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COMPLETE MAXIMAL SPACELIKE SUBMANIFOLDS

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Abstract

We generalize Simons' method to spacelike submanifolds of $M_q^{n+p}(c)$ $(1\leqq q\leqq p)$ and characterize the totally geodesic submanifolds of $S_q^{n+p}(c)(1\leqq q\leqq p)$ under the pinching conditions on scalar curvature, Ricci curvature and sectional curvature, respectively.

1. Introduction

Let $M_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite Riemannian manifold of index $q(1\leqq q\leqq p)$ and of constant curvature c, which is called an indefiite space form of index q. According to $c>0$, $c=0$ and $c<0$, it is denoted by $S_q^{n+p}(c)$, $\boldsymbol{R}_q^{n+p}$ or $H_q^{n+p}(c)$. A submanifold M^n of an indefinite space form $M_q^{n+p}(c)$ is said to be *spacelike* if the induced metric on M^n from that of $M_q^{n+p}(c)$ is positive definite. $\boldsymbol{R}^n$ can be embedded in $S_1^{n+1}(c)$ as a complete totally umbilical spacelike submanifold. But it can not be embedded in the unit sphere $S^m(c)$ as a totally umbilical submanifold. Hence it is very interesting to investigate complete spacelike submanifolds in $M_q^{n+p}(c)$.

When $p=q$, we know that complete maximal spacelike submanifolds in $S_p^{n+p}(c)$ or $\boldsymbol{R}_p^{n+p}$ are totally geodesic (cf. [3]). Hence the class of all such submanifolds are very small. But if $q<p$ we shall see that the class of complete maximal spacelike submanifolds is very large. In fact, if M^n is a complete minimal submanifold in sphere $S^m(c)(m>n)$ of constant curvature c embeded in $S_q^{m+q}(c)$ as a totally geodesic spacelike submanifold where $m-n+q=p$, then M^n is a complete maximal spacelike submanifold in $S_q^{n+p}(c)$. In [1], Alias and Romero studied the compact maximal spacelike submanifolds in $S_q^{n+p}(c)$. They proved that if M^n is a compact maximal spacelike submanifold in $S_q^{n+p}(c)$ with Ricci curvature $\mathrm{Ric}(M^n)\geqq(n-1)c$, then M^n is totally geodesic. And they indicated that to get a Bernstein type result, the bound on the Ricci curvature

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is best possible. But their statement can not guarantee this fact. In fact, according to the theory of minimal submanifold in sphere, we know that there are no n-dimensional compact minimal submanifold in $S^m(c)$ of which the Ricci curvature satisfies $(n-2)c<\mathrm{Ric}(M^n)<(n-1)c$. Hence the set of examples which they supposed is empty if $(n-2)c<\mathrm{Ric}(M^n)<(n-1)c$.

The purpose of this paper is to generalize the Simons' method to complete spacelike submanifolds in $M_q^{n+p}(c)$ and to get the following theorems. In particular, we obtain the best possible bound on the Ricci curvature of a complete maximal spacelike submanifold in the de Sitter space $S_1^{n+2}(c)$.

THEOREM 1. *Let M^n be an n-dimensional compact maximal spacelike submanifold in the de Sitter space $S_q^{n+p}(c)(1\leqq q\leqq p)$. If*

$$S\leqq\max\left\{\frac{nc}{2-(1/(p-q))},\ \frac{2nc}{3}\right\},$$

then

(1) *M^n is the totally geodesic submanifold in $S_q^{n+p}(c)$, or*

(2) *$p-q=1$, M^n lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S_q^{n+q+1}(c)$ and is isometric to the Clifford torus $S^k((n/k)c)\times S^{n-k}((n/(n-k))c)$ or*

(3) *$n=2$ and $p-q=2$, M^2 lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$ and is isometric to the Veronese surface where S is the squared norm of the second fundamental form of M^n.*

Remark 1. When M^n is an n-dimensional complete maximal spacelike submanifold in the de Sitter space $S_q^{n+p}(c)(1\leqq q\leqq p)$, and S satisfies the condition

$$\sup S<\max\left\{\frac{nc}{2-(1/(p-q))},\ \frac{2nc}{3}\right\},$$

we can prove that M^n is the totally geodesic submanifold in $S_q^{n+p}(c)$.

THEOREM 2. *Let M^n be an n-dimensional compact maximal spacelike submanifold in the de Sitter space $S_q^{n+q+1}(c)$. If the sectional curvature K of M^n is positive, then M^n is the totally geodesic submanifold in $S_q^{n+q+1}(c)$.*

Remark 2. The Clifford torus $S^{n-k}((n/(n-k))c)\times S^k((n/k)c)$ in $S^{n+1}(c)$ can be embedded in $S_q^{n+q+1}(c)$ as a compact maximal spacelike submanifold with nonnegative curvature and it is not totally geodesic. Hence, the bound on the sectional curvature is best possible.

THEOREM 3. *Let M^n be an n-dimensional complete maximal spacelike submanifold in the de Sitter space $S_1^{n+2}(c)$. If* $\mathrm{Ric}(M^n)\geqq(n-2)c$, *then M^n is totally geodesic submanifold in $S_1^{n+2}(c)$ or M^n is a maximal spacelike Einstein submanifold with* $\mathrm{Ric}(M^n)=(n-2)c$ *and the parallel second fundamental form.*

Remark 3. Let $n=2k$. The Clifford torus $S^k(2c)\times S^k(2c)$ of $S^{n+1}(c)$ can be embedded in $S_1^{n+2}(c)$ as a compact spacelike maximal submanifold with $\mathrm{Ric}(M^n)=(n-2)c$ and the parallel second fundamental form. It is open for authors whether there exist the other compact maximal spacelike submanifolds in $S_1^{n+2}(c)$ with $\mathrm{Ric}(M^n)=(n-2)c$ and the parallel second fundamental form except the Clifford torus $S^k(2c)\times S^k(2c)$.

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2. Preliminaries

Let $M_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of constant curvature c whose index is $q(1\leqq q\leqq p)$ and M^n an n-dimensional connected Riemannian manifold immersed in $M_q^{n+p}(c)$. We choose a local frame of orthonormal vector fields $\{e_1, \dots, e_{n+p}\}$ adapted to the indefinite Riemannian metric of $M_q^{n+p}(c)$ and the dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$ in such a way that, restricted to the submanifold M^n, $\{e_1, \dots, e_n\}$ are tangent to M^n. Then the connection forms $\{\omega_{AB}\}$ of $M_q^{n+p}(c)$ are characterized by the structure equations

$$
(2.1)\qquad
\begin{cases}
d\omega_A=-\sum_{B=1}^{n+p}\varepsilon_B\omega_{AB}\wedge\omega_B, \qquad \omega_{AB}+\omega_{BA}=0 \\
d\omega_{AB}=-\sum_{C=1}^{n+p}\varepsilon_C\omega_{AC}\wedge\omega_{CB}-\frac{1}{2}\sum_{C,D=1}^{n+p}\varepsilon_C\varepsilon_D K_{ABCD}\omega_C\wedge\omega_D,
\end{cases}
$$

$$
K_{ABCD}=c\varepsilon_A\varepsilon_B(\delta_{AD}\delta_{BC}-\delta_{AC}\delta_{BD}),
$$

where $\varepsilon_A=1$ for $1\leqq A\leqq n+p-q$, $\varepsilon_A=-1$ for $n+p-q+1\leqq A\leqq n+p$ and K_{ABCD} denotes the components of indefinite Riemannian curvature tensor of $M_q^{n+p}(c)$.

The canonic forms $\{\omega_A\}$ and connection forms $\{\omega_{AB}\}$ restricted to M^n are also denoted by the same symbols. We then see

$$
(2.2)\qquad \omega_\alpha=0, \qquad \alpha=n+1, \dots, n+p,
$$

and $\{e_1, \dots, e_n\}$ is a local frame of orthonormal vector fields adapted to the induced Riemannian metric on M^n and $\{\omega_1, \dots, \omega_n\}$ is its dual coframe on M^n. It follows from (2.1), (2.2) and Cartan's Lemma that

$$
(2.3)\qquad \omega_{\alpha i}=\sum_{j=1}^{n}h_{ij}^\alpha\omega_j, \qquad h_{ij}^\alpha=h_{ji}^\alpha .
$$

The second fundamental form Π and the mean curvature vector $\boldsymbol{h}$ of M^n are defined by

$$
(2.4)\qquad \Pi=\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}\varepsilon_\alpha h_{ij}^\alpha\omega_i\omega_j e_\alpha ,
$$

and

(2.5) $$h=\frac{1}{n}\sum_{\alpha=n+1}^{n+p}\varepsilon_\alpha\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)e_\alpha$$

respectively. The mean curvature H of M^n is defined by

(2.7) $$H=\frac{1}{n}\sqrt{\sum_{\alpha=n+1}^{n+p}\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)^2}.$$

If $H=0$, we recall that M^n is *maximal.* Let

$$S=\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2$$

denote the squared norm of the second fundamental form II of M^n. The connection forms of M^n are characterized by the structure equations

(2.8) $$d\omega_i=-\sum_{j=1}^{n}\omega_{ij}\wedge\omega_j,\qquad \omega_{ij}+\omega_{ji}=0,$$

(2.9) $$d\omega_{ij}=-\sum_{k=1}^{n}\omega_{ik}\wedge\omega_{kj}-\frac{1}{2}\sum_{k,l=1}^{n}R_{ijkl}\omega_k\wedge\omega_l$$

where R_{ijkl} are the components of the curvature tensor of M^n, that is,

(2.10) $$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+\sum_{\alpha=n+1}^{n+p-q}(h_{il}^\alpha h_{jk}^\alpha-h_{ik}^\alpha h_{jl}^\alpha)$$
$$-\sum_{\alpha=n+p-q+1}^{n+p}(h_{il}^\alpha h_{jk}^\alpha-h_{ik}^\alpha h_{jl}^\alpha).$$

Letting R_{ij} and r denote the components of the Ricci curvature and the scalar curvature of M^n respectively, we have from (2.10)

(2.11) $$R_{jk}=(n-1)c\delta_{jk}+\sum_{\alpha=n+1}^{n+p-q}\Big(\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)h_{jk}^\alpha-\sum_{i=1}^{n}h_{ik}^\alpha h_{ji}^\alpha\Big)$$
$$-\sum_{\alpha=n+p-q+1}^{n+p}\Big(\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)h_{jk}^\alpha-\sum_{i=1}^{n}h_{ik}^\alpha h_{ji}^\alpha\Big)$$

and

(2.12) $$r=n(n-1)c+\sum_{\alpha=n+1}^{n+p-q}\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)^2-\sum_{\alpha=n+1}^{n+p-q}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2$$
$$-\sum_{\alpha=n+p-q+1}^{n+p}\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)^2+\sum_{\alpha=n+p-q+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2$$

respectively. We also have

(2.13) $$d\omega_{\alpha\beta}=-\sum_{\gamma=n+1}^{n+p}\varepsilon_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\frac{1}{2}\sum_{i,j=1}^{n}R_{\alpha\beta ij}\omega_i\wedge\omega_j,$$

(2.14) $$R_{\alpha\beta ij}=-\sum_{l=1}^{n}(h_{il}^{\alpha}h_{lj}^{\beta}-h_{jl}^{\alpha}h_{li}^{\beta}).$$

By taking the exterior differentiation of (2.3) and defining h_{ijk}^{α} by

(2.15) $$\sum_{k=1}^{n}h_{ijk}^{\alpha}\omega_k=dh_{ij}^{\alpha}-\sum_{k=1}^{n}h_{ik}^{\alpha}\omega_{kj}-\sum_{k=1}^{n}h_{jk}^{\alpha}\omega_{ki}-\sum_{\beta=n+1}^{n+p}\varepsilon_\beta h_{ij}^{\beta}\omega_{\beta\alpha},$$

we get the Codazzi equation

(2.16) $$h_{ijk}^{\alpha}=h_{ikj}^{\alpha}=h_{jik}^{\alpha}.$$

We take the exterior differentiation of (2.15) and define h_{ijkl}^{α} by

(2.17) $$\sum_{l=1}^{n}h_{ijkl}^{\alpha}\omega_l=dh_{ijk}^{\alpha}-\sum_{l=1}^{n}h_{ljk}^{\alpha}\omega_{li}-\sum_{l=1}^{n}h_{ilk}^{\alpha}\omega_{lj}$$
$$-\sum_{l=1}^{n}h_{ijl}^{\alpha}\omega_{lk}-\sum_{\beta=n+1}^{n+p}\varepsilon_\beta h_{ijk}^{\beta}\omega_{\beta\alpha}.$$

Hence, the Ricci formula for the second fundamental form is given by

(2.18) $$h_{ijkl}^{\alpha}-h_{ijlk}^{\alpha}=-\sum_{m-1}^{n}h_{mj}^{\alpha}R_{mikl}-\sum_{m=1}^{n}h_{im}^{\alpha}R_{mjkl}-\sum_{\beta=n+1}^{n+p}\varepsilon_\beta h_{ij}^{\beta}R_{\beta\alpha kl}.$$

The Laplacian Δh_{ij}^{α} of h_{ij}^{α} is defined by

$$\Delta h_{ij}^{\alpha}=\sum_{k=1}^{n}h_{ijkk}^{\alpha}.$$

From the Codazzi equation (2.16) and the Ricci formula (2.18) we get, for the maximal submanifold M^n in $M_q^{n+p}(c)$,

(2.19) $$\Delta h_{ij}^{\alpha}=\sum_{k=1}^{n}h_{kijk}^{\alpha}=\sum_{k=1}^{n}h_{kikj}^{\alpha}-\sum_{k,m=1}^{n}h_{km}^{\alpha}R_{mijk}-\sum_{k,m=1}^{n}h_{mi}^{\alpha}R_{mkjk}$$
$$-\sum_{k=1}^{n}\sum_{\beta=n+1}^{n+p}\varepsilon_\beta h_{ki}^{\beta}R_{\beta\alpha jk}$$
$$=-\sum_{k,m=1}^{n}h_{km}^{\alpha}R_{mijk}-\sum_{k,m=1}^{n}h_{mi}^{\alpha}R_{mkjk}-\sum_{k=1}^{n}\sum_{\beta=n+1}^{n+p}\varepsilon_\beta h_{ki}^{\beta}R_{\beta\alpha jk}.$$

Thus we get

LEMMA. *For the squared norm S of the second fundamental form of the maximal submanifold M^n in $M_q^{n+p}(c)$, we have*

$$\frac{1}{2}\Delta S=\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2+\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$
$$=\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2-\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k,m=1}^{n+p}h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}$$

$$-\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k,m=1}^{n} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}-\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}.$$

3. Proofs of theorems

We define S_1 and S_2 by

$$S_1:=\sum_{\alpha=n+1}^{n+p-q}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2,\qquad S_2:=\sum_{\alpha=n+p-q+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2,$$

respectively. Then

$$S=S_1+S_2.$$

Proof of Theorem 1. Since

$$\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha}h_{kl}^{\alpha}R_{lijk}=-c\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2$$
$$+\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{kl}^{\alpha}(h_{lk}^{\beta}h_{ij}^{\beta}-h_{lj}^{\beta}h_{ik}^{\beta}).$$

$$\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk}=c\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2-nc\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2$$
$$+\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{li}^{\alpha}(h_{lk}^{\beta}h_{kj}^{\beta}-h_{lj}^{\beta}h_{kk}^{\beta})$$

and

$$\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}=-\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}(h_{lk}^{\alpha}h_{lj}^{\beta}-h_{lj}^{\alpha}h_{lk}^{\beta}),$$

we conclud, by using Lemma in the section 2,

$$\frac{1}{2}\Delta S=\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2+nc\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}(h_{ij}^{\alpha})^2$$
$$-\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{kl}^{\alpha}(h_{lk}^{\beta}h_{ij}^{\beta}-h_{lj}^{\beta}h_{ik}^{\beta})$$
$$-\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{li}^{\alpha}(h_{lk}^{\beta}h_{kj}^{\beta}-h_{lj}^{\beta}h_{kk}^{\beta})$$
$$+\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}(h_{lk}^{\alpha}h_{lj}^{\beta}-h_{lj}^{\alpha}h_{lk}^{\beta})$$
$$=\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2+nc\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k,l=1}^{n}(h_{ij}^{\alpha})^2$$
$$-\sum_{\alpha,\beta=n+1}^{n+p}\varepsilon_{\beta}[\mathrm{trace}(H_{\alpha}H_{\beta})]^2-\sum_{\alpha,\beta=n+1}^{n+p}\varepsilon_{\beta}N(H_{\alpha}H_{\beta}-H_{\beta}H_{\alpha})$$

where $H_\alpha=(h_{ij}^\alpha)$. Here we denote $N(A)=\mathrm{trace}(A^tA)$ for the $n\times n$-matrix $A=(a_{ij})$ and the transposed matrix A^t of A. Then we know $N(H_\alpha H_\beta-H_\beta H_\alpha)\geqq 0$ for any α and β. Moreover, we put $S_{\alpha\beta}=\sum_{i,j=1}^n h_{ij}^\alpha h_{ij}^\beta$, then the $(p\times p)$-matrix $(S_{\alpha\beta})$ is symmetric. So we can choose $\{e_{n+1},\dots,e_{n+p}\}$ such that $(S_{\alpha\beta})$ is diagonal.

Now we divide the proof of Theorem 1 into two cases.

Case 1. $p-q=1$. From (3.1), we have

$$
\begin{aligned}
(3.2)\quad \frac{1}{2}\Delta S=&\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^\alpha)^2+nc\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2+\sum_{\alpha,\beta=n+2}^{n+p}N(H_\alpha H_\beta-H_\beta H_\alpha)\\
&+\sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{trace}(H_\alpha H_\beta)]^2-N(H_{n+1})^2\\
\geqq& nc\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2-N(H_{n+1})^2+\sum_{\alpha=n+2}^{n+p}N(H_\alpha)^2\\
\geqq& ncS-S^2.
\end{aligned}
$$

From the assumptions in Theorem 1 and the Stokes formula, we get $S=0$ or $S=nc$. If $S=0$, then M^n is totally geodesic. If $S=nc$, from the above (3.2), we know $S_2=0$ on M^n, i.e., $h_{ij}^\alpha=0$ on M^n for $\alpha=n+2,\dots,n+p$. Hence M^n lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S_q^{n+q+1}(c)$ (see Theorem 1 in [6]). Thus M^n becomes a compact minimal hypersurface in $S^{n+1}(c)$ such that the squared norm S of the second fundamental form is equal to nc. From the result due to Chern-do Carmo and Kobayashi [2], we know that M^n is isometric to the Clifford torus. We complete the proof of Theorem 1 in this case.

Case 2. $p-q>1$. In this case, we have

$$
\begin{aligned}
\frac{1}{2}\Delta S=&\sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^\alpha)^2+nc\sum_{\alpha=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2\\
&+\sum_{\alpha,\beta=n+p-q+1}^{n+p}N(H_\alpha H_\beta-H_\beta H_\alpha)+\sum_{\alpha,\beta=n+p-q+1}^{n+p}[\mathrm{trace}(H_\alpha H_\beta)]^2\\
&-\sum_{\alpha,\beta=n+1}^{n+p-q}N(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_{\alpha,\beta=n+1}^{n+p-q}[\mathrm{trace}(H_\alpha H_\beta)]^2.
\end{aligned}
$$

From a Lemma due to Li-Li in [4], we get

$$
-\sum_{\alpha,\beta=n+1}^{n+p-q}N(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_{\alpha,\beta=n+1}^{n+p-q}[\mathrm{trace}(H_\alpha H_\beta)]^2\geqq-\frac{3}{2}\Big[\sum_{\alpha=n+1}^{n+p-q}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2\Big]^2.
$$

Hence, we get

$$
(3.3)\qquad \frac{1}{2}\Delta S\geqq\Big(ncS-\frac{3}{2}S^2\Big)+\sum_{\alpha=n+p-q+1}^{n+p}N(H_\alpha)^2.
$$

From the Stokes formula, the assumptions in Theorem 1 and (3.3), we get $S=(2/3)nc$ or $S=0$. If $S=0$, then M^n is totally geodesic. If $S=(2/3)nc$, we know

$h_{ij}^{\alpha}=0$ on M^n for $\alpha=n+p-q, \dots, n+p$. Hence M^n lies in the totally geodesic spacelike submanifold $S^{n+p-q}(c)$ of $S_q^{n+p}(c)$ (see Theorem 1 in [6]). Thus M^n becomes a compact minimal submanifold in $S^{n+p-q}(c)$ such that the squared norm S of the second fundamental form is equal to $(2/3)nc$. From the result due to Li-Li [4], we know that $n=p-q=2$ and M^n is isometric to a Veronese surface. Theorem 1 holds in this case. We complete the proof of Theorem 1.

PROPOSITION. *Let M^n be an n-dimensional compact maximal spacelike submanifold in the de Sitter space $S_q^{n+q+1}(c)$. It the sectional curvature K of M^n is nonnegative, then M^n is totally geodesic or M^n is a compact maximal spacelike submanifold with parallel second fundamental form.*

Proof of Proposition. For any fixed α, we can choose $e_1, \dots, e_n$ such that $h_{ij}^{\alpha}=\lambda_i^{\alpha}\delta_{ij}$. Then we have

$$\begin{aligned}
&-\sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha}h_{kl}^{\alpha}R_{lijk}-\sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk}\\
&=-\sum_{i,k=1}^{n}\lambda_i^{\alpha}\lambda_k^{\alpha}R_{kiik}-\sum_{i,k=1}^{n}(\lambda_i^{\alpha})^2R_{ikik}\\
&=\frac{1}{2}\sum_{i,k=1}^{n}(\lambda_i^{\alpha}-\lambda_k^{\alpha})^2R_{kiik}\\
&\geqq\frac{1}{2}\sum_{i,k=1}^{n}(\lambda_i^{\alpha}-\lambda_k^{\alpha})^2K_0=nK_0\sum_{i,k=1}^{n}(h_{ik}^{\alpha})^2,
\end{aligned}$$

where K_0 denotes the infimum of the sectional curvature of M^n. Since the both sides of the above inequality do not depend on the choice of the orthonormal frame $\{e_1, \dots, e_n\}$, we have

$$\begin{aligned}
(3.4)\qquad &-\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha}h_{kl}^{\alpha}R_{lijk}-\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk}\\
&\geqq nK_0\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2\geqq nK_0S.
\end{aligned}$$

From Lemma and (3.4), we get

$$\begin{aligned}
\frac{1}{2}\Delta S=&\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2-\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k,m=1}^{n} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}\\
&-\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k,m=1}^{n} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}-\sum_{\alpha,\beta=n+1}^{n+q+1}\sum_{i,j,k=1}^{n}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}\\
=&\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2-\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k,m=1}^{n} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}\\
&-\sum_{\alpha=n+1}^{n+q+1}\sum_{i,j,k,m=1}^{n} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}-\frac{1}{2}\sum_{\alpha,\beta=n+1}^{n+q+1}\varepsilon_{\beta}N(H_{\alpha}H_{\beta}-H_{\beta}H_{\alpha})
\end{aligned}$$

$$\geqq nK_0S+\frac{1}{2}\sum_{\alpha,\beta=n+2}^{n+q+1}N(H_\alpha H_\beta-H_\beta H_\alpha)$$
$$\geqq nK_0S.$$

Since the sectional curvature of M^n is nonnegative, we have $K_0 \geqq 0$. Hence, from the Stokes formula, we obtain $S=0$, i.e., M^n is totally geodesic or S is constant and $h^\alpha_{ijk}=0$. We complete the proof of Proposition.

Proof of Theorem 2. From Proposition and its proof, it is obvious that Theorem 2 holds.

Proof of Theorem 3. From the assumptions of Theorem 3 and Myers Theorem, we know that M^n is compact. According to (3.1), we get

$$\frac{1}{2}\Delta S=\sum_{\alpha=n+1}^{n+2}\sum_{i,j,k=1}^{n}(h^\alpha_{ijk})^2+nc\sum_{\alpha=n+1}^{n+2}\sum_{i,j=1}^{n}(h^\alpha_{ij})^2$$
$$-\sum_{\alpha,\beta=n+1}^{n+2}\varepsilon_\beta N(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_{\alpha,\beta=n+1}^{n+2}\varepsilon_\beta[\mathrm{trace}(H_\alpha H_\beta)]^2$$
$$\geqq nc\sum_{\alpha=n+1}^{n+2}\sum_{i,j=1}^{n}(h^\alpha_{ij})^2-N(H_{n+1})^2+N(H_{n+2})^2.$$

Hence we have

$$\frac{1}{2}\Delta S \geqq (nc-S_1+S_2)S \tag{3.5}$$

where $S_1=N(H_{n+1})$, $S_2=N(H_{n+2})$ and $S=S_1+S_2$. By using (2.11) and the assumption $\mathrm{Ric}(M^n)\geqq(n-2)c$ in Theorem 3, we have

$$c-\sum_{i=1}^{n}(h^{n+1}_{ij})^2+\sum_{i=1}^{n}(h^{n+2}_{ij})^2\geqq 0.$$

Thus

$$nc-S_1+S_2\geqq 0. \tag{3.6}$$

From (3.5) and (3.6), we conclude

$$\sum_{\alpha=n+1}^{n+2}\sum_{i,j,k=1}^{n}(h^\alpha_{ijk})^2=0$$

and $S=0$ or $nc-S_1+S_2=0$ and S is constant. If $S=0$, then M^n is totally geodesic. If $S\neq 0$, then all of the above inequalities become equalities. Hence, the Ricci curvature is equal to $(n-2)c$. We complete the proof of Theorem 3.

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