^n be an oriented closed submanifold immersed into the unit sphere S^{n+p}, with nonzero parallel mean curvature vector field. If $S < C$, then S is constant and M^n is a small sphere $S^n(1/\sqrt{1+S/n})$ in S^{n+p}.*

THEOREM 2. *Let M^n be an oriented closed submanifold immersed into the unit sphere S^{n+p}, with nonzero parallel mean curvature vector field. Suppose $n > 2$ and $S = C$. Then:*

(i) *If $p = 1, 2$ or $n \geq 8$, then $C = 2\sqrt{n-1}$ and M^n is either a small sphere $S^n(r_0)$ in S^{n+p} or a torus $S^1(r) \times S^{n-1}(s)$ in a totally geodesic sphere S^{n+1} of S^{n+p}, where $r_0^2 = n/(n + 2\sqrt{n-1})$, $r^2 = 1/(1+\sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$;*

(ii) *If $p > 2$ and $n \leq 7$, then $C = 2n/3$ and M^n is a small n-sphere $S^n(\sqrt{3/5})$ in S^{n+p}.*

THEOREM 3. (i) *Let M^2 be an oriented closed surface immersed into the unit sphere S^{2+p}, with nonzero parallel mean curvature vector field. If $p > 1$ and $S = C$, then M^2 is a small sphere $S^2(1/\sqrt{2})$ $(p = 2)$ or $S^2(\sqrt{3/5})$ $(p \geq 3)$ in S^{2+p}.*

(ii) *For any $\varepsilon > 0$, there exists a pseudo-umbilical surface M_ε^2 in S^{2+p} such that:*

(a) *M_ε^2 is not totally umbilical;*

(b) *M_ε^2 is one with nonzero parallel mean curvature vector field;*

(c) $C < S_\varepsilon < C + \varepsilon$, *where* S_ε *is the square of the length of the second fundamental form of* M_ε^2.

COROLLARY 2. *Above theorems show that, when* $p = 1, 2, n = 2$ *or* $n \geq 8$, C *is the best possible pinching constant of* S *depending only on* n *and* p.

We also get a result concerning Erbacher's problem discussed in [3].

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2. Formulas of Simons' type

Let M^n be a closed oriented submanifold with nonzero parallel mean curvature vector field immersed into the unit sphere S^{n+p}. From now on, we identify M^n with its immersed image and agree on the following index ranges:

$$1 \leq i, j, k, \ldots \leq n; \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq p; \quad q \leq A, B, C, \ldots \leq n + p.$$

Take a local orthonormal frame $\{e_A\}_{A=1}^{n+p}$ in $\mathscr{T}(S^{n+p})$ on M such that $\{e_i\}_{i=1}^{n}$ lies in the tangent bundle $\mathscr{T}(M)$ and $\{e_\alpha\}_{\alpha=n+1}^{n+p}$ in the normal bundle $\mathscr{N}(M)$. Let $\{\omega_A\}_{A=1}^{n+p}$ be the dual coframe of $\{e_A\}_{A=1}^{n+p}$. Let $(\omega_{AB})_{A,B=1}^{n+p}$ denote the Riemannian connection matrix associated with $\{\omega_A\}_{A=1}^{n+p}$. Then $(\omega_{ij})_{i,j=1}^{n}$ defines a Riemannian connection in $\mathscr{T}(M)$ and $(\omega_{\alpha\beta})_{\alpha,\beta=n+1}^{n+p}$ defines a normal connection in $\mathscr{N}(M)$.

It follows that the second fundamental form of M can be expressed as

$$II = \sum_{(i,\alpha)} \omega_i \otimes \omega_{i\alpha} \otimes e_\alpha = \sum_{(i,j,\alpha)} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

where $\omega_{i\alpha} = \sum_{(j)} h_{ij}^\alpha \omega_j$ and $h_{ij}^\alpha = h_{ji}^\alpha$ for all $\alpha = n+1, \ldots, n+p$ and $i, j = 1, \ldots, n$.

Denote $L_\alpha = (h_{ij}^\alpha)_{n\times n}$ and $H_\alpha = (1/n)\sum_{(i)} h_{ii}^\alpha$ for $\alpha = n+1, \ldots, n+p$. Then the mean curvature vector field ξ is expressed as $\xi = \sum_{(\alpha)} H_\alpha e_\alpha$. We denote by H the length of ξ and by S the square of the length of the second fundamental form, i.e., $H = \|\xi\|$ and $S = \sum_{(\alpha,i,j)} (h_{ij}^\alpha)^2$. The Riemannian curvature tensor $\{R_{ijkl}\}$ and the normal curvature tensor $\{R_{\alpha\beta kl}\}$ are expressed as

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha, \quad R_{\alpha\beta kl} = h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta.$$

Define the first and the second covariant derivatives of $\{h_{ij}^\alpha\}$, say $\{h_{ijk}^\alpha\}$ and $\{h_{ijkl}^\alpha\}$ by

$$\nabla h_{ij}^{\alpha} = h_{ijk}^{\alpha}\omega_k \equiv dh_{ij}^{\alpha} + h_{mj}^{\alpha}\omega_{mi} + h_{im}^{\alpha}\omega_{mj} + h_{ij}^{\beta}\omega_{\beta\alpha}, \tag{1}$$

$$\nabla h_{ijk}^{\alpha} = h_{ijkl}^{\alpha}\omega_l \equiv dh_{ijk}^{\alpha} + h_{mjk}^{\alpha}\omega_{mi} + h_{imk}^{\alpha}\omega_{mj} + h_{ijm}^{\alpha}\omega_{mk} + h_{ijk}^{\beta}\omega_{\beta\alpha}. \tag{2}$$

It follows from Ricci's identity that

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \quad h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = h_{mj}^{\alpha}R_{mikl} + h_{im}^{\alpha}R_{mjkl} + h_{ij}^{\beta}R_{\beta\alpha kl}. \tag{3}$$

The Laplacian of h_{ij}^{α} is defined by $\Delta h_{ij}^{\alpha} = \sum_{(k)} h_{ijkk}^{\alpha}$. Using (3), we have

$$\begin{aligned}
\Delta h_{ij}^{\alpha} &= h_{km}^{\alpha}R_{mijk} + h_{im}^{\alpha}R_{mkjk} + h_{ik}^{\beta}R_{\beta\alpha jk} \\
&= h_{km}^{\alpha}(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij} + h_{mj}^{\beta}h_{ik}^{\beta} - h_{mk}^{\beta}h_{ij}^{\beta}) \\
&\quad + h_{im}^{\alpha}(\delta_{mj}\delta_{kk} - \delta_{mk}\delta_{kj} + h_{mj}^{\beta}h_{kk}^{\beta} - h_{mk}^{\beta}h_{kj}^{\beta}) \\
&\quad + h_{ik}^{\beta}(h_{jm}^{\beta}h_{mk}^{\alpha} - h_{km}^{\beta}h_{mj}^{\alpha}) \\
&= nh_{ij}^{\alpha} - nH_{\alpha}\delta_{ij} + nH_{\beta}h_{im}^{\alpha}h_{mj}^{\beta} - S_{\alpha\beta}h_{ij}^{\beta} \\
&\quad + 2h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta} - h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta} - h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{\alpha},
\end{aligned}$$

where we denote $S_{\alpha\beta} = \sum_{(i,j)} h_{ij}^{\alpha}h_{ij}^{\beta}$ for $\alpha, \beta = n+1, \dots, n+p$.

Define $N(A) = \sum_{(i,j)} a_{ij}^2$ for a matrix $A = (a_{ij})_{i,j=1}^{n}$ and denote $S_{\alpha} = S_{\alpha\alpha}$ for all α. Then we have, for every fixed α,

$$\begin{aligned}
\sum_{(i,j)} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} =& nS_{\alpha} - n^2H_{\alpha}^2 + n\sum_{(\beta)} H_{\beta}\operatorname{Tr}(L_{\alpha}^2L_{\beta}) - S_{n+1\alpha}^2 - \sum_{(\beta>n+1)} S_{\alpha\beta}^2 \\
&- N(L_{\alpha}L_{n+1} - L_{n+1}L_{\alpha}) - \sum_{(\beta>n+1)} N(L_{\alpha}L_{\beta} - L_{\beta}L_{\alpha}).
\end{aligned} \tag{4}$$

Choose e_{n+1} to have the same direction as ξ such that $\xi = He_{n+1}$. Then we have

$$H_{n+1} = H; \quad H_{\alpha} = 0, \quad \alpha = n+2, \dots, n+p. \tag{5}$$

Since ξ is nonzero and parallel, we have that $H \neq 0$ is constant and e_{n+1} is parallel. It follows that $L_{n+1}L_{\alpha} = L_{\alpha}L_{n+1}$. From (4), we obtain

$$\sum_{(i,j)} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1} = nS_{n+1} + nH\operatorname{Tr}(L_{n+1})^3 - n^2H^2 - S_{n+1}^2 - \sum_{(\beta>n+1)} (S_{n+1\beta})^2, \tag{6}$$

$$\begin{aligned}
\sum_{(i,j)} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} &= nS_{\alpha} + nH\operatorname{Tr}L_{n+1}(L_{\alpha})^2 - (S_{n+1\alpha})^2 - \sum_{(\beta>n+1)} (S_{\alpha\beta})^2 \\
&\quad - \sum_{(\beta>n+1)} N(L_{\beta}L_{\alpha} - L_{\alpha}L_{\beta}), \quad \alpha > n+1.
\end{aligned} \tag{7}$$

We recall that a submanifold is said to be *pseudo-umbilical* if the mean curvature vector field is nonzero and lies in an umbilical direction of the fundamental form. Define $\tilde{S}_{n+1}$ by

$$(**)\qquad \tilde{S}_{n+1} = \sum_{(i,j)} (h_{ij}^{n+1} - H\delta_{ij})^2.$$

It is easy to get the following

LEMMA 1. *Let $\tilde{S}_{n+1}$ be defined as in (**). Then $\tilde{S}_{n+1} = S_{n+1} - nH^2 \geq 0$ and the equality holds if and only if M^n is pseudo-umbilical.*

We denote $f = \mathrm{Tr}(L_{n+1})^3$ and $S_I = \sum_{(\beta>n+1)} S_\beta$. It follows from (6) that

$$(8)\qquad \sum_{(i,j)} h_{ij}^{n+1} \Delta h_{ij}^{n+1} = nS_{n+1} + nHf - n^2H^2 - S_{n+1}^2 - \sum_{(\beta>n+1)} (S_{n+1\beta})^2.$$

Using the same arguments as in [5], we have

$$(9)\qquad \sum_{(i,j)} h_{ij}^{n+1} \Delta h_{ij}^{n+1} \geq \tilde{S}_{n+1} \left\{ n - (\tilde{S}_{n+1} - nH^2) - \frac{n-2}{\sqrt{n-1}} H \sqrt{n\tilde{S}_{n+1}} \right\} - \sum_{(\beta>n+1)} (S_{n+1\beta})^2.$$

It follows from (5) that

$$(10)\qquad \sum_{(\beta>n+1)} (S_{n+1\beta})^2 = \sum_{(\beta>n+1)} \left\{ \sum_{(i,j)} (h_{ij}^{n+1} - H\delta_{ij}) h_{ij}^\beta \right\}^2.$$

By applying Schwarz's inequality to the right hand-side of (10), we have

$$(11)\qquad \sum_{(\beta>n+1)} (S_{n+1\beta})^2 \leq \tilde{S}_{n+1} S_I.$$

Substituting (11) into (9), we have

$$\begin{aligned}(12)\qquad \sum_{(i,j)} h_{ij}^{n+1} \Delta h_{ij}^{n+1} &\geq \tilde{S}_{n+1} \left\{ n - (\tilde{S}_{n+1} - nH^2) - S_I - \frac{n-2}{\sqrt{n-1}} H \sqrt{n\tilde{S}_{n+1}} \right\} \\ &= \tilde{S}_{n+1} \left\{ n - \tilde{S} + nH^2 - \frac{n-2}{\sqrt{n-1}} H \sqrt{n\tilde{S}_{n+1}} \right\} \\ &\geq \tilde{S}_{n+1} \left\{ n - \tilde{S} + nH^2 - \frac{n-2}{\sqrt{n-1}} H \sqrt{n\tilde{S}} \right\},\end{aligned}$$

where $\tilde{S} = \tilde{S}_{n+1} + S_I = S - nH^2$.

Using the same arguments as in [5] to the last term of (12), we obtain

$$\sum_{(i,j)} h_{ij}^{n+1} \Delta h_{ij}^{n+1} \geq \tilde{S}_{n+1} \left(n - \frac{n}{2\sqrt{n-1}} S \right).$$

It follows that

$$(13)\qquad \frac{1}{2}\Delta S_{n+1} = \sum_{(i,j,k)} (h_{ijk}^{n+1})^2 + \sum_{(i,j)} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \geq \tilde{S}_{n+1}\left(n - \frac{n}{2\sqrt{n-1}}\, S\right).$$

Taking integrations on both sides of (13) on M^n, we obtain

$$(14)\qquad 0 \geq \int_{M^n} \tilde{S}_{n+1}\left(n - \frac{n}{2\sqrt{n-1}}\, S\right).$$

If $S \leq 2\sqrt{n-1}$, we have from (13), (14) and Hopf's Lemma that S_{n+1} is constant and

$$\tilde{S}_{n+1}\left(n - \frac{n}{2\sqrt{n-1}}\, S\right) = 0.$$

It follows from Lemma that $\tilde{S}_{n+1}$ is also a constant. Therefore we obtain the following

PROPOSITION 1. *Let M^n be a closed submanifold immersed into a unit sphere S^{n+p}, with nonzero parallel mean curvature vector field. If $S \leq 2\sqrt{n-1}$, then we have*:

(i) $S = 2\sqrt{n-1}$; *or*

(ii) $\tilde{S}_{n+1} = 0$ *and M^n is pseudo-umbilical.*

If M^n is not pseudo-umbilical, we have from (12) that $S_I \equiv 0$. It follows that M^n lies in a totally geodesic sphere S^{n+1} of S^{n+p}. From a result in [5], we get the following

COROLLARY 1. *Under the same assumptions as in Proposition* 1, *we have*:

(i) *Suppose $n > 2$. If M^n is not pseudo-umbilical and $S \leq 2\sqrt{n-1}$, then $S = 2\sqrt{n-1}$ and M^n is a torus $S^1(r) \times S^{n-1}(s)$ in a totally geodesic sphere S^{n+1} of S^{n+p}, where $r^2 = 1/(1+\sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$;*

(ii) *Suppose $n = 2$. If $S \leq 2$, then M^2 is pseudo–umbilical.*

Sketch of the proof of Corollary 1. (i) is obvious from [5]. To prove (ii), we need only to consider the case $S = 2$. Supposing that M^2 is not umbilical, we have that M^2 can be immersed as a flat torus $S^1(r) \times S^1(s)$ into a totally geodesic sphere S^3 in S^{p+2}. But the only flat torus with $S = 2$ in S^3 is the Clifford torus, which is minimal. This contradicts the assumption $H \neq 0$. We complete the proof. Q.E.D.

From now on, we suppose that M^n is pseudo-umbilical and $p \geq 2$. In this case, we know that M^n can be minimally immersed into a hypersphere $S^{n+p-1}(1/\sqrt{1+H^2})$ of S^{n+p}.

Chen [2] proved the following classification result (see also Santos [10, pp. 411]): Let M^n be a compact pseudo-umbilical submanifold of $S^{n+p}, p \geq 2$, with parallel mean curvature vector field. If $S \leq n(1+H^2)/\{2-1/(p-1)\}$, then either (i) $S=0$ and M^n is totally umbilical; or (ii) $S = n(1+H^2)/\{2-1/(p-1)\}$ and M^n is a minimal Clifford hypersurface in $S^{n+1}(1/\sqrt{1+H^2}) \hookrightarrow S^{n+2}$ or M^2 is a Veronese surface in $S^4(1/\sqrt{1+H^2}) \hookrightarrow S^5$.

We propose to give an improvement to this result.

Since $L_{n+1} = HI_n$ in this case, we have, from (10),

$$\sum_{(\beta > n+1)} (S_{n+1\beta})^2 = 0. \tag{15}$$

It follows from (7) and (15) that

$$\sum_{(i,j;\alpha>n+1)} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} = n(1+H^2)S_I - \sum_{(\alpha,\beta>n+1)} (S_{\alpha\beta})^2 - \sum_{(\alpha,\beta>n+1)} N(L_\beta L_\alpha - L_\alpha L_\beta). \tag{16}$$

We have to estimate the sum of the last two terms in the right-hand side of (16).

Li's [7] proved the following

LEMMA 2. *Let $A_1, A_2, \ldots, A_q$ be symmetric $(n \times n)$-matrices, where $q \geq 2$. We denote $S_{\alpha\beta} = \operatorname{Tr} A_\alpha^T A_\beta, S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$ and $S = S_1 + \cdots + S_q$. Then*

$$\sum_{(\alpha,\beta)} S_{\alpha\beta}^2 + \sum_{(\alpha,\beta)} N(A_\beta A_\alpha - A_\alpha A_\beta) \leq \frac{3}{2} S^2, \tag{17}$$

and the equality holds if and only if one of the following conditions holds:

(i) $A_1 = \cdots = A_q = 0$;

(ii) *only two of the matrices $A_1, A_2, \ldots, A_q$ are different from zero. Moreover, assuming $A_1 \neq 0, A_2 \neq 0$ and $A_3 = \cdots = A_q = 0$, we have $S_1 = S_2$ and there exists an orthogonal $(n \times n)$-matrix U such that*

$$UA_1U^T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right), \quad UA_2U^T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right).$$

Together with the case $q = 1$, we obtain

$$\sum_{(\alpha,\beta)} S_{\alpha\beta}^2 + \sum_{(\alpha,\beta)} N(A_\beta A_\alpha - A_\alpha A_\beta) \leq [1 + (1/2)\operatorname{sgn}(q-1)] S^2. \tag{18}$$

Replacing A_α's in (18) by L_β's with $\beta > n+1$, we have

$$\sum_{(\alpha,\beta<n+1)} S_{\alpha\beta}^2 + \sum_{(\alpha,\beta>n+1)} N(L_\beta L_\alpha - L_\alpha L_\beta) \leq [1 + (1/2)\operatorname{sgn}(p-2)] S_I^2.$$

Substituting above inequality into (16), we obtain

$$\sum_{(i,j;\alpha>n+1)} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \ge S_I\{n(1+H^2) - [1 + (1/2)\,\mathrm{sgn}(p-2)]S_I\}.$$

It follows that

$$\text{(19)} \qquad \frac{1}{2}\Delta S_I = \sum_{(i,j,k;\alpha>n+1)} (h_{ijk}^{\alpha})^2 + \sum_{(i,j;\alpha>n+1)} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$
$$\ge S_I\{n(1+H^2) - [1 + (1/2)\,\mathrm{sgn}(p-2)]S_I\}.$$

Using (19), we can improve Chen's result as follows:

PROPOSITION 2. *Let M^n be a closed pseudo-umbilical submanifold with nonzero parallel mean curvature vector field immersed into a unit sphere S^{n+p} with $p \ge 2$. Then M^n can be minimally immersed into a hypersphere $S^{n+p-1}(1/\sqrt{1+H^2})$* in S^{n+p}. *Furthermore, if $S_I \le n(1+H^2)/[1+(1/2)\,\mathrm{sgn}(p-2)]$, we have*:

(i) *$S_I = 0$ and M^n is a small sphere $S^n(1/\sqrt{1+H^2})$ in S^{n+p};*

(ii) *$S_I = n(1+H^2)$ and M^n is a Clifford torus $S^k(r) \times S^{n-k}(s)$ in a hypersphere $S^{n+1}(1/\sqrt{1+H^2})$ in a totally geodesic sphere S^{n+2} of S^{n+p};*

(iii) *$S_I = (4/3)(1+H^2)$ and M^2 is the Veronese surface in a hypersphere $S^4(1/\sqrt{1+H^2})$ in a totally geodesic sphere S^5 of S^{n+p}.*

Proof. It is clear that M^n can be minimally immersed into a hypersphere $S^{n+p-1}(1/\sqrt{1+H^2})$ in S^{n+p}. Let us prove assertions (i)–(iii).

First, we have to show that (19) works on the reduced immersion.

Recall that the normal connection matrix of M^n in S^{n+p} is $(\omega_{\alpha\beta})_{\alpha,\beta=n+1}^{n+p}$. Hence the normal connection matrix of M^n in $S^{n+p-1}(1/\sqrt{1+H^2})$ can be expressed as $(\omega_{\alpha\beta})_{\alpha,\beta=n+2}^{n+p}$ and the square of the length of the second fundamental form of M^n in $S^{n+p-1}(1/\sqrt{1+H^2})$ is the same as the S_I of M^n in S^{n+p}. On the other hand, we have $\omega_{n+1\alpha} = 0$, $\alpha = n+1, \dots, n+p$, since the mean curvature vector field $\xi = He_{n+1}$ is parallel in the normal bundle $\mathcal{N}(M)$. Hence the covariant derivatives of $\{h_{ij}^{\alpha}\}$ in $S^{n+p-1}(1/\sqrt{1+H^2})$ are the same as that of $\{h_{ij}^{\alpha}\}$ in S^{n+p}. And so is the Laplacian of $\{h_{ij}^{\alpha}\}$.

Therefore (19) can also be considered as being computed on the minimal immersion from M^n into $S^{n+p-1}(1/\sqrt{1+H^2})$ of constant curvature $(1+H^2)$.

Taking integration on both-sides of (19) on M^n, we have

$$\text{(19')} \qquad 0 \ge \int_{M^n} S_I\{n(1+H^2) - [1 + (1/2)\,\mathrm{sgn}(p-2)]S_I\}.$$

From (19′) and the assumption, we have $S_I = 0$ or $S_I = n(1+H^2)/[1+(1/2)\,\mathrm{sgn}(p-2)]$, on M^n. Assertion (i) follows directly from $S_I = 0$. If

$p=2$, then $S_I = n(1+H^2)$. Assertion (ii) follows from the result of Chern-do Carmo-Kobayashi [3]. If $p \geq 3$, then $S_I = (4/3)(1+H^2)$. Following the same arguments as in Li's [7], we obtain assertion (iii). This completes the proof. Q.E.D.

Remark 1. From (19′), we can immediately get

$$0 \geq \int_{M^n} S_I\{n - [1 + (1/2)\operatorname{sgn}(p-2)]S\}, \tag{20}$$

which will be used in next section.

3. Proof of the theorems

Let C be defined as in $(*)$. It is easy to see that

$$4(n-1) - \frac{4}{9}n^2 = -\frac{4}{9}[(n-2)(n-7)-5]\begin{cases} >0, & 2 \leq n \leq 7, \\ <0, & 8 \leq n. \end{cases}$$

Therefore we have

$$C = \begin{cases} 2\sqrt{n-1}, & p=2 \quad \text{or} \quad 8 \leq n, \\ 2n/3, & p>2 \quad \text{and} \quad 2 \leq n \leq 7. \end{cases} \tag{21}$$

If $S < C$, then $S < 2\sqrt{n-1}$. From Proposition 1, we have that M^n is pseudo-umbilical. In this case, it follows from (20) that $S_I \equiv 0$. Hence $S = S_I + nH^2 = nH^2$. Using Proposition 2, we can see that M^n is a small sphere $S^n(1/\sqrt{1+S/n})$ in S^{n+p}.

Therefore we obtain the following

THEOREM 1. *Let M^n be an oriented closed submanifold immersed into the unit sphere S^{n+p}, with nonzero parallel mean curvature vector field. If $S < C$, then S is constant and M^n is a small sphere $S^n(1/\sqrt{1+S/n})$ in S^{n+p}.*

Now let us consider the case that M^n is one with $S = C$.

Case 1. *Suppose* $n > 2$. If $p=2$ or $n \geq 8$, then $C = 2\sqrt{n-1} < n/[1+1/2\operatorname{sgn}(p-2)]$. From (20) we have $S_I \equiv 0$. If M^n is not pseudo-umbilical, it follows from Corollary 1 that M^n is a torus $S^1(r) \times S^{n-1}(s)$ in a totally geodesic sphere S^{n+1} of S^{n+p} where $r^2 = 1/(1+\sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$. If M^n is pseudo-umbilical, it follows from Proposition 2 that M^n is a small sphere $S^n(r_0)$ in S^{n+p} where $r_0^2 = n/(n+2\sqrt{n-1})$.

If $p > 2$ and $n \leq 7$, we have $C = n/[1+1/2\operatorname{sgn}(p-2)] < 2\sqrt{n-1}$. Therefore M^n is pseudo-umbilical. In this case $S_I < S = 2n/3 < n(1+H^2)/[1+1/2\operatorname{sgn}(p-2)]$. It follows from Proposition 2 that $S_I = 0$ and M^n is a small sphere $S^n(\sqrt{3/5})$ in S^{n+p}.

From the above discussions plus the case $p = 1$, we obtain the following

THEOREM 2. *Let M^n be an oriented closed submanifold immersed into the unit sphere S^{n+p}, with nonzero parallel mean curvature vector field. Suppose $n > 2$ and $S = C$. Then:*

(i) *If $p = 1, 2$ or $n \geq 8$, then $C = 2\sqrt{n-1}$ and M^n is either a small sphere $S^n(r_0)$ in S^{n+p} or a torus $S^1(r) \times S^{n-1}(s)$ in a totally geodesic sphere S^{n+1} of S^{n+p}, where $r_0^2 = n/(n + 2\sqrt{n-1})$, $r^2 = 1/(1 + \sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1 + \sqrt{n-1})$;*

(ii) *If $p > 2$ and $n \leq 7$, then $C = 2n/3$ and M^n is a small n-sphere $S^n(\sqrt{3/5})$ in S^{n+p}.*

Case 2. *Suppose $n = 2$ and $p \geq 2$.* If $p = 2$, then $C = 2 < 2(1 + H^2)$. From (i) of Proposition 2 we have that M^2 is a sphere $S^2(1/\sqrt{2})$.

If $p > 2$, then $C = 4/3 < 2$. It follows from Corollary 1 that the M^2 with $S = C$ is pseudo-umbilical. Thus M^2 can be minimally immersed into a hypersphere $S^{p+1}(1/\sqrt{1+H^2})$ of S^{p+2}. Note that $S_I < S = 4/3 < 4(1 + H^2)/3$. From (i) of Proposition 2 we have that M^2 is a sphere $S^2(\sqrt{3/5})$.

For an arbitrary $\varepsilon > 0$, we can choose H_ε small enough such that

$$\left(1 + 1/\left[1 + \frac{1}{2}\operatorname{sgn}(p-2)\right]\right) \cdot 2H_\varepsilon^2 < \varepsilon.$$

Then we have

$$C < S_\varepsilon = S_I + 2H_\varepsilon^2 = 2(1 + H_\varepsilon^2)/\left[1 + \frac{1}{2}\operatorname{sgn}(p-2)\right] + 2H_\varepsilon^2 < C + \varepsilon.$$

From (ii) of Proposition 2, we can see that the only minimal surface with $S_I = 2(1 + H_\varepsilon^2)$, in a hypersphere $S^3(1/\sqrt{1+H_\varepsilon^2})$ of a totally geodesic sphere S^4 of S^{2+p}, is the Clifford torus $S^1(1/\sqrt{2(1+H_\varepsilon^2)}) \times S^1(1/\sqrt{2(1+H_\varepsilon^2)})$. From (iii) of Proposition 2, we can see that the only minimal surface with $S_I = 4(1 + H_\varepsilon^2)/3$, in a hypersphere $S^4(1/\sqrt{1+H_\varepsilon^2})$ of a totally geodesic sphere S^5 of S^{2+p}, is the Veronese surface.

Therefore we obtain the following.

THEOREM 3. (i) *Let M^2 be an oriented closed surface immersed into the unit sphere S^{2+p}, with nonzero parallel mean curvature vector field. If $p > 1$ and $S = C$, then M^2 is a small sphere $S^2(1/\sqrt{2})$ $(p = 2)$ or $S^2(\sqrt{3/5})$ $(p \geq 3)$ in S^{2+p}.*

(ii) *For any $\varepsilon > 0$, there exists a pseudo-umbilical surface M_ε^2 in S^{2+p} such that:*

(a) *M_ε^2 is not totally umbilical;*

(b) *M_ε^2 is one with nonzero parallel mean curvature vector field;*

(c) *$C < S_\varepsilon < C + \varepsilon$, where S_ε is the square of the length of the second fundamental form of M_ε^2.*

Therefore we can claim our desired conclusion:

COROLLARY 2. *Above theorems show that, when $p=1, 2$, $n=2$ or $n \geq 8$, C is the best possible pinching constant of S depending only on n and p.*

Erbacher [4] suggested the following problem:

When can we reduce the codimension of an isometric immersion into a space form of constant curvature?
and got a result under an assumption on the first normal space of the isometric immersion.

From Theorems 1–3, we can get the following

PROPOSITION 3. *Let M^n be a closed submanifold immersed into the unit sphere S^{n+p} with nonzero parallel mean curvature vector field. If $S \leq C$, then the codimension p of M^n can be reduced to one.*

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU
TOKYO 152-8551, JAPAN
E-mail: hou@math.titech.ac.jp