C. JANG
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SURFACES WITH 1-TYPE GAUSS MAP

CHANGRIM JANG

0. Introduction

Submanifolds of finite type were introduced by B.-Y. Chen about thirteen years ago [2]. Many works have been done in characterizing or classifying submanifolds in Euclidean space with this notion. On the other hand, several authors studied submanifolds with finite type Gauss map. B.-Y. Chen and P. Piccinni studied compact submanifolds with finite type Gauss map [3]. And C. Baikoussis, B.-Y. Chen and L. Verstraelen classified ruled surfaces and tubes with finite-type Gauss map [1]. Recently Y. H. Kim studied surfaces in 3-dimensional Euclidean space E^3 with 1-type Gauss map and he proved that the only co-closed surfaces in E^3 with 1-type Gauss map are spheres and circular cylinders [6]. In this paper we study surfaces in E^3 with 1-type Gauss map without the assumption of co-closedness and obtain the following theorem.

THEOREM. *Let M be an orientable, connected surface in E^3. Then M has 1-type Gauss map if and only if M is an open part of a sphere or an open part of a circular cylinder.*

1. Preliminaries

Let M be an orientable, connected surface in E^3. We now choose e_1 and e_2 as principal normal vectors of M and let x and y the corresponding principal curvatures of the shape operator S associated with a unit normal vector e_3. Let ω^1, ω^2, ω^3 be the dual 1-forms to e_1, e_2 and e_3 and ω_A^B the connection forms associated with ω^1, ω^2, ω^3 satisfying $\omega_A^B+\omega_B^A=0$ and

$$\bar{\nabla}_{e_i} e_j=\sum_k \omega_j^k(e_i)e_k+h(e_i, e_j)e_3, \quad \nabla_{e_i} e_j=\sum_k \omega_j^k(e_i)e_k,$$

$$\bar{\nabla}_{e_i} e_3=\sum_k \omega_3^k(e_i)e_k=-Se_i,$$

$$x=\omega_1^3(e_1)=h(e_1, e_1), \quad y=\omega_2^3(e_2)=h(e_2, e_2), \quad h(e_1, e_2)=h(e_2, e_1)=0,$$

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where $\bar{\nabla}$ and ∇ are the Levi-Civita connections of E^3 and M respectively and h the second fundamental form of M. The indices A, B run over the range $\{1, 2, 3\}$ and i, j, k over $\{1, 2\}$. The covariant derivative of the second fundamental form h of M is given by

$$(\nabla_{e_k} h)(e_i, e_j)=e_k h(e_i, e_j)-h(\nabla_{e_k} e_i, e_j)-h(e_i, \nabla_{e_k} e_j).$$

We will use abbreviations h_{ij}, $h_{ij,k}$ for $h(e_i, e_j)$ and $(\nabla_{e_k} h)(e_i, e_j)$ respectively. The Codazzi's equation $h_{ij,k}=h_{ik,j}$ implies that

$$h_{11,1}=e_1 x, \quad h_{22,2}=e_2 y, \tag{1.1}$$

$$h_{12,1}=h_{21,1}=h_{11,2}=e_2 x=(y-x)\omega_2^1(e_1), \tag{1.2}$$

$$h_{12,2}=h_{21,2}=h_{22,1}=e_1 y=(x-y)\omega_1^2(e_2). \tag{1.3}$$

We now give the definition of co-closed surface introduced by Y.H. Kim for later use.

DEFINITION [6]. *A surface of Euclidean 3-space is called co-closed if the connection form ω_1^2 is co-closed, that is, trace $(\nabla\omega_1^2)=0$.*

For a smooth function f on M, ∇f, the gradient f and Δf, the Laplacian of f are given by

$$\nabla f=\sum_i (e_i f)e_i,$$

$$\Delta f=\sum_i \{e_i e_i f-(\nabla_{e_i} e_i f)\}.$$

The Laplacian Δ can be extended in a natural way to E^3-valued smooth maps on M. In fact, if ν is an E^3-valued smooth map on M. Then

$$\Delta\nu=\sum_i \{\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\nu-(\bar{\nabla}_{\nabla_{e_i} e_i}\nu)\}. \tag{1.4}$$

Applying (1.4) to the unit normal vector e_3, we find

$$\Delta e_3=-\nabla H-\mathrm{tr}\, S^2 e_3, \tag{1.5}$$

where H and $\mathrm{tr}\, S^2$ denote the mean curvature function of M in E^3 and the square length of the second fundamental form h respectively. A smooth map ν is said to be of *k-type* if ν can be written as

$$\nu=\nu_0+\nu_1+\cdots+\nu_k,$$

where ν_0 is a constant vector, $\nu_1, \nu_2, \cdots, \nu_k$ are non-constant maps satisfying $\Delta\nu_i=\lambda_i\nu_i$, $i=1, 2, \cdots, k$ and all eigen values $\{\lambda_1, \cdots, \lambda_k\}$ are mutually different. Suppose that the Gauss map $e_3: M\to S_0^2(1)\subseteq E^3$ of M is of 1-type. Then there exist a constant a and a constant vector c such that

$$\Delta e_3=a(e_3-c). \tag{1.6}$$

Then, (1.5) and (1.6) imply that

$$a(e_3-c)=-\nabla H-\mathrm{tr}\,S^2e_3. \tag{1.7}$$

So we have

$$\langle ac,\ ac\rangle=\langle\nabla H,\ \nabla H\rangle+(\mathrm{tr}\,S^2+a)^2, \tag{1.8}$$

where $\langle\,,\rangle$ means the Euclidean metric of E^3. Comparing the tangential and normal components in (1.7), we obtain the followings

$$\nabla H=ac^T, \tag{1.9}$$

$$\mathrm{tr}\,S^2=a\langle c,\ e_3\rangle-a, \tag{1.10}$$

where $(\ )^T$ means the projection to the tangent space of M. By (1.9) and (1.10), differentiating the mean curvature H in the principal normal vectors e_1, e_2 on M, we obtain

$$\begin{aligned} &e_jH=a\langle e_j,\ c\rangle,\\ &e_ie_jH-(\nabla_{e_i}e_jH)=h_{ij}(\mathrm{tr}\,S^2+a). \end{aligned} \tag{1.11}$$

So we have

$$e_1e_2(x+y)+e_1(x+y)\omega_1^2(e_1)=0, \tag{1.12}$$

$$e_2e_1(x+y)+e_2(x+y)\omega_2^1(e_2)=0, \tag{1.13}$$

$$e_1e_1(x+y)+e_2(x+y)\omega_2^1(e_1)-x(x^2+y^2+a)=0, \tag{1.14}$$

$$e_2e_2(x+y)+e_1(x+y)\omega_1^2(e_2)-y(x^2+y^2+a)=0, \tag{1.15}$$

since $H=x+y$ and $\mathrm{tr}\,S^2=x^2+y^2$. From (1.10), we get

$$\nabla\,\mathrm{tr}\,S^2=-S(\nabla H).$$

This imply

$$3xe_1x+(x+2y)e_1y=0, \tag{1.16}$$

$$(2x+y)e_2x+3ye_2y=0. \tag{1.17}$$

From (1.5) and (1.10), we find

$$\Delta\,\mathrm{tr}\,S^2=-\langle\nabla H,\ \nabla H\rangle-\mathrm{tr}\,S^2(\mathrm{tr}\,S^2+a).$$

This and (1.18) imply

$$\Delta\,\mathrm{tr}\,S^2=a(\mathrm{tr}\,S^2+a)-\alpha, \tag{1.18}$$

where $\alpha=\langle ac,\ ac\rangle$. We need to mention a well known identity.

LEMMA 1 (Simons' identity) [5]. *Let M is a surface in E^3 with the induced metric. Let H, S and h be the mean curvature function of M in E^3, the shape operator of M and the second fundamental form of M, respectively. Then, for given orthonormal frame e_1, e_2, the value $\Delta \operatorname{tr} S^2$ is calculated as follows.*

$$(1.19) \qquad \Delta \operatorname{tr} S^2 = 2\sum_{i,j} h_{ij}(e_i e_j H - (\nabla_{e_i} e_j H)) + 2|\nabla S|^2 + 2H \operatorname{tr} S^3 - 2(\operatorname{tr} S^2)^2,$$

where $|\nabla S|^2$ means $\sum_{i,j,k}(h_{ij,k})^2$.

From (1.11), (1.18) and (1.19), we get

$$(1.20) \qquad 2|\nabla S|^2 = -a \operatorname{tr} S^2 - 2H \operatorname{tr} S^3 + a^2 - \alpha .$$

2. Proof of Theorem

At first we will prove that the mean curvature function H of M is constant. We need the following lemmas.

LEMMA 2. *If $(e_1x, e_1y)=(0, 0)$ or $(e_2x, e_2y)=(0, 0)$ in an open subset U of M, then H is constant in U.*

Proof. Suppose that $(e_1x, e_1y)=(0, 0)$ in U. Then from (1.13), we find $e_2H\omega_1^2(e_2)=0$. So we may assume that $\omega_1^2(e_2)=0$. Differentiating (1.14) in the direction e_1 and using (1.12), we obtain $e_1(\omega_1^2(e_1))=0$. So we have that

$$\operatorname{tr}(\nabla\omega_1^2) = e_1(\omega_1^2(e_1)) - \omega_1^2(e_2)\omega_1^2(e_1) + e_2(\omega_1^2(e_2)) - \omega_1^2(e_1)\omega_2^1(e_2) = 0 .$$

This imply that U is a co-closed surface with 1-type Gauss map. Hence, due to the result of Kim [6], we see that H is constant in U. In case that $(e_2x, e_2y)=(0, 0)$ we can get the same conclusion by similar computation.

LEMMA 3. *Let $f(u, v)$ be a nonconstant real polynomial in two variables u, v. If the principal curvatures x, y satisfy $f(u, v)$, that is $f(x, y)=0$, in a open subset U of M, then H is constant in U.*

Proof. Suppose that H is nonconstant in U. Then

$$V = \{p \in U \mid \nabla H(p) \neq 0\}$$

is a nonempty open subset. Since the real polynomial ring $R[u, v]$ is a UFD, the polynomial $f(u, v)$ can be factored as $f = f_1 f_2 \cdots f_k$, where f_i are irreducible polynomials in $R[u, v]$. From the condition $f(x, y)=0$ on U, we can guarantee the existence of a nonempty open subset W of U, where x, y satisfy a non-constant irreducible polynomial f_i. We may assume $f_i = f_1$ without loss of generality. Differentiating $f_1(x, y)=0$ in the direction e_i, we have

$$(2.1) \qquad (f_1)_u(x, y)e_i x + (f_1)_v(x, y)e_i y = 0$$

where $(f_1)_u$ and $(f_1)_v$ mean partial derivatives of f_1 with respect to u and v. If $(e_1x, e_1y)=(0, 0)$ or $(e_2x, e_2y)=(0, 0)$ holds in W, then H is constant in W by Lemma 2. So we may assume that $(e_1x, e_1y)\neq(0, 0)$ and $(e_2x, e_2y)\neq(0, 0)$ in W. From (1.16), (1.17) and (2.1) we get

$$3x(f_1)_v(x, y)-(x+2y)(f_1)_u(x, y)=0,$$

$$(2x+y)(f_1)_v(x, y)-3y(f_1)_u(x, y)=0.$$

If $(3x)(-3y)+(x+2y)(2x+y)=2(x-y)^2=0$ in W, then W is totally umbilical and hence H is constant in W, which contradicts to the assumption. So we get $(f_1)_u(x, y)=(f_1)_v(x, y)=0$ in W. Since $f_1(u, v)$ is a nonconstant polynomial, both of $(f_1)_u$ and $(f_1)_v$ are not zero polynomials. Assume that $(f_1)_u$ is not a zero polynomial. Since f_1 and $(f_1)_u$ are relatively prime, the system

$$f_1(u, v)=0$$

$$(f_1)_u(u, v)=0$$

has only finitely many zeros [4, page 18]. But x, y satisfy this system in W. Hence x and y must be constant, which contradicts to the assumption. So we can conclude that H is constant in U.

Suppose that the mean curvature function H of M is nonconstant. Then there exists an open subset U of M where ∇H never vanishes. By Lemma 3, we also assume that $y\neq 0$, $x+2y\neq 0$, $x-y\neq 0$ and $x+y\neq 0$ in U. We will work in U. By (1.1), (1.2) and (1.3) we see that $|\nabla S|^2=(e_1x)^2+3(e_2x)^2+3(e_1y)^2+(e_2y)^2$. So from (1.20) we have

$$2\{(e_1x)^2+3(e_2x)^2+3(e_1y)^2+(e_2y)^2\}=-a(x^2+y^2)-2(x+y)(x^3+y^3)+a^2-\alpha.$$

From (1.16), (1.17) and this we find

$$2\left\{(e_1x)^2+3(e_2x)^2+3\left(\frac{3x}{x+2y}\right)^2(e_1x)^2+\left(\frac{2x+y}{3y}\right)^2(e_2x)^2\right\}$$
$$=-a(x^2+y^2)-2(x+y)(x^3+y^3)+a^2-\alpha.$$

After some calculation we get

$$8(3y^2)(7x^2+xy+y^2)(e_1x)^2+8(x+2y)^2(7y^2+xy+x^2)(e_2x)^2$$
$$=\{-a(x^2+y^2)-2(x+y)(x^3+y^3)+a^2-\alpha\}(x+2y)^2(3y)^2. \tag{2.2}$$

From (1.8) we obtain

$$\alpha-(\mathrm{tr}\, S^2+a)^2=(e_1x)^2+(e_2x)^2+2(e_1x)(e_1y)+2(e_2x)(e_2y)+(e_1y)^2+(e_2y)^2.$$

Using (1.16) and (1.17) and after some computations, we get

(2.3)
$$4(3y)^2(x-y)^2(e_1x)^2+4(x+2y)^2(x-y)^2(e_2x)^2$$
$$=\{\alpha-(x^2+y^2+a)^2\}(x+2y)^2(3y)^2.$$

From (2.2) and (2.3) we obtain

(2.4)
$$(e_1x)^2=\frac{1}{48(x^2-y^2)(x-y)^2}F(x,\ y),$$

where

$$F(x,\ y)=(x+2y)^2[(x-y)^2\{a(a-x^2-y^2)-2(x+y)(x^3+y^3)-\alpha\}$$
$$-2(7y^2+xy+x^2)\{\alpha-(x^2+y^2+a)^2\}]$$
$$=(4y)x^7+\sum_{j<7}f_j(y)x^j,$$

where $f_j(y)$ are real polynomials in one variable y. From (2.4) we see that

$$e_1x=\pm\sqrt{\frac{1}{48(x^2-y^2)(x-y)^2}F(x,\ y)}.$$

We will denote e_1x by $G(x,\ y)$. Substituting

$$\omega_1^2(e_2)=\frac{e_1y}{x-y},\quad e_1y=-\frac{3x}{x+2y}e_1x\quad \text{and}\quad e_2y=-\frac{2x+y}{3y}e_2x$$

into (1.13), and after some computations we have

$$\left[\frac{3y^2+3xy+3x^2}{x+2y}G+\frac{(x-y)\{3yG_x-(2x+y)G_y\}}{3}\right]e_2x=0,$$

where G_x and G_y are partial derivatives of $G(x,\ y)$ with respect to x and y. So the following holds

$$[9(y^2+xy+x^2)G+(x+2y)(x-y)\{3yG_x-(2x+y)G_y\}]e_2x=0.$$

Suppose $e_2x=0$ locally. Then it follows that $e_2y=0$ from (1.17). This is a contradiction to the assumption by Lemma 2. Thus the following holds in U,

(2.5)
$$9(y^2+xy+x^2)G+(x+2y)(x-y)\{3yG_x-(2x+y)G_y\}=0.$$

From (2.4), we have

(2.6)
$$48(x^2-y^2)(x-y)^2G^2=4yx^7+\sum_{j<7}f_j(y)x^j.$$

Differentiating this with respect to x, we get

$$96x(x-y)^2G^2+96(x^2-y^2)(x-y)G^2+96(x^2-y^2)(x-y)^2GG_x$$
$$=28yx^6+\sum \text{ lower degree terms with respect to } x.$$

Multiplying $(x^2-y^2)(x-y)$ at both sides of this, we get

$$(2.7)\qquad 96(x^2-y^2)(x-y)^3(2x+y)G^2+96(x^2-y^2)^2(x-y)^3GG_x$$
$$=28yx^9+\sum \text{ lower degree terms with respect to } x.$$

Similarly we get

$$(2.8)\qquad -96(x^2-y^2)(x-y)^3(x+2y)G^2+96(x^2-y^2)^2(x-y)^3GG_y$$
$$=4x^{10}+\sum \text{ lower degree terms with respect to } x.$$

From (2.6), (2.7) and (2.8) we find

$$(2.9)\qquad 96(x^2-y^2)^2(x-y)^3GG_x=12yx^9+\sum \text{ lower degree terms to } x,$$

$$(2.10)\qquad 96(x^2-y^2)^2(x-y)^3GG_y=4x^{10}+\sum \text{ lower degree terms to } x.$$

Multiplying $96(x^2-y^2)^2(x-y)^3G$ at both sides of (2.5), we get

$$(2.11)\qquad 9(x^2+xy+y^2)\{96(x^2-y^2)^2(x-y)^3G^2\}$$
$$+(x+2y)(3y)(x-y)\{96(x^2-y^2)^2(x-y)^3GG_x\}$$
$$-(x+2y)(2x+y)(x-y)\{96(x^2-y^2)^2(x-y)^3GG_y\}=0.$$

Substituting (2.6), (2.9) and (2.10) into (2.11), we can see that the heighest degree term with respect to x in (2.11) is $-8x^{13}$. So x and y satisfy a non-constant polynomial (2.11). Hence, by Lemma 3, H is constant in U, which contradicts to our assumption. Consequently H is constant in M. And from (1.14) and (1.15) we can see that x and y are constant. So M is an open part of a plane or a sphere or a circular cylinder. But the Gauss map of a plane is constant. Hence a plane has not 1-type Gauss map. The converse is an easy computation.

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Department of Mathematics
University of Ulsan
Ulsan, Korea 680-749