EXPLICIT REPRESENTATION OF SUPER-MINIMAL J-HOLOMORPHIC CURVES IN A 6-DIMENSIONAL SPHERE

HIDEYA HASHIMOTO

Abstract

We give the differential equation of first order with respect to a super-minimal J-holomorphic curves of S^6 and write down the holomorphic horizontal curve in the complex quadric corresponds to the Boruvka sphere in S^6.

1. Introduction

Let S^6 be a 6-dimensional unit sphere centered at the origin in the purely imaginary octonians $\mathrm{Im}\,O$ which is identified with a 7-dimensional Euclidean space. It is known that there exists the canonical nearly Kähler structure on S^6 which is not integrable. Let M^2 be a Riemann surface. We call a immersion from M^2 into S^6 a J-holomorphic curve of S^6 if each tangent space of M^2 is invariant under the almost complex structure. In ([Br1]), R. L. Bryant has constructed super-minimal J-holomorphic curves of S^6 as follows;

Let G_2 be the exceptional simple Lie group of automorphisms of the octonions. It is known that the homogeneous space $G_2/U(2)$ coincide with the Grassmannian manifold $G_2(\mathrm{Im}\,O)$ of oriented 2-planes in $\mathrm{Im}\,O$. Then we have the CP^2-bundle over S^6 such that $\pi: G_2/U(2) \; (=G_2(\mathrm{Im}\,O)) \to G_2/SU(3) \; (=S^6)$. Let $H(q)$ be the horizontal subspace of $G_2(\mathrm{Im}\,O)$ at $q \in G_2(\mathrm{Im}\,O)$ with respect to the $SU(3)$ connection of S^6. Let $L_+(q)$ be the subspace (of $H(q)$) of the vectors on which π_* is complex linear with respect to the canonical complex structure of $G_2(\mathrm{Im}\,O)$ and the nearly Kähler structure on S^6. Then it can be shown that the dimension of the space $L_+(q)$ is four and $\cup_{q \in G_2(\mathrm{Im}\,O)} L_+(q)$ is the subbundle of rank 4 of the tangent bundle of $G_2(\mathrm{Im}\,O)$. Let $\phi: M^2 \to G_2(\mathrm{Im}\,O)$ be a holomorphic map from a Riemann surface M^2 to the Grassmaniann manifold $G_2(\mathrm{Im}\,O)$ whose differential image $d\phi(T_p M^2)$ is included in $L_+(p)$ for any point $p \in M$. Then we have the superminimal J-holomorphic curve $\pi \circ \phi: M^2 \to S^6$, locally.

This construction is to solve the first order differential equations by twistor

and moving frame methods. In this paper, we give another proof of the R. L. Bryant's construction by purely algebraic direct calculation. As an application, we write down the holomorphic horizontal curve with respect to the Boruvka sphere $S^2(1/6)$ to $S^6(1)$, explicitly. Our method is essentially same as that R. L. Bryant's one, but its idea is based on S. S. Chern's method ([Ch]). We use the same notational convention to the previous paper ([H1]).

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2. Preliminaries

Let M^2 be a J-holomorphic curves in a 6-dimensional sphere S^6, then it is a minimal surface in S^6. It is known that there exists the following fibre bundle structure;

$$
\begin{array}{ccc}
T^{(1,0)}M^2 \oplus \nu_1^{(1,0)} \oplus \nu_2^{(1,0)} & \longrightarrow & T^{(1,0)}S^6 \\
C \oplus C \oplus C \downarrow & & \downarrow C^3 \\
M^2 & \underset{x}{\longrightarrow} & S^6
\end{array}
$$

where $T^{(1,0)}M^2$ is a complexified tangent bundle of type $(1, 0)$, $\nu^{(1,0)}$ is a complexified 1st normal bundle of type $(1, 0)$ and $\nu_2^{(1,0)}$ is a complexified 2nd normal bundle of type $(1, 0)$ (for details in ([B2] or [H1])). Let $\{U_\alpha\}_{\alpha \in A}$ be a sufficiently fine finite covering of M^2. Let (f_3, f_2, f_1) be a local section of the bundle $T^{(1,0)}M^2 \oplus \nu_1^{(1,0)} \oplus \nu_2^{(1,0)}$ on U_α, and (f_3', f_2', f_1') another section on U_β. We assume that $U_\alpha \cap U_\beta \neq \emptyset$, then we have

$$
(2.1) \qquad (f_3', f_2', f_1') = (f_3, f_2, f_1) \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{2i\theta} & 0 \\ 0 & 0 & e^{-3i\theta} \end{pmatrix},
$$

where θ is a real valued function on $U_\alpha \cap U_\beta$. By (2.1), we can define a map from M^2 to a (complex) 5-dimensional quardrics $Q^5 \subset P^6(C)$ such that

$$
[f_1]: M^2 \longrightarrow Q^5
$$

where $[\ ]$ denotes the homogeneous coordinate of $P^6(C)$. From the above relations of the transition functions, this map is well defined on whole on M^2

Remark. Here f_1 is considered as a $C^7 \setminus \{0\}$ valued function; $f_1: U_\alpha \to C^7 \setminus \{0\}$.

Unfortunately, the map $[f_1]$ is not holomorphic map, but we can derive a holomorphic data from $[f_1]$. Next we give the fundamental relations of the immersion $x: M^2 \to S^6$ and $[f_1]$. Since the octonians is a non-commutative normed algebra, there exists the exterior product $u \times v = (1/2)(\bar{v}u - \bar{u}v)$ for any

$u, v \in O$. Let $C \otimes_R O$ be the complexification of the octonions O. We extend the exterior product complex linearly on $C \otimes_R O$ and denote the same symbol. Then we see that

$$2\sqrt{-1}f_1 \times \bar{f}_1 = x.$$

In fact, since $f_1 = (1/2)(a - \sqrt{-1}a \times x)$, for some $a \in \mathrm{Im}\, O$, $|a|^2 = 1$, we get

$$2\sqrt{-1}f_1 \times \bar{f}_1 = 2\sqrt{-1}\left(\frac{1}{2}(a - \sqrt{-1}a \times x) \times \frac{1}{2}(a + \sqrt{-1}a \times x)\right)$$

$$= a \times (a \times x) = -a^2 x = x.$$

3. Fundamental properties of super-minimal J-holomorphic curves

First, we derive the holomorphic data from f_1. Let U be a sufficiently small open subset of M^2. Then f_1 can be considered as a vector valued function on U, $f_1: U \to C^7 \backslash \{0\}$, which satisfy $\langle f_1, f_1 \rangle = 0$ and $\langle f_1, \bar{f}_1 \rangle = 1/2$. Taking account of structure equations of the simple Lie group G_2, (Weingarten formula (5.8) in [H1], or (2.16) in [Br1]), we have

$$(3.1) \qquad df_1 = f_1 \kappa_1{}^2 + f_2 \kappa_1{}^2 + f_3 \kappa_1{}^3 - \bar{f}_2 \theta^3.$$

Since we take the special unitary frame $\{f_1, f_2, f_3\}$, we have $\kappa_1{}^3 \equiv 0$. Now we assume a J-holomorphic curve is superminimal; $\kappa_1{}^2 \equiv 0$. By (3.1), we have

$$(3.2) \qquad df_1 = f_1 \kappa_1{}^1 - \bar{f}_2 \theta^3,$$

where $\theta^3 \in \Lambda^{(1, 0)}(U)$. By (3.2), we have

$$(3.3) \qquad \frac{\partial f_1}{\partial \bar{z}} = f_1 \kappa_1{}^1 \left(\frac{\partial}{\partial \bar{z}}\right).$$

If we put $\xi = \gamma \cdot f_1$ such that $\gamma: U \to C$ is a complex valued function on U. By (3.3), we have

$$\frac{\partial \xi}{\partial \bar{z}} = \frac{\partial \gamma}{\partial \bar{z}} \cdot f_1 + \gamma \cdot \frac{\partial f_1}{\partial \bar{z}}$$

$$= f_1 \left(\frac{\partial \gamma}{\partial \bar{z}} + \gamma \cdot \kappa_1{}^1 \left(\frac{\partial}{\partial \bar{z}}\right)\right).$$

From this we see that $\xi: U \to C^7 / \{0\}$ is the holomorphic map if and only if γ is a solution of the following differential equation;

$$(3.4) \qquad \frac{\partial \gamma}{\partial \bar{z}} + \gamma \cdot \kappa_1{}^1 \left(\frac{\partial}{\partial \bar{z}}\right) = 0.$$

In ([G-H: p. 5]), by $\bar{\partial}$-Poincaré Lemma in one variable, we can solve the differential equation (3.4). By using this solution, we can get a holomorphic map $\xi: U \to C^7 \backslash \{0\}$ locally. Since f_1 satisfies $\langle f_1, f_1 \rangle = 0$, by direct calculation, we

get $\langle \xi, \xi \rangle = 0$ and $\xi \neq 0$. The above argument we get the following

PROPOSITION 3.1. *Let* $x : M^2 \to S^6$ *be a super minimal* J-*holomorphic curve of* S^6. *Then the map* $\xi : M^2 \to Q^5$ *is a holomorphic map.*

Next, our purpose is to prove the converse of Proposition 3.1. We shall prove the following

THEOREM 3.2. *Let* $\xi : M^2 \to C^7 \backslash \{0\} \cong C \otimes_R \mathrm{Im} O \backslash \{0\}$ *is a holomorphic map satisfying the following conditions;*

$$(1) \qquad\qquad\qquad \langle \xi, \xi \rangle = 0,$$

$$(2) \qquad\qquad\qquad \frac{d\xi}{dz} \neq 0,$$

$$(3) \qquad\qquad\qquad \frac{d\xi}{dz} \times \bar{\xi} \neq 0,$$

$$(4) \qquad\qquad\qquad \left\langle \frac{d\xi}{dz}, \xi \times \bar{\xi} \right\rangle = 0.$$

Then the map

$$\frac{\sqrt{-1}\xi \times \bar{\xi}}{|\xi|^2} : M^2 \longrightarrow S^6$$

is a super minimal J-*holomorphic curve of* S^6.

Remark. The condition (1) is equivalent that ξ is contained in a 5-dimensional complex quadric. The condition (2) is equivalent that ξ is an immersion. The condition (3) means that the map $(\sqrt{-1}\xi \times \bar{\xi})/|\xi|^2 : M^2 \to S^6$ is an immersion. The condition (4) means that the map $(\sqrt{-1}\xi \times \bar{\xi})/|\xi|^2$ is a J-holomorphic curve in S^6. In particular, the condition (4) is important to construct the immersion. In order to prove Theorem 3.2, we prepare some facts. We assume that ξ satisfy the conditions (1)–(4). We may show that the condition (4) is equivalent to the immersion x is a superminimal J-holomorphic curve in S^6. If we put

$$(3.5) \qquad\qquad\qquad \frac{\xi}{|\xi|} = \alpha + \sqrt{-1}\beta$$

where α and β are $\mathrm{Im} O$-valued functions, and $|\xi| = \sqrt{\langle \xi, \bar{\xi} \rangle}$. Then we have

$$0 = \left\langle \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \right\rangle = \langle \alpha, \alpha \rangle - \langle \beta, \beta \rangle + 2\sqrt{-1}\langle \alpha, \beta \rangle$$

and

$$1 = \left\langle \frac{\xi}{|\xi|}, \frac{\bar{\xi}}{|\xi|} \right\rangle = \langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle.$$

Hence we have

(3.6) $$\langle \alpha,\ \alpha \rangle = \langle \beta,\ \beta \rangle = \frac{1}{2},$$

and

(3.7) $$\langle \alpha,\ \beta \rangle = 0.$$

We need two identities to calculate exterior product of $\mathrm{Im}\,O$ as follows;
    Moufang identity

$$(uv)(wu) = u(vw)u$$

for any $u,\ v,\ w \in O$
    The identity of exterior product

$$u \times v = uv + \langle u,\ v \rangle 1$$

$$(u \times v) \times v = \langle u,\ v \rangle v - \langle v,\ v \rangle u,$$

for any $u,\ v \in \mathrm{Im}\,O$.
    Also we need the following two lemmas

LEMMA 3.3.   *The map $\xi\colon M^2 \to Q^5$ is holomorphic if and only if*

(3.8) $$\frac{1}{2}\Big(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} \Big) = \mu \alpha - \nu \beta$$

(3.9) $$\frac{1}{2}\Big(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \Big) = \nu \alpha + \mu \beta$$

where μ and ν are some real valued functions on an open subset $U(\subset M^2)$ and $z = x + \sqrt{-1}\,y$ is an isothermal coordinates on U.

Proof of Lemma 3.3.   Since ξ is a holomorphic, we have

(3.10) $$\frac{\partial}{\partial \bar{z}}\Big(\frac{\xi}{|\xi|} \Big) = \frac{\partial}{\partial \bar{z}}\Big(\frac{1}{|\xi|} \Big)\xi.$$

If we put $\mu + \sqrt{-1}\,\nu = (\partial/\partial \bar{z})(1/|\xi|)$, then we get

$$\text{L.H.S. of (3.10)} = \frac{1}{2}\Big(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} \Big) + \sqrt{-1}\Big(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \Big).$$

On the other hand, we have

$$\text{R.H.S. of (3.10)} = \mu \alpha - \nu \beta + \sqrt{-1}(\nu \alpha + \mu \beta).$$

So we get the desired result.   ∎

LEMMA 3.4.   *Let $\alpha,\ \beta$ be functions defined by (3.5), then we have*

(3.11)
$$\left(\frac{\partial\alpha}{\partial x}\times\beta\right)\times(\alpha\times\beta)=\frac{1}{2}\,\alpha\times\frac{\partial\beta}{\partial y}+2\left\langle\frac{\partial\beta}{\partial y},\,\alpha\times\beta\right\rangle\beta$$

(3.12)
$$\left(\alpha\times\frac{\partial\beta}{\partial x}\right)\times(\alpha\times\beta)=\frac{1}{2}\,\frac{\partial\alpha}{\partial y}\times\beta+2\left\langle\frac{\partial\alpha}{\partial y},\,\alpha\times\beta\right\rangle\beta$$

(3.13)
$$\left(\frac{\partial\alpha}{\partial y}\times\beta\right)\times(\alpha\times\beta)=-\frac{1}{2}\,\alpha\times\frac{\partial\beta}{\partial x}-2\left\langle\frac{\partial\beta}{\partial x},\,\alpha\times\beta\right\rangle\beta$$

(3.14)
$$\left(\alpha\times\frac{\partial\beta}{\partial y}\right)\times(\alpha\times\beta)=-\frac{1}{2}\,\frac{\partial\alpha}{\partial y}\times\beta-2\left\langle\frac{\partial\alpha}{\partial x},\,\alpha\times\beta\right\rangle\beta.$$

Proof of Lemma 3.4. By Moufang identity and (3.6)-(3.9), we have

$$\text{L.H.S. of } (3.11)=-\left(\beta\times\frac{\partial\alpha}{\partial x}\right)\times(\alpha\times\beta)$$

$$=-\left(\left\langle\beta,\,\frac{\partial\alpha}{\partial x}\right\rangle1+\beta\cdot\frac{\partial\alpha}{\partial x}\right)\times(\alpha\cdot\beta)$$

$$=-\left\langle\beta,\,\frac{\partial\alpha}{\partial x}\right\rangle\alpha\cdot\beta-\left\langle\beta\cdot\frac{\partial\alpha}{\partial x},\,\alpha\cdot\beta\right\rangle-\left(\beta\cdot\frac{\partial\alpha}{\partial x}\right)\cdot(\alpha\cdot\beta).$$

By (3.6), we get

$$\left\langle\beta\cdot\frac{\partial\alpha}{\partial x},\,\alpha\cdot\beta\right\rangle=0.$$

By Moufang identity, we have

$$\text{L.H.S. of } (3.11)=-\left\langle\beta,\,\frac{\partial\alpha}{\partial x}\right\rangle\alpha\cdot\beta-\beta\cdot\left(\frac{\partial\alpha}{\partial x}\cdot\alpha\right)\cdot\beta.$$

By (3.6) and (3.8), we get

(3.15)
$$\left\langle\beta,\,\frac{\partial\alpha}{\partial x}\right\rangle=-\nu.$$

Also, by (3.8), we have

(3.16)
$$\frac{\partial\alpha}{\partial x}\cdot\alpha=\frac{\partial\beta}{\partial y}\cdot\alpha-\mu-2\nu\beta\alpha.$$

By (3.15) and (3.16), we get

(3.17)
$$\text{L.H.S. of } (3.11)=-\beta\left(\frac{\partial\beta}{\partial y}\cdot\alpha\right)\beta-\frac{1}{2}\,\mu.$$

Now we calculate $\beta((\partial\beta/\partial y)\cdot\alpha)\beta$. By (3.6), we get

$$\beta\left(\frac{\partial\beta}{\partial y}\cdot\alpha\right)\beta=\beta\left(\frac{\partial\beta}{\partial y}\cdot\alpha-2\left\langle\frac{\partial\beta}{\partial y}\cdot\alpha,\,\beta\right\rangle\beta+2\left\langle\frac{\partial\beta}{\partial y}\cdot\alpha,\,\beta\right\rangle\beta\right)\beta$$

$$=\beta\left(\frac{\partial\beta}{\partial y}\cdot\alpha-2\left\langle\frac{\partial\beta}{\partial y}\cdot\alpha,\,\beta\right\rangle\right)\beta+2\left\langle\frac{\partial\beta}{\partial y}\cdot\alpha,\,\beta\right\rangle\beta^3.$$

Since we have

$$\left\langle \frac{\partial \beta}{\partial y} \cdot \alpha - 2 \left\langle \frac{\partial \beta}{\partial y} \cdot \alpha, \beta \right\rangle \beta, \beta \right\rangle = 0, \quad \text{and} \quad \beta^3 = -\frac{1}{2} \beta.$$

Hence we have

$$(3.18) \quad \beta \left(\frac{\partial \beta}{\partial y} \cdot \alpha \right) \beta = -\beta \left(\left(\frac{\partial \beta}{\partial y} \cdot \alpha - 2 \left\langle \frac{\partial \beta}{\partial y} \cdot \alpha, \beta \right\rangle \beta \right) \bar{\beta} \right) - \left\langle \frac{\partial \beta}{\partial y} \cdot \alpha, \beta \right\rangle \beta$$

$$= \beta^2 \left(\overline{\frac{\partial \beta}{\partial y} \cdot \alpha - 2 \left\langle \frac{\partial \beta}{\partial y} \cdot \alpha, \beta \right\rangle \beta} \right) - \left\langle \frac{\partial \beta}{\partial y}, \alpha \times \beta \right\rangle \beta$$

$$= -\frac{1}{2} \left(\bar{\alpha} \cdot \overline{\frac{\partial \beta}{\partial y}} - 2 \left\langle \frac{\partial \beta}{\partial y} \cdot \alpha, \beta \right\rangle \bar{\beta} \right) - \left\langle \frac{\partial \beta}{\partial y}, \alpha \times \beta \right\rangle \beta$$

$$= -\frac{1}{2} \alpha \cdot \frac{\partial \beta}{\partial y} - 2 \left\langle \frac{\partial \beta}{\partial y}, \alpha \times \beta \right\rangle \beta$$

$$= -\frac{1}{2} \mu - \frac{1}{\alpha} \alpha \times \frac{\partial \beta}{\partial y} - 2 < \frac{\partial \beta}{\partial y}, \alpha \times \beta \rangle \beta$$

By (3.17) and (3.18), we have (3.15). Similarly we get (3.12)-(3.14). ∎

Now we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. We put the map $\tilde{x} = (\sqrt{-1} \xi \times \bar{\xi})/|\xi|^2$. We can easily see that

$$(3.19) \qquad\qquad \tilde{x} = 2\alpha \times \beta.$$

Also, by direct calculations, we get

$$d\tilde{x} \left(\frac{\partial}{\partial z} \right) = \frac{\sqrt{-1}}{|\xi|^2} \frac{d\xi}{dz} \times \bar{\xi} + \frac{d}{dz} \left(\frac{1}{|\xi|^2} \right) \sqrt{-1} \xi \times \bar{\xi}.$$

Hence, if $|\xi|^2 \neq 0$ and $(d\xi/dz) \times \bar{\xi} \neq 0$ then x is an immersion from M^2 to S^6. Next we may show that the following identity under the conditions (1) to (4) are satisfied:

$$(3.20) \qquad\qquad d\tilde{x} \left(\frac{\partial}{\partial z} \right) \times \tilde{x} = \sqrt{-1} d\tilde{x} \left(\frac{\partial}{\partial z} \right)$$

on M^2. In fact, by (3.19), we get

$$\text{L.H.S. of } (3.20) = d\tilde{x} \left(\frac{\partial}{\partial z} \right) \times \tilde{x} = 4 \frac{\partial(\alpha \times \beta)}{\partial z} \times (\alpha \times \beta)$$

$$= 2 \frac{\partial(\alpha \times \beta)}{\partial x} \times (\alpha \times \beta) - 2\sqrt{-1} \frac{\partial(\alpha \times \beta)}{\partial y} \times (\alpha \times \beta).$$

On the other hand, we have

$$\text{R.H.S. of } (3.20) = \sqrt{-1}\, d\tilde{x}\left(\frac{\partial}{\partial z}\right) = \sqrt{-1}\,\frac{\partial(\alpha\times\beta)}{\partial x} + \frac{\partial(\alpha\times\beta)}{\partial y}.$$

Therefore the identity (3.20) is satisfied if and only if

(3.21)
$$\frac{\partial(\alpha\times\beta)}{\partial x}\times(\alpha\times\beta) = \frac{1}{2}\frac{\partial(\alpha\times\beta)}{\partial y},$$

(3.22)
$$\frac{\partial(\alpha\times\beta)}{\partial y}\times(\alpha\times\beta) = -\frac{1}{2}\frac{\partial(\alpha\times\beta)}{2\partial x}.$$

Therefore, by Lemma 3.4, the identities (3.21) and (3.22) are holds, if and only if

$$\left\langle\frac{\partial\beta}{\partial y}, \alpha\times\beta\right\rangle = \left\langle\frac{\partial\alpha}{\partial y}, \alpha\times\beta\right\rangle = \left\langle\frac{\partial\beta}{\partial x}, \alpha\times\beta\right\rangle = \left\langle\frac{\partial\alpha}{\partial x}, \alpha\times\beta\right\rangle = 0.$$

From this and (3.5) the identities (3.21) and (3.22) are satisfied if and only if the condition (4) holds. We get the desired result. ∎

4. Holomorphic horizontal curves in $G_2(\operatorname{Im}O)$

The purpose of this section is to prove the following Theorem 4.1, by applying Theorem 3.2. We set a basis of $C\otimes_R \operatorname{Im}O$ by $\varepsilon=(0, 1)\in H\oplus H$, $E_1=iN$, $E_2=jN$, $E_3=-kN$, $\bar{E}_1=i\bar{N}$, $\bar{E}_2=j\bar{N}$ and $\bar{E}_3=-k\bar{N}$, where $N=(1-\sqrt{-1}\varepsilon)/2$, $\bar{N}=(1+\sqrt{-1}\varepsilon)/2\in C\otimes_R O$ and $\{1, i, j, k\}$ is the canonical basis of H. Then we have $\langle E_i, E_j\rangle=\langle\bar{E}_i, \bar{E}_j\rangle=\langle N, N\rangle=\langle\bar{N}, \bar{N}\rangle=0$, $\langle E_i, \bar{E}_j\rangle=(1/2)\delta_{ij}$, and $\langle N, \bar{N}\rangle=1/2$.

THEOREM 4.1. *Let* $\xi: C\to Q^5$ *be the holomorphic map defined as follows;*

$$\xi = E, 2 + \bar{E}_2(2\sqrt{6}z) + \bar{E}_3(2\sqrt{-1}\sqrt{15}z^2) + \varepsilon(2\sqrt{10}z^3)$$
$$+ E_3(2\sqrt{-1}\sqrt{15}z^4) + E_2(2\sqrt{6}z^5) + \bar{E}_1(-2z^6),$$

where $\{\varepsilon, E_1, E_2, E_3, \bar{E}_1, \bar{E}_2, \bar{E}_3\}$ *is the basis of* $C\otimes_R \operatorname{Im}O$ *(defined above). Then the map*

$$\frac{\sqrt{-1}\xi\times\bar{\xi}}{|\xi|^2}: P^1(C)\longrightarrow S^6$$

coincide with the Boruvka sphere (superminimal J-holomorphic curve of S^6*)* ψ_0: $S^2(1/6)\to S^6(1)$.

Proof of Theorem 4.1. We may show that ξ satisfies 4-conditions in Theorem 3.2. By direct calculation, we get

(4.1) $\langle\xi, \xi\rangle = 2(-2z^6) + (2\sqrt{6}z)(2\sqrt{6}z^5) - (2\sqrt{15}z^2)(2\sqrt{15}z^4) + (2\sqrt{10}z^3)^2 = 0$

So we have $\xi: C\to Q^5$. Hence the condition (1) of Theorem 3.2 is satisfied.

Since the ξ is a polynomial of order 6, we may check that the corresponding infinitesimal conditions at $z=0$. We have

$$\xi(0)=2E_1,$$

$$\sqrt{-1}\xi(0)\times\overline{\xi(0)}=4\sqrt{-1}E_1\times\bar{E}_1=2\varepsilon,$$

$$\xi'(0)=\bar{E}_2(2\sqrt{6}).$$

Hence we have

$$\langle\xi'(0),\ \xi(0)\times\overline{\xi(0)}\rangle=\langle\bar{E}_2(2\sqrt{6}),\ -2\sqrt{-1}\varepsilon\rangle=0.$$

Also we have

$$\xi''(0)=\bar{E}_3(4\sqrt{-1}\sqrt{15}),\qquad\langle\xi''(0),\ -2\sqrt{-1}\varepsilon\rangle=0.$$

$$\xi'''(0)=\varepsilon(12\sqrt{10}),$$

$$\langle\xi'''(0),\ -2\sqrt{-1}\varepsilon\rangle+\langle\xi''(0),\ \xi'(0)\times\overline{\xi(0)}\rangle$$

$$=\langle\varepsilon(12\sqrt{10}),\ -2\sqrt{-1}\varepsilon\rangle$$

$$+\langle\bar{E}_3(4\sqrt{-1}\sqrt{15}),\ \bar{E}_2(2\sqrt{6})\times2\bar{E}_1\rangle=0.$$

$$\langle\xi^{(4)}(0),\ \xi(0)\times\overline{\xi(0)}\rangle+2\langle\xi'''(0),\ \xi'(0)\times\overline{\xi(0)}\rangle=0,$$

$$\langle\xi^{(5)}(0),\ \xi(0)\times\overline{\xi(0)}\rangle+3\langle\xi^{(4)}(0),\ \xi'(0)\times\overline{\xi(0)}\rangle$$

$$+2\langle\xi'''(0),\ \xi''(0)\times\overline{\xi(0)}\rangle=0,$$

$$\langle\xi^{(6)}(0),\ \xi(0)\times\overline{\xi(0)}\rangle+4\langle\xi^{(5)}(0),\ \xi'(0)\times\overline{\xi(0)}\rangle$$

$$+5\langle\xi^{(4)}(0),\ \xi''(0)\times\overline{\xi(0)}\rangle=0.$$

Hence at $z=0$, ξ satisfy condition (4). Next we shall calculate $\xi\times\bar{\xi}$, by using the following table of the exterior product of $C\otimes_R O$. (See the next page.)

We can easily see that $\bar{\xi}$ is given by;

$$\bar{\xi}=N(2\sqrt{-1}\sqrt{10}\bar{z}^3)+\bar{N}(-2\sqrt{-1}\sqrt{10}\bar{z}^3)$$

$$+E_1(-2\bar{z}^3)+\bar{E}_1 2$$

$$+E_2(2\sqrt{6}\bar{z})+\bar{E}_2(2\sqrt{6}\bar{z}^5)$$

$$+E_3(-2\sqrt{-1}\sqrt{15}\bar{z}^2)+\bar{E}_3(-2\sqrt{-1}\sqrt{15}\bar{z}^4).$$

columm×row	N	E_1	E_2	E_3	$\bar{N}$	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$
N	0	$-E_1$	$-E_2$	$-E_3$	$-\dfrac{\sqrt{-1}}{2}\varepsilon$	0	0	0
E_1	E_1	0	$-\bar{E}_3$	$\bar{E}_2$	0	$-\dfrac{\sqrt{-1}}{2}\varepsilon$	0	0
E_2	E_2	$\bar{E}_3$	0	$-\bar{E}_1$	0	0	$-\dfrac{\sqrt{-1}}{2}\varepsilon$	0
E_3	E_3	$-\bar{E}_2$	$\bar{E}_1$	0	0	0	0	$-\dfrac{\sqrt{-1}}{2}\varepsilon$
$\bar{N}$	$\dfrac{\sqrt{-1}}{2}\varepsilon$	0	0	0	0	$-\bar{E}_1$	$-\bar{E}_2$	$-\bar{E}_3$
$\bar{E}_1$	0	$\dfrac{\sqrt{-1}}{2}\varepsilon$	0	0	$\bar{E}_1$	0	$-E_3$	E_2
$\bar{E}_2$	0	0	$\dfrac{\sqrt{-1}}{2}\varepsilon$	0	$\bar{E}_2$	E_3	0	$-E_1$
$\bar{E}_3$	0	0	0	$\dfrac{\sqrt{-1}}{2}\varepsilon$	$\bar{E}_3$	$-E_2$	E_1	0

Hence we get

$$
\begin{aligned}
\sqrt{-1}\xi\times\bar{\xi}=\;& i\,(-2\sqrt{10})(z^3+\bar{z}^3)(1+|z|^2)^3 \\
&+j\,(2\sqrt{15})(z^2+\bar{z}^2)(1+2|z|^2-2|z|^6-|z|^8) \\
&+k\,(-2\sqrt{6}\sqrt{-1}(z-\bar{z}))(1-5|z|^4-5|z|^6+|z|^{10}) \\
&+\varepsilon 2(1-6|z|^2-15|z|^4+15|z|^8+6|z|^{10}-|z|^{12}) \\
&+i\varepsilon\,(-2\sqrt{10})(\sqrt{-1}(z^3-\bar{z}^3))(1+|z|^2)^3 \\
&+j\varepsilon\,(2\sqrt{15})(\sqrt{-1}(z^2-\bar{z}^2))(1+2|z|^2-2|z|^6-|z|^8) \\
&+k\varepsilon\,(-2\sqrt{6}(z+\bar{z}))(1-5|z|^4-5|z|^6+|z|^{10}).
\end{aligned}
$$

Next we show that the map $(\sqrt{-1}\xi\times\bar{\xi})/|\xi|^2$ coincide with the Boruvka sphere. To prove this, we take a coordinate as follows;

$$
(4.2) \qquad y_1=\frac{|z|^2-1}{|z|^2+1}, \quad y_2+\sqrt{-1}\,y_3=\frac{2z}{|z|^2+1}.
$$

Then we have $y_1^2+y_2^2+y_3^2=1$. By (4.2), we get

$$
\begin{aligned}
\sqrt{-1}\xi\times\bar{\xi}=\frac{2^5}{(1-y)^6}\,\{ & i\,(-\sqrt{10})y_2(y_2^2-3y_3^2) \\
&+j\,(-2\sqrt{15})y_1(y_2^2-y_3^2)
\end{aligned}
$$

$$+k(\sqrt{6})y_3(y_2{}^2+y_3{}^2-4y_1{}^2)$$

$$+\varepsilon 2y_1(3y_2{}^2+3y_3{}^2-2y_1{}^2)$$

$$+i\varepsilon(\sqrt{10})y_2(3y_2{}^2-y_3{}^2)$$

$$+j\varepsilon(4\sqrt{15})y_1y_2y_3)$$

$$+k\varepsilon(\sqrt{6})y_2(y_2{}^2+y_3{}^2-4y_1{}^2)\}$$

$$=\frac{2^5}{(1-y)^6}\phi_1.$$

Then we have $\langle\phi_1, \phi_1\rangle=16$ so we get

$$\frac{\sqrt{-1}\xi\times\bar{\xi}}{|\xi|^2}=\frac{1}{4}\phi_1.$$

From this representation, we obtain the map $(1/4)\phi_1: S^2(1)\to S^6(1)$ which is not isometric immersion. By direct calculation, the Gauss curvature of the induced metric of $(1/4)\phi_1$ is identically $1/6$. We get the desired result. ■

Remark. More precisely, S^2 is simply connected and Theorem 9.3 in [H1], we get the desired conclusion by standard monodromy argument.

REFERENCES

[Br1] R. L. BRYANT, Submanifolds and special structures on the octonians, J. Differential Geom., 17 (1982), 185-232.

[Br2] R. L. BRYANT, Minimal surfaces of constant curvature in S^n, Trans. Amer. Math. Soc., 290 (1985), 259-271.

[Ch] S. S. CHERN, On minimal spheres in the four spheres, Selected papers, Springer Verlag, New York, 1978, 421-434.

[G-H] G. GRIFFITHS AND J. HARRIS, Principles of algebraic geometry, Wiley-Interscience, New York, 1978.

[H-L] R. HARVEY AND H. B. LAWSON, Calibrated geometries, Acta Math., 148 (1982), 47-157.

[H-S] H. HASHIMOTO AND K. SEKIGAWA, Minimal surfaces in a 4-dimensional sphere, Houston J. Math., 21 (1995), 449-464.

[H1] H. HASHIMOTO, *J*-holomorphic curves in a 6-dimensional sphere (to appear).

[Se] K. SEKIGAWA, Almost complex submanifolds of a 6-dimensioanl sphere, Kodai Math. J., 6 (1983), 174-185.

[Sp] M. SPIVAK, A Comprehensive Introduction to Differential Geometry **IV**, Publish or Perish., 1975.

NIPPON INSTITUTE OF TECHNOLOGY
4-1, GAKUENDAI, MIYASHIRO
MINAMI-SAITAMA GUN
SAITAMA, 345, JAPAN
E-mail: hideya@nit.ac.jp