T.NODA
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ON THE GOLDBACH PROBLEM IN ALGEBRAIC NUMBER FIELDS AND THE POSITIVITY OF THE SINGULAR INTEGRAL

TAKUMI NODA

Abstract

We show the positivity of the singular integral which arises in the Goldbach problem in algebaic number fields.

Introduction

In this paper our main purpose is to show the positivity of the singular integral under a sufficient condition. The singular integral is the generalized Dirichlet integral which appears in the coefficient of asymptotic formula in the Waring problem and the Goldbach problem in algebraic number fields (see Y. Wang [14]).

In §1 and §2 we shall define a singular integral which arises in the Goldbach problem in algebraic number fields and show the positivity as Theorem 1. Here we notice that the positivity is not trivial if the algebraic number field K has the complex conjugates. In §3 and §4 we shall derive an asymptotic formula as Theorem 2 following the generalized Vinogradov-Vaughan method introduced by Mitsui ([4], [5]). The asymptotic formula is a generalization of Sultanova [10] and Theorem 1 shows that the leading term of this formula has a positive coefficient.

1. Statement of results

Let K be an algebraic number field of degree n. Let $K^{(q)}$ $(q=1, 2, \ldots, r_1)$ be the real conjugates of K and $K^{(p)}$, $K^{(p+r_2)}$ $(p=r_1+1, \ldots, r_1+r_2)$ be the complex conjugates of K with $K^{(p+r_2)}=\bar{K}^{(p)}$. Let $\mathfrak{d}$ denote the different of K and $D=N(\mathfrak{d})$ (norm of $\mathfrak{d}$) the absolute value of the discriminant of K. Further, h denotes the ideal class number of K and R the regulator of K. Let γ be a number of K and put $\mathfrak{d}\gamma=\mathfrak{b}/\mathfrak{a}$ with integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $(\mathfrak{a}, \mathfrak{b})=1$. We write this relation by $\gamma \rightarrow \mathfrak{a}$.

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Let μ be a number of K; μ also denotes an n-dimensional complex vector $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$ with $\mu^{(i)} \in K^{(i)}$ $(i=1, 2, \ldots, n)$. More generally we consider any n-dimensional complex vector $\xi=(\xi_1, \xi_2, \ldots, \xi_n)$ with real ξ_q $(q=1, 2, \ldots, r_1)$ and complex $\xi_{p+r_2}=\bar{\xi}_p$ $(p=r_1+1, \ldots, r_1+r_2)$. We denote the set of ξ by E^{r_1,r_2}. For $\xi \in E^{r_1,r_2}$, we write

$$N(\xi) = \prod_{i=1}^{n} \xi_i, \qquad S(\xi) = \sum_{j=1}^{n} \xi_j \qquad \text{and} \qquad E(\xi) = e^{2\pi i S(\xi)}.$$

Let $x(\xi)$ denote the n-dimensional real vector $x(\xi)=(X_1(\xi), X_2(\xi), \ldots, X_n(\xi))$ with $X_q(\xi)=\xi_q$, $X_p(\xi)=(\xi_p+\bar{\xi}_p)/2$ and $X_{p+r_2}(\xi)= (\xi_p-\bar{\xi}_p)/2\sqrt{-1}$. We denote the map from E^{r_1,r_2} into $\boldsymbol{R}^n$ such that the image of ξ is $x(\xi)$ by ϕ.

Let $D(t)$ $(t>0)$ be a set of $\xi \in E^{r_1,r_2}$ such that $0<\xi_q \le t$ $(q=1, \ldots, r_1)$ and $|\xi_p| \le t$ $(p=r_1+1, \ldots, r_1+r_2)$. Regarding $X_1(\xi), \ldots, X_n(\xi)$ as variables we define an integral

$$\Phi_k(z) = \frac{2^{r_2}}{\sqrt{D}} \int_{D(1)} E(z\xi^k)dx(\xi)$$

where k is a positive rational integer, $z\xi^k=(z_1\xi_1^k, \ldots, z_n\xi_n^k)$ with $z \in E^{r_1,r_2}$ and $dx(\xi)=dX_1(\xi) \cdots dX_n(\xi)$.

In the following we let μ be a totally positive integer and $a_k=(a_k^{(1)}, a_k^{(2)}, \ldots, a_k^{(n)})$ $(k=1, 2, \ldots, s)$ be a point of E^{r_1,r_2} which satisfy the condition:

$$a_k^{(i)} \in \boldsymbol{R}, \qquad a_k^{(i)} > 0 \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, s),$$
$$0 < a_1^{(i)} \le a_2^{(i)} \le \cdots \le a_s^{(i)} \le 1 < 1 + c^{(i)} = a_1^{(i)} + a_2^{(i)} + \cdots + a_s^{(i)}$$

with a positive constant $c^{(i)}$. We define a singular integral as follows:

$$\Psi_1(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s) = 2^{r_2}\sqrt{D} \int_{\boldsymbol{R}^n} \prod_{k=1}^{s} \Phi_1(\lambda_k z)\, E(-\mu z)dx(z)$$

with

$$\lambda_k = a_k\mu \qquad (k = 1, 2, \ldots, s).$$

We shall prove the following theorem:

THEOREM 1. *There is a positive constant c_1 which depends on $a_k^{(i)}$ $(i=1, 2, \ldots, n;\ k=1, 2, \ldots, s)$ such that*

$$\Psi_1(\mu; \lambda_1\lambda_2, \ldots, \lambda_s) \ge \frac{c_1}{N(\mu)}.$$

In this paper we call an integer ω of K a *prime number*, if the principal ideal (ω) is a prime ideal. Let $\Omega(\lambda_k)$ be a set of prime numbers ω_k of K such that

$0 < \omega_k^{(q)} \leq \lambda_k^{(q)}$ $(q = 1, 2, \ldots, r_1)$, $|\omega_k^{(p)}| \leq |\lambda_k^{(p)}|$ $(p = r_1 + 1, \ldots, r_1 + r_2)$.

We define a sum $R(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s)$ as follows:

$$R(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s) = \sum_{\mu = \omega_1 + \cdots + \omega_s,\ \omega_k \in \Omega(\lambda_k)} \log N(\omega_1) \cdots \log N(\omega_s),$$

where the sum is taken over all the s-tuples $(\omega_1, \omega_2, \ldots, \omega_s)$ of prime numbers such that

$$\mu = \omega_1 + \omega_2 + \cdots + \omega_s, \qquad \omega_k \in \Omega(\lambda_k) \qquad (k = 1, 2, \ldots, s).$$

Then we have

THEOREM 2. *Let μ be a totally positive integer of K and s be a rational integer with $s \geq 3$. Then*

$$R(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s) = \frac{\Psi_1(1; a_1, a_2, \ldots, a_s)}{W^s} \mathfrak{S}_G(\mu) \prod_{k=1}^{s} N(a_k) N(\mu)^{s-1} + O\left(\frac{N^{(s-1)n}}{(\log N)^{s+1}}\right),$$

where $N = \max\{|\lambda_s^{(i)}|\}_{(1 \leq i \leq n)}$, $W = 2^{r_1 + r_2} \pi^{r_2} hR / \omega\sqrt{D}$ with ω the number of the roots of unity in K and $\mathfrak{S}_G(\mu)$ is the singular series which is written as an infinite product:

$$\mathfrak{S}_G(\mu) = \prod_{\mathfrak{p} | \mu} \left(1 + \frac{(-1)^s}{(N(\mathfrak{p}) - 1)^{s-1}}\right) \prod_{\mathfrak{p} \nmid \mu} \left(1 + \frac{(-1)^{s+1}}{(N(\mathfrak{p}) - 1)^s}\right).$$

2. Singular integral

First, we recall a fundamental lemma for $\Phi_k(z)$.

LEMMA 1 ([5] Theorem 4.6.1, [14] Lemma 5.3). *For $k \geq 2$ we have*

$$\Phi_k(z) \ll \prod_{\iota=1}^{n} \min(1, |z_j|^{-\frac{1}{k}}).$$

In the case $k = 1$, we have

$$\Phi_1(z) \ll \prod_{\iota=1}^{r_1 + r_2} \min(1, |z_j|^{-1}).$$

If $s > 2k$

$$\int_{R^n} |\Phi_k(z)|^s dx(z)$$

is convergent.

We now prove the positivity of a singular integral with a linear restriction on the integral $\Phi_k(z)$.

Proof of Theorem 1. $\Psi_1(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s)$ is written as follows:

$$(2.1)\quad \Psi_1(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s)$$
$$= \frac{2^{r_2(1+s)}}{D^{\frac{s-1}{2}}} \prod_{q=1}^{r_1} \int_{-\infty}^{\infty} \prod_{k=1}^{s} \left\{ \int_0^1 \exp(2\pi i \lambda_k^{(q)} z_q \xi_q) dX_q(\xi_q) \right\} \exp(-2\pi i \mu^{(q)} z_q) dX_q(z)$$
$$\times \prod_{p=r_1+1}^{r_1+r_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{k=1}^{s} \left\{ \iint_{|\xi_p| \le 1} \exp(2\pi i (\lambda_k^{(p)} z_p \xi_p + \bar{\lambda}_k^{(p)} \bar{z}_p \bar{\xi}_p)) \right.$$
$$\left. dX_p(\xi) dX_{p+r_2}(\xi) \right\} \times \exp(-2\pi i (\mu^{(p)} z_p + \bar{\mu}^{(p)} \bar{z}_p)) dX_p(z) dX_{p+r_2}(z).$$

We consider the factors of product $\prod_q$ and $\prod_p$ on the right hand side of (2.1). Firstly, the factor of product $\prod_p$ is

$$(2.2)\quad \int_{-\infty}^{\infty} \prod_{k=1}^{s} \left\{ \int_0^1 \exp(2\pi i \lambda_k^{(q)} z_q \xi_q) d\xi_q \right\} \exp(-2\pi i \mu^{(q)} z_q) dz_q$$
$$= \frac{1}{\mu^{(q)}} \int_{-\infty}^{\infty} \prod_{k=1}^{s} \left\{ \int_0^1 \exp(2\pi i a_k^{(q)} z_q \xi_q) d\xi_q \right\} \exp(-2\pi i z_q) dz_q$$
$$= \frac{1}{\mu^{(q)}} \int_{-\infty}^{\infty} \left\{ \int_0^1 \cdots \int_0^1 e^{2\pi i z_q (a_1^{(q)} \xi_1 + a_2^{(q)} \xi_2 + \cdots + a_s^{(q)} \xi_s)} d\xi_1 d\xi_2 \cdots d\xi_s \right\} e^{-2\pi i z_q} dz_q$$
$$= \frac{1}{\mu^{(q)} \prod_{k=1}^{s} a_k^{(q)}} \int_{-\infty}^{\infty} \left\{ \int_0^{a_1^{(q)}} \int_0^{a_2^{(q)}} \cdots \int_0^{a_s^{(q)}} e^{2\pi i z_q (u_1 + u_2 + \cdots + u_s)} du_1 du_2 \cdots du_s \right\}$$
$$e^{-2\pi i z_q} dz_q.$$

Then, putting $w = u_1 + \cdots + u_s$, we have

$$= \frac{1}{\mu^{(q)} \prod_{k=1}^{s} a_k^{(q)}} \int_{-\infty}^{\infty} \left\{ \int_{A^{(q)}} e^{2\pi i z_q w} du_1 du_2 \cdots du_{s-1} dw \right\} e^{-2\pi i z_q} dz_q$$
$$= \frac{1}{\mu^{(q)} \prod_{k=1}^{s} a_k^{(q)}} \int_{-\infty}^{\infty} \left\{ \int_0^{a_1^{(q)} + \cdots + a_s^{(q)}} \left(\int_{B^{(q)}} du_1 \cdots du_{s-1} \right) e^{2\pi i z_q w} dw \right\}$$
$$e^{-2\pi i z_q} dz_q,$$

where $A^{(q)}$ and $B^{(q)}$ are defined as follows:

$$A^{(q)} = \left\{ (u_1, u_2, \ldots, u_{s-1}, w) \in \boldsymbol{R}^s \;\middle|\; \begin{array}{l} 0 \le u_1 \le a_1^{(q)}, \ldots, 0 \le u_{s-1} \le a_{s-1}^{(q)} \\ 0 \le w \le a_1^{(q)} + \cdots + a_s^{(q)} \\ 0 \le w - u_1 - \cdots - u_{s-1} \le a_s^{(q)} \end{array} \right\},$$

$$B^{(q)} = \left\{ (u_1, u_2, \ldots, u_{s-1}) \in \boldsymbol{R}^{s-1} \;\middle|\; \begin{array}{l} 0 \le u_1 \le a_1^{(q)}, \ldots, 0 \le u_{s-1} \le a_{s-1}^{(q)} \\ 0 \le w - u_1 - \cdots - u_{s-1} \le a_s^{(q)} \end{array} \right\}.$$

Now we put

$$F_0^{(q)}(w) = \int_{B^{(q)}} du_1 du_2 \cdots du_{s-1}.$$

Then we have $F_0^{(q)}(w) = 0$ for $w < 0$ or $w > a_1^{(q)} + \cdots + a_s^{(q)}$ and $F_0^{(q)}(w)$ is a continuous function of w. Therefore, applying the theory of Fourier integrals, we find that (2.2) is equal to

$$(2.3) \qquad \frac{1}{\mu^{(q)} \prod_{k=1}^{s} a_k^{(q)}} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F_0^{(q)}(w) e^{2\pi i z_q w} dw \right\} e^{-2\pi i z_q \times 1} dz_q$$
$$= \frac{1}{\mu^{(q)} \prod_{k=1}^{s} a_k^{(q)}} F_0^{(q)}(1).$$

In a similar way, the factor of product $\prod_p$ is

$$(2.4) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{k=1}^{s} \left\{ \int \int \int_{|\xi_p| \le 1} \exp(2\pi i(\lambda_k^{(p)} z_p \xi_p + \bar{\lambda}_k^{(p)} \bar{z}_p \bar{\xi}_p)) dX_p(\xi) dX_{p+r_2}(\xi) \right\}$$
$$\times \exp(-2\pi i(\mu^{(p)} z_p + \bar{\mu}^{(p)} \bar{z}_p)) dX_p(z) dX_{p+r_2}(z)$$
$$= \frac{1}{2^2 \mu^{(p)} \mu^{(p+r_2)} \prod_{k=1}^{s} a_k^{(p)} a_k^{(p+r_2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(p)}(U, V) e^{2\pi i(uU + vV)} dU dV \right\}$$
$$\times e^{-2\pi i(u \times 1 + v \times 0)} du dv$$
$$= \frac{1}{2^2 \mu^{(p)} \mu^{(p+r_2)} \prod_{k=1}^{s} a_k^{(p)} a_k^{(p+r_2)}} G^{(p)}(1, 0),$$

where

$$G^{(p)}(U, V) = \int_{D^{(p)}} dx_1 dx_2 \cdots dx_{s-1} dy_1 dy_2 \cdots dy_{s-1}$$

is a $2(s-1)$-fold integral with

$$D^{(p)} = \left\{ \begin{matrix} x_1, x_2, \ldots, x_{s-1} \\ y_1, y_2, \ldots, y_{s-1} \end{matrix} \;\middle|\; \begin{matrix} x_1^2 + y_1^2 \le (a_1^{(p)})^2, \ldots, x_{s-1}^2 + y_{s-1}^2 \le (a_{s-1}^{(p)})^2 \\ (U - x_1 - \cdots - x_{s-1})^2 + (V + y_1 + \cdots + y_{s-1})^2 \\ \le (a_s^{(p)})^2 \end{matrix} \right\}.$$

By (2.1), (2.3) and (2.4) we obtain

$$\text{(2.5)} \quad \Psi_1(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s) = \frac{2^{r_2(s-1)} D^{\frac{1-s}{2}}}{N(\mu) \prod_{k=1}^{s} N(a_k)} \prod_{q=1}^{r_1} F_0^{(q)}(1) \prod_{p=r_1+1}^{r_1+r_2} G^{(p)}(1, 0).$$

Here $F_0^{(q)}(1)$ and $G^{(q)}(1.0)$ denote the volumes of domains $B_0^{(q)}$ and $D_0^{(p)}$ in $(s-1)$ and $2(s-1)$-dimensional euclidian space, where $B_0^{(q)}$ and $D_0^{(p)}$ are given as follows:

$$B_0^{(q)} = \left\{ (u_1, u_2, \ldots, u_{s-1}) \in \boldsymbol{R}^{s-1} \;\middle|\; \begin{matrix} 0 \le u_1 \le a_1^{(q)}, \ldots, 0 \le u_{s-1} \le a_{s-1}^{(q)} \\ 0 \le 1 - u_1 - \cdots - u_{s-1} \le a_s^{(q)} \end{matrix} \right\},$$

$$D_0^{(p)} = \left\{ \begin{matrix} x_1, x_2, \ldots, x_{s-1} \\ y_1, y_2, \ldots, y_{s-1} \end{matrix} \;\middle|\; \begin{matrix} x_1^2 + y_1^2 \le (a_1^{(p)})^2, \ldots, x_{s-1}^2 + y_{s-1}^2 \le (a_{s-1}^{(p)})^2 \\ (1 - x_1 - \cdots - x_{s-1})^2 + (y_1 + \cdots + y_{s-1})^2 \\ \le (a_s^{(p)})^2 \end{matrix} \right\}.$$

We shall show the existence of domains $B_1^{(q)}$ and $D_1^{(p)}$ such that $B_1^{(q)} \subset B_0^{(q)}$, $D_1^{(p)} \subset D_0^{(p)}$ and that the volumes of $B_1^{(q)}$ and $D_1^{(p)}$ are positive in each euclidian space. We now consider two cases to define $B_1^{(q)}$.

Case 1. $a_1^{(q)} + a_2^{(q)} + \cdots + a_{s-1}^{(q)} < 1$. Let us define

$$B_1^{(q)} = \{u_1, u_2, \ldots, u_{s-1} \mid a_i^{(q)} - \delta^{(q)} \le u_i \le a_i^{(q)} \quad (i = 1, 2, \ldots, s-1)\},$$

where

$$\delta^{(q)} = \min(a_1^{(q)}, c^{(q)}/(s-1)).$$

Case 2. $1 \le a_1^{(q)} + a_2^{(q)} + \cdots + a_{s-1}^{(q)}$. Let us define

$$B_1^{(q)} = \left\{ u_1, u_2, \ldots, u_{s-1} \;\middle|\; \begin{matrix} a_i^{(q)} - h_i^{(q)} \le u_i \le a_i^{(q)} - h_i^{(q)} + \delta_i^{(q)} \\ (i = 1, 2, \ldots, s-1) \end{matrix} \right\}$$

with

$$h_i^{(q)} = s_i^{(q)} c^{(q)},$$
$$\delta_i^{(q)} h_i^{(q)} + s_i^{(q)} (1 - a_1^{(q)} - \cdots - a_{s-1}^{(q)}),$$

where we take positive constants $s_i^{(q)}$ $(i = 1, 2, \ldots, s-1)$ which satisfy following conditions:

$$s_1^{(q)} + s_2^{(q)} + \cdots + s_{s-1}^{(q)} = 1,$$
$$s_i^{(q)} c^{(q)} \le a_i^{(q)}.$$

It is easy to see that such constants $s_i^{(q)}$ exist for all $q=1, 2, \ldots, r_1$ and $B_1^{(q)} \subset B_0^{(q)}$ in each case. Then we have

$$(2.6) \qquad F_0^{(q)}(1) \geq \int_{B_1(q)} du_1 \cdots du_{s-1} = \prod_{i=1}^{s-1} \delta_i^{(q)}.$$

Now we consider two cases to define $D_1^{(p)}$.

Case 1. Suppose $2a_s^{(q)} \geq c^{(q)}$. Let us define

$$D_1^{(p)} = \left\{ \begin{matrix} x_1, x_2, \ldots, x_{s-1} \\ y_1, y_2, \ldots, y_{s-1} \end{matrix} \;\middle|\; \begin{matrix} a_j^{(p)} - s\delta^{(p)}/(s-1) \leq x_j \leq a_j^{(p)} - \delta^{(p)} \\ 0 \leq y_j \leq \delta^{(p)}/(s-1) \; (j = 1, 2, \ldots, s-1) \end{matrix} \right\},$$

where

$$\delta^{(p)} = \min(a_1^{(p)}/2, c^{(p)}/2s).$$

For $z=(x_1, x_2, \ldots, x_{s-1}, y_1, y_2, \ldots, y_{s-1}) \in D_1^{(p)}$, the coordinates $x_1, \ldots, x_{s-1}$, $y_1, \ldots, y_{s-1}$ satisfy the condition

$$(2.7) \qquad x_j^2 + y_j^2 \leq (a_j^{(p)})^2 \qquad (j = 1, 2, \ldots, s-1).$$

In order to show $z \in D_0^{(p)}$, we shall prove

$$(2.8) \; (1 - x_1 - x_2 - \cdots - x_{s-1})^2 + (y_1 + y_2 + \cdots + y_{s-1})^2 \leq (a_s^{(p)})^2.$$

We consider two cases to show (2.8).

First, we take the point $z=(x_1, x_2, \ldots, d_{s-1}, y_1, y_2, \ldots, y_{s-1}) \in D_1^{(p)}$ with $x_j = a_j^{(p)} - \delta^{(p)}$, $y_j = \delta^{(p)}/(s-1)$ $(j=1, 2, \ldots, s-1)$.
Then we have

$$\begin{aligned} &(1 - x_1 - x_2 - \cdots - x_{s-1})^2 + (y_1 + y_2 + \cdots + y_{s-1})^2 \\ &= \{a_s^{(p)} - c^{(p)} + (s-1)\delta^{(p)}\}^2 + (\delta^{(p)})^2 \\ &= (a_s^{(p)})^2 - 2a_s^{(p)} \{c^{(p)} - (s-1)\delta^{(p)}\} + \{c^{(p)} - (s-1)\delta^{(p)}\}^2 + (\delta^{(p)})^2 \\ &\leq (a_s^{(p)})^2 - c^{(p)} \{c^{(p)} - (s-1)\delta^{(p)}\} + \{c^{(p)} - (s-1)\delta^{(p)}\}^2 + (\delta^{(p)})^2 \\ &= (a_s^{(p)})^2 - (s-1)\delta^{(p)} c^{(p)} + \{(s-1)^2 + 1\} (\delta^{(p)})^2 \\ &< (a_s^{(p)})^2, \end{aligned}$$

where the last inequality is followed by using $\delta^{(p)} \leq c^{(p)}/2s$.

Secondly, we take the point $z=(x_1, x_2, \ldots, x_{s-1}, y_1, y_2, \ldots, y_{s-1}) \in D_1^{(p)}$ with $x_j = a_j^{(p)} - s\delta^{(p)}/(s-1)$, $y_j = \delta^{(p)}/(s-1)$ $(j=1, 2, \ldots, s-1)$. In a similar way we can see that the condition (2.8) is satisfied also on this point. Therefore the condition (2.8) is satisfied on all points of $D_1^{(p)}$.

Case 2. Suppose $c^{(p)} > 2a_s^{(p)}$. Let $t_j^{(p)}$ be positive constants which satisfy the following conditions:

$$t_1^{(p)} + t_2^{(p)} + \cdots + t_{s-1}^{(p)} = 1, \qquad 0 < t_1^{(p)} < t_2^{(p)} < \cdots < t_{s-1}^{(p)} < 1,$$
$$t_j^{(p)} \leq sa_j^{(p)}/(s+1) \qquad (j = 1, 2, \ldots, s-1).$$

We define

$$D_1^{(p)} = \left\{ \begin{matrix} x_1, x_2, \ldots, x_{s-1} \\ y_1, y_2, \ldots, y_{s-1} \end{matrix} \,\middle|\, \begin{matrix} t_j^{(p)} - \delta^{(p)} \leq x_j \leq t_j^{(p)} \\ 0 \leq y_j \leq \delta^{(p)} \quad (j = 1, 2, \ldots, s-1) \end{matrix} \right\}$$

with

$$\delta^{(p)} = \min(a_1^{(p)}/(s+1),\ t_1^{(p)}).$$

Under the condition $c^{(p)} > 2a_s^{(p)}$ such constants $t_j^{(p)}$ exist for all $p = r_1 + 1, \ldots, r_1 + r_2$ and we see that $D_1^{(p)} \subset D_0^{(p)}$. Then we have

$$\text{(2.9)} \quad D^{(p)}(1, 0) \geq \int_{D_1^{(p)}} dx_1 \cdots dx_{s-1} dy_1 \cdots dy_{s-1} \geq \{\delta^{(p)}/(s-1)\}^{2(s-1)}.$$

By (2.5), (2.6) and (2.9) we have

$$\Psi_1(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s) \geq \frac{2^{r_2(s-1)} D^{\frac{1-s}{2}}}{N(\mu) \prod_{k=1}^{s} N(a_k)} \prod_{q=1}^{r_1} \left\{ \prod_{i=1}^{s-1} \delta_i^{(q)} \right\} \prod_{p=r_1+1}^{r_1+r_2} \{\delta^{(p)}/(s-1)\}^{2(s-1)}.$$

Now we consider the constant c_1 defined by

$$c_1 = \frac{2^{r_2(s-1)} D^{\frac{1-s}{2}}}{\prod_{k=1}^{s} N(a_k)} \prod_{q=1}^{r_1} \left\{ \prod_{i=1}^{s-1} \delta_i^{(q)} \right\} \prod_{p=r_1+1}^{r_1+r_2} \{\delta^{(p)}/(s-1)\}^{2(s-1)},$$

which allow us to establish Theorem 1. □

3. Farey dissection

Let $\{\delta_1, \delta_2, \ldots, \delta_n\}$ be a basis of $\mathfrak{d}^{-1}$. We put

$$z_j = x_1\delta_1^{(j)} + x_2\delta_2^{(j)} + \cdots + x_n\delta_n^{(j)} \qquad (j = 1, 2, \ldots, n)$$

for real numbers $x_1, x_2, \ldots, x_n$ and define a set E of $z = (z_1, z_2, \ldots, z_n)$ as follows:

$$E = \left\{ z \,\middle|\, z_j = \sum_{i=1}^{n} x_i\delta_i^{(j)},\ -1/2 < x_i \leq 1/2 \qquad (j = 1, 2, \ldots, n) \right\}$$

Let H and T be real numbers such that

$$H > 2DT \qquad \text{and} \qquad T > 1,$$

and let Γ be a set of numbers γ of K such that $\gamma \in E$ and $\gamma \to \mathfrak{a}$ with $N(\mathfrak{a}) \leq T^n$. In the following we let γ_1 be a number of K and $N(\max(|z|, T)) = \prod_{i=1}^{n} \max(|z_i|, T)$.

Now we define divisions of E in two ways.

(i) For every $\gamma \in \Gamma$ with $\gamma \to \mathfrak{a}$ we define a subset E_γ of E by

$E_\gamma = \{z \in E \mid \exists\gamma_1 \equiv \gamma \pmod{\mathfrak{d}^{-1}}$ such that $N(\max(H|z - \gamma_1|, T^{-1})) \leq N(\mathfrak{a})^{-1}\}$

and put

$$E^0 = E - \cup_{\gamma\in\Gamma} E_\gamma.$$

(ii) For every $\gamma \in \Gamma$ with $\gamma \to \mathfrak{a}$ we define a subset B_γ of E by

$$B_\gamma = \left\{ z \in E \;\middle|\; \begin{array}{l} \exists\gamma_1 \equiv \gamma \pmod{\mathfrak{d}^{-1}} \text{ such that} \\ |z_j - \gamma_1^{(j)}| \leq T^{n-1}/H \quad (j = 1, 2, \ldots, n) \end{array} \right\}$$

and put

$$B^0 = E - \cup_{\gamma\in\Gamma} B_\gamma.$$

These division of E depend on the pair (H, T). We shall call these divisions *Farey dissection of E with respect to (H, T)*. The Farey dissection (i) was introduced by Siegel [8] and the Farey dissection (ii) by Mitsui ([4] §4).

Now we take positive constants σ, σ_1, σ_2, u and v such that

$$\begin{array}{l} \sigma \geq 5, \ \sigma_2 > v, \ u - 1 \geq \sigma, \\ \min(\sigma_1 - v, \sigma_1 - v + v/n - 1) \geq \sigma + u + v/n, \\ \min(\sigma_2, v/n, \sigma_1 - 1, (u - v)/n - 1) \geq \sigma + 2, \\ \min(\sigma_2, v/n, \sigma_1 - 1) - 2r - 20 \geq 4\sigma \qquad (r = r_1 + r_2 - 1). \end{array}$$

We put

$$N = \max\{|\lambda_s^{(i)}|\}_{(1\leq i\leq n)}$$

and

$$H = N/(\log N)^{\sigma_1}, \qquad T = (\log N)^{\sigma_2}. \tag{3.1}$$

The following Lemma 2 was proved by Siegel and Lemma 3 is due to Mitsui.

LEMMA 2 (Siegel [9]). *If γ_1 and γ_2 belong to Γ and $\gamma_1 \neq \gamma_2$, then we have*

$$E_{\gamma_1} \cap E_{\gamma_2} = \emptyset.$$

LEMMA 3 (Mistui [4] §4). *We have*

$$B_\gamma \supset E_\gamma, \qquad B^0 \subset E^0$$

and if γ_1 and γ_2 belong to Γ and $\gamma_1 \neq \gamma_2$, then we have

$$B_{\gamma_1} \cap B_{\gamma_2} = \emptyset.$$

4. Asymptotic formula

Let μ be a totally positive integer of K with sufficiently large norm $N(\mu)$. For $z \in E$ we put

$$(4.1) \qquad S(z;\lambda_k) = \sum_{\omega_k \in \Omega(\lambda_k)} E(\omega_k z)\log N(\omega_k) \qquad (k = 1, 2, \ldots, s).$$

Then the integral

$$(4.2) \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{k=1}^{s} S(z;\lambda_k)E(-\mu z)dx_1dx_2 \cdots dx_n = 2^{r_2}\sqrt{D}\int_{\phi(E)} \prod_{k=1}^{s} S(z;\lambda_k)E(-\mu z)dx(z)$$

is equal to $R(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s)$. To get an asymptotic formula of $R(\mu;\lambda_1,\lambda_2,\ldots\lambda_s)$ we shall quote the estimate of $S(z;\lambda_k)$ in Mitsui [5]. In the case of rational field, we can find the estimate in Vaughan [11].

First, for any totally positive unit η we have

$$R(\eta\mu;\eta\lambda_1,\eta\lambda_2,\ldots,\eta\lambda_s) = R(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s).$$

The theory of units allow us to take a totally positive unit η_0 satisfying

$$c'N(v)^{\frac{1}{n}} < |\eta_0^{(i)}\mu^{(i)}| < c''N(v)^{\frac{1}{n}} \qquad (i = 1, 2, \ldots, n).$$

Taking $\eta_0\mu$ instead of μ, we shall assume that μ in (4.1) satisfies the inequalities

$$c'N(v)^{\frac{1}{n}} < |\mu^{(i)}| < c''N(v)^{\frac{1}{n}} \qquad (i = 1, 2, \ldots, n).$$

Then $N = \max\{|\lambda_s^{(i)}|\}_{(1\le i\le n)}$ is sufficiently large and the inequalities

$$cN < |\lambda_k^{(i)}| \le N \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, s)$$

are satisfied.

LEMMA 4. *Let z be a point of E. For a positive constant b we put*

$$\phi(B_0) = \{x(z) \in \mathbf{R}^n |\ \ |z_i| \le (\log N)^b/N \qquad (i = 1, 2, \ldots, n)\}$$

Then we have

$$(4.3) \qquad 2^{r_2}\sqrt{D}\int_{\phi(B_0)} \prod_{k=1}^{s} \Psi_1(\lambda_k z)E(-\mu z)dx(z) = \Psi_1(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s) + O(1/N^n(\log N)^{b(s-2)}).$$

Proof. By Lemma 1 we have

$$\Psi_1(\lambda_k z) \ll \prod_{j=1}^{r_1+r_2} \min(1, |\lambda_k^{(j)} z j|^{-1})$$
$$\ll N^{-(r_1+r_2)} \prod_{j=1}^{r_1+r_2} \min(N, |z_j|^{-1}).$$

Hence

$$\Psi_1(\mu; \lambda_1, \lambda_2, \ldots, \lambda_s) - 2^{r_2}\sqrt{D} \int_{\phi(B_0)} \prod_{k=1}^{s} \Psi_1(\lambda_k z) E(-\mu z) dx(z)$$
$$= 2^{r_2}\sqrt{D} \int_{R^n - \phi(B_0)} \prod_{k=1}^{s} \Psi_1(\lambda_k z) E(-\mu z) dx(z)$$
$$\ll N^{-(r_1+r_2)s} \int_{R^n - \phi(B_0)} \left\{ \prod_{j=1}^{r_1+r_2} \min(N, |z_j|^{-1}) \right\}^s dx(z).$$

Here, we see

$$\int_{(\log N)^b/N}^{\infty} \min(N, |x|^{-1})^s \, dx \ll \frac{N^{(s-1)}}{(\log N)^{b(s-1)}},$$
$$\int_{(\log N)^b/N}^{\infty} \int_0^{2\pi} \min(N, |r|^{-1})^s \, r d\theta dr \ll \frac{N^{(s-2)}}{(\log N)^{b(s-2)}}$$

thus we obtain (4.3). □

Under the notations of §3, we quote the following Theorems:

THEOREM A([4] Theorem 5.1, [5] Theorem 6.6.2). *If z belongs to E^0,*

$$S(z, \lambda) \ll \frac{N^n}{(\log N)^{\sigma}}.$$

THEOREM B([5] Theorem 6.2.1). *Let γ_0 be a number of K such that*

$$\gamma_0 \equiv \gamma \pmod{\mathfrak{b}^{-1}},$$
$$|z - \gamma_0^{(i)}| \leq T^{n-1}/H \qquad (i = 1, 2, \ldots, n).$$

Then for $z \in B_\gamma$ $(\gamma \to \mathfrak{a})$,

$$S(z, \lambda) = \frac{w\sqrt{D}\, \mu(\mathfrak{a}) N(\lambda)}{2^{r_1+r_2}\pi^{r_2} h \varphi(\mathfrak{a})\, R} \Phi_1(\lambda(z - \gamma_0)) + O\left(\frac{N^n}{(\log N)^{a-b+1}}\right)$$

where

$$b = (n-1)\sigma_2 + \sigma_1, \qquad a > b,$$

and

$$\mu(\mathfrak{a}) = \sum_{\gamma \to \mathfrak{a},\, \gamma \bmod \mathfrak{b}^{-1}} E(\gamma), \quad \varphi(\mathfrak{a}) = \sum_{\gamma \to \mathfrak{a},\, \gamma \bmod \mathfrak{b}^{-1}} E(\rho\gamma) \text{ with } \rho \in \mathfrak{a}.$$

Now we prove the Theorem 2.

Proof of Theorem 2. By the Farey dissection of E defined in §3, we have

$$(4.4)\qquad R(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s)=2^{r_2}\sqrt{D}\left\{\int_{\phi(B^0)}+\sum_{\gamma\in\Gamma}\int_{\phi(B_\gamma)}\right\}\prod_{k=1}^{s}S(z;\lambda_k)E(-\mu z)dx(z).$$

We shall estimate the right hand side of (4.4) on $\phi(B^0)$ and $\phi(B_\gamma)$ ($\gamma\in\Gamma$) respectively;

Let us consider the integral on $\phi(B^0)$. By Theorem A we have

$$R(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s)=2^{r_2}\sqrt{D}\int_{\phi(B^0)}\prod_{k=1}^{s}S(z;\lambda_k)E(-\mu z)dx(z).$$

$$\ll\frac{N^{(s-2)}}{(\log N)^{b(s-2)}}\int_{\phi(E)}|S(z;\lambda_s)|^2dx(z).$$

Applying Parseval's identity and the prime ideal theorem, we find that this quantity is equal to

$$\frac{N^{(s-2)}}{(\log N)^{b(s-2)}}\sum_{\omega\in\Omega(\lambda_k)}\log^2 N(\omega)$$

$$\ll\frac{N^{(s-2)}}{(\log N)^{b(s-2)}}N^{n}\log N.$$

Thus we have

$$(4.5)\qquad 2^{r_2}\sqrt{D}\int_{\phi(B^0)}\prod_{k=1}^{s}S(z;\lambda_k)E(-\mu z)dx(z)\ll\frac{N^{(s-1)}}{(\log N^{(s+1)})}.$$

Now we look at the integral on $\phi(B_\gamma)$ ($\gamma\in\Gamma$). By Theorem B we have

$$(4.6)\qquad 2^{r_2}\sqrt{D}\int_{\phi(B_\gamma)}\prod_{k=1}^{s}S(z;\lambda_k)E(-\mu z)dx(z)$$

$$=\frac{2^{r_2}\sqrt{D}\,\mu\,(\mathfrak{a})^s}{W^s\varphi(\mathfrak{a})^s}\prod_{k=1}^{s}N(\lambda_k)\int_{\phi(B_\gamma)}\prod_{k=1}^{s}\Phi_1(\lambda_k(z-\gamma_0))E(-\mu z)dx(z)$$

$$+\,O\left(\frac{N^{sn}}{(\log N)^{a-b+1}}\right)\int_{\phi(B_\gamma)}|E(-\mu z)|dx(z).$$

By putting z instead of $z-\gamma_0$ and applying the estimate

$$\int_{\phi(B_\gamma)}dx(z)\ll\frac{(\log N)^{bn}}{N^n},$$

on the error term, we see that this is equal to

$$\frac{2^{r_2}\sqrt{D}\,\mu(\mathfrak{a})^s}{W^s\varphi(\mathfrak{a})^s}\prod_{k=1}^{s} N(\lambda_k)E(-\mu\gamma)\int_{\phi(B_\gamma)}\prod_{k=1}^{s}\Phi_1(\lambda_k(z))E(-\mu z)dx(z) + O(N^{(s-1)n}/(\log N)^{a-b(n+1)+1})$$

with

$$\phi(B_\gamma) = \{x(z) \in \boldsymbol{R}^n \mid |z_j| \le (\log N)^b/N \qquad (j = 1, 2, \ldots, n)\}.$$

Taking summation of the both sides of (4.6) over all $\gamma \in \Gamma$, and together with the estimate

$$\sum_{\gamma\in\Gamma} 1 \ll \sum_{N(\mathfrak{a})\le T^n} N(\mathfrak{a}) \le T^{2n},$$

we have

$$\begin{aligned}(4.7)\qquad &\sum_{\gamma\in\Gamma} 2^{r_2}\sqrt{D}\int_{\phi(B_\gamma)}\prod_{k=1}^{s} S(z;\lambda_k)E(-\mu z)dx(z)\\ &= \frac{2^{r_2}\sqrt{D}}{W^s}\prod_{k=1}^{s} N(\lambda_k)\sum_{\gamma\in\Gamma}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}E(-\mu\gamma)\int_{\phi(B_\gamma)}\prod_{k=1}^{s}\Phi_1(\lambda_k z)E(-\mu z)dx(z)\\ &\qquad + O(N^{(s-1)n}/(\log N)^{a-b(n+1)+1-2n\sigma_2}).\end{aligned}$$

By Lemma 4, we find the right hand side of (4.7) is

$$(4.8)\quad \frac{2^{r_2}\sqrt{D}}{W^s}\prod_{k=1}^{s} N(\lambda_k)\sum_{\gamma\in\Gamma}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}E(-\mu\gamma)\Psi_1(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s) + O\left(\frac{N^{(s-1)n}}{(\log N)^{s+1}}\right)$$

with

$$a \ge b(n+1) + 1 - 2n\sigma_2 + s + 1.$$

Now we apply the following property of the singular series $\mathfrak{S}_G(\mu)$ (see Rademacher [7], Mitsui ([4] § 10)):

$$\begin{aligned}(4.9)\qquad \mathfrak{S}_G(\mu) &= \sum_{\mathfrak{a}}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}\sum_{\gamma\in\Gamma,\gamma \bmod \mathfrak{b}^{-1}} E(-\mu\gamma)\\ &= \sum_{N(\mathfrak{a})\le T^n}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^a}\sum_{\gamma\in\Gamma,\gamma \bmod \mathfrak{b}^{-1}} E(-\mu\gamma) + O\left(\frac{1}{(\log N)^{\sigma_2 n/2}}\right).\end{aligned}$$

By (4.4), (4.5), (4.8) and (4.9) we finally obtain

$$\begin{aligned}&R(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s)\\ &= \frac{1}{W^s}\mathfrak{S}_G(\mu)\Psi_1(\mu;\lambda_1,\lambda_2,\ldots,\lambda_s)\prod_{k=1}^{s} N(\lambda_k) + O\left(\frac{N^{(s-1)n}}{(\log N)^{s+1}}\right)\\ &= \frac{\Psi_1(1;a_1,a_2,\ldots,a_s)}{W^s}\mathfrak{S}_G(\mu)\prod_{k=1}^{s} N(a_k)N(\mu)^{s-1} + O\left(\frac{N^{(s-1)n}}{(\log N)^{s+1}}\right).\end{aligned}$$

This completes the proof of Theorem 2. □

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Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152
Japan