Z.-X. CHEN
KODAI MATH. J.
19 (1996), 341—354

THE ZERO, POLE AND ORDER OF MEROMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS*

ZONG-XUAN CHEN

Abstract

In this paper, we investigate the complex oscillation of non-homogeneous linear differential equations with meromorphic coefficients under substracting the condition that all solutions of differential equation are meromorphic functions.

1. Introduction and results

Consider non-homogeneous linear differential equations of the form

$$f^{(k)}+b_{k-1}f^{(k-1)}+\cdots+b_0 f=H(z)\quad (k\geqq 1) \tag{1.1}$$

where b_{k-j} $(j=1, \cdots, k)$ are rational functions, $H(z)$ is a meromorphic function. Z.-X. Chen and S.-A. Gao proved in [3].

THEOREM A. *Let b_{k-j} $(j=1, \cdots, k)$ be rational functions having a pole at ∞ of order $n_{k-j}\geqq 0$, $k\geqq 1$, $H(z)$ be a meromorphic function, $\sigma(H)=\beta$ satisfying*

$$1+\max_{1\leq j\leq k} n_{k-j}/j<\beta<\infty . \tag{1.2}$$

If all solutions f of the differential equation (1.1) *are meromorphic functions, then*

(a) $\sigma(f)=\beta$.

(b) $\lambda(1/f)=\lambda(1/H)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/H)$. If $\lambda(H)>\lambda(1/H)$, then $\lambda(f)\geqq\lambda(H)$.

(c) *If $\beta>\max\{\lambda(H), \lambda(1/H)\}$, then all solutions of* (1.1) *satisfy $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta$, except at most a possible one. The possible exceptional one f_0 satisfies $\lambda(f_0)<\beta$.*

---

* Project supported by the National Natural Science Foundation of China.
Keywords: non-homogeneous linear differential equation, meromorphic function, zero, pole, order.
AMS No.: 34A20, 30D35.
Received June 9, 1995; revised April, 19, 1996.

THEOREM B. *Let b_{k-j} $(j=1, \cdots, k)$ be rational functions having a pole at ∞ of order $n_{k-j} \geqq 0$, $k \geqq 1$, $H(z) \not\equiv 0$ be a meromorphic function satisfying $\sigma(H)=\beta \leqq 1+\max_{1 \leqq j \leqq k} n_{k-j}/j$. If all solutions f of* (1.1) *are meromorphic functions, then*

(a) $\beta \leqq \sigma(f) \leqq 1+\max_{1 \leqq j \leqq k} n_{k-j}/j$.

(b) $\lambda(1/f)=\lambda(1/H)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/H)$. *If* $\lambda(H)>\lambda(1/H)$ *then* $\lambda(f) \geqq \lambda(H)$.

(c) *If* $\sigma(f)>\beta$, *then* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

In this paper, we use the same notations as in [1], i.e. $\lambda(f)$ and $\bar{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$, $\lambda(1/f)$ and $\bar{\lambda}(1/f)$ to denote respectively the exponents of convergence of the pole-sequence and the sequence of distinct poles of a meromorphic function $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$. And we use the standard notations of the Nevanlinna theory (e.g. see [5]).

By a fundamental theory of the differential equation with complex coefficients, we know that all solutions of linear differential equation with entire coefficients are entire functions. But a solution of linear differential equation with meromorphic coefficients is not always a meromorphic function. For example, $f_1=\exp\{1/z\}+e^z$ and $f_2=e^z$ are all solutions of the equation

$$f''+(z^3+z^2)f'+\left(z+1-\frac{1}{z^4}-\frac{2}{z^3}\right)f=\left(z^3+z^2+z+2-\frac{1}{z^4}-\frac{2}{z^3}\right)e^z$$

but f_1 is not a meromorphic function. Therefore in Theorems A and B, the condition that all solutions of (1.1) are meromorphic functions is very rigorous. In this paper, we will substract this condition in Theorems A and B to generalize Theorems A and B.

We will prove the following theorems.

THEOREM 1. *Suppose that b_{k-j} $(j=1, \cdots, k)$ are rational functions having a pole at ∞ of order $n_{k-j} \geqq 0$, $k \geqq 1$, $H(z)$ is a meromorphic function, $\sigma(H)=\beta$ satisfying* (1.2). *If* (1.1) *has a meromorphic solution f, then*

(a) $\sigma(f)=\beta$.

(b) $\lambda(1/f)=\lambda(1/H)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/H)$. *If* $\lambda(H)>\lambda(1/H)$, *then* $\lambda(f) \geqq \lambda(H)$.

(c) *If* $\beta>\max\{\lambda(H), \lambda(1/H)\}$, *then all meromorphic solutions of* (1.1) *satisfy* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta$, *except at most one* f_0 *satisfying* $\lambda(f_0)<\beta$.

THEOREM 2. *Suppose that b_{k-j} $(j=1, \cdots, k)$ are rational functions having a pole at ∞ of order $n_{k-j} \geqq 0$, $k \geqq 1$, $H(z) \not\equiv 0$ is a meromorphic function satisfying $\sigma(H)=\beta \leqq 1+\max_{1 \leqq j \leqq k} n_{k-j}/j$. If* (1.1) *has a meromorphic solution f, then*

(a) $\beta \leqq \sigma(f) \leqq 1+\max_{1 \leqq j \leqq k} n_{k-j}/j$.

(b) $\lambda(1/f)=\lambda(1/H)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/H)$. *If* $\lambda(H)>\lambda(1/H)$, *then* $\lambda(f) \geqq \lambda(H)$.

(c) *If* $\sigma(f)>\beta$, *then* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

Example having an exceptional solution in Theorem 1(c).

The equation

$$f''+f'-2(z+1)f=\left(\frac{1+\sin^2 z}{\cos^3 z}+\frac{(4z+1)\sin z}{\cos^2 z}+\frac{4z^2}{\cos z}\right)e^{z^2}$$

satisfies the additional hypothesis of Theorem 1 (c). And the equation has exceptional solution $f_0=(1/\cos z)e^{z^2}$, where $\sigma(f_0)=2$, $\lambda(1/f_0)=1$, $\lambda(f_0)=0<\sigma(f_0)$.

2. Lemmas and preliminaries

THEOREM C (Borel, see Theorem 5.13 in [7] or 2.6.18. Lemma in [2, P. 21]). *Suppose that $Q(z)$ is canonical product formed with $\{z_n;\ n=1, 2, \cdots\}$ $(z_n\neq 0)$ and $\lambda(Q)=\beta<\infty$. Set $O_n=\{z: |z-z_n|<|z_n|^{-\alpha}\}$. ($\alpha(>\beta)$ is a constant) then for any given $\varepsilon>0$,*

$$|Q(z)|\geqq\exp\{-|z|^{\beta+\varepsilon}\}$$

holds for $z\notin\bigcup_{n=1}^{\infty}O_n$.

THEOREM D (See [6, P. 19] or 2.3.6* in [2, P. 13]). *Suppose that $w(z)$ is a finite order entire function, $\mu(r)$ is the maximum term of the power series of $w(z)$, then*

$$\lim_{r\to\infty}\log M(r, w)/\log\mu(r)=1.$$

LEMMA 1. *Suppose that $H(z)$ is a meromorphic function, $\sigma(H)=\beta<\infty$, then for any given $\varepsilon>0$, there is a set $E_1\subset(1, \infty)$ that has finite linear measure and finite logarithmic measure, such that*

$$|H(z)|\leqq\exp\{r^{\beta+\varepsilon}\}$$

holds for $|z|=r\notin[0, 1]\cup E_1$, $r\to\infty$.

Proof. If H has only finitely many poles, then Lemma 1 holds obviously. Now assume that $H(z)$ has infinitely many poles. Set $H(z)=h(z)/[z^{k_1}\cdot Q(z)]$, where k_1 is nonnegative integer, $h(z)$ is an entire function, $Q(z)$ is the canonical product formed with the nonzero poles $\{z_j: j=1, 2, \cdots;\ |z_j|=r_j,\ 0<r_1\leqq r_2\leqq\cdots\}$ of $H(z)$, hence $\sigma(h)\leqq\sigma(H)=\beta$, $\sigma(Q)=\lambda(Q)\leqq\beta$.

For any given $\varepsilon>0$, set $O_j=\{z: |z-z_j|\leqq r_j^{-(\beta+\varepsilon/2)}\}$ $(j=1, 2, \cdots\}$ and $O=\bigcup_{j=1}^{\infty}O_j$. Set

$$E_1=\bigcup_{j=1}^{\infty}(r_j-r_j^{-(\beta+\varepsilon/2)},\ r_j+r_j^{-(\beta+\varepsilon/2)}).$$

Since

$$\sum_{j=1}^{\infty}1/r_j^{\beta+\varepsilon/2}=d<\infty, \tag{2.1}$$

we know that linear measure of E_1, $mE_1=2d<\infty$. For $|z|=r\notin E_1\cup[0, 1]$, we have from Theorem C

$$|Q(z)|\geqq\exp\{-r^{\beta+\varepsilon/2}\}.$$

Hence

$$|H(z)| \leqq \exp\{2r^{\beta+\varepsilon/2}\}/r^{k_1} \leqq \exp\{r^{\beta+\varepsilon}\}$$

holds for $|z|=r \notin E_1 \cup [0, 1]$, $r \to \infty$.

Now we prove the logarithmic measure of E_1, $\operatorname{lm} E_1 < \infty$. From

$$\begin{aligned}\operatorname{lm} E_1 &= \sum_{j=1}^{\infty} [\log(r_j + r_j^{-(\beta+\varepsilon/2)}) - \log(r_j - r_j^{-(\beta+\varepsilon/2)})] \\ &= \sum_{j=1}^{\infty} \log\left(1 + \frac{2r_j^{-(\beta+\varepsilon/2)}}{r_j - r_j^{-(\beta+\varepsilon/2)}}\right),\end{aligned}$$

and for sufficiently large r_j

$$\log\left(1 + \frac{2r_j^{-(\beta+\varepsilon/2)}}{r_j - r_j^{-(\beta+\varepsilon/2)}}\right) \leqq \frac{2r_j^{-(\beta+\varepsilon/2)}}{r_j - r_j^{-(\beta+\varepsilon/2)}} \leqq 2r_j^{-(\beta+\varepsilon/2)},$$

we have $\operatorname{lm} E_1 < \infty$ by (2.1).

LEMMA 2. *Suppose that $g(z)$ is a transcendental entire function, $\sigma(g)=\alpha<\infty$, then there is a set $E_2 \subset (1, \infty)$ that has infinite logarithmic measure such that*

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log\log M(r, g)}{\log r} = \lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log \nu_g(r)}{\log r} = \alpha$$

where $\nu_g(r)$ denotes the centralindex of $g(z)$.

Proof. By $\sigma(g)=\alpha$, there exists $\{r_n\}$ $(r_n \to \infty)$, such that

$$\lim_{r_n\to\infty} \frac{\log\log M(r_n, g)}{\log r_n} = \alpha. \tag{2.2}$$

Setting $E_2 \subset (1, +\infty)$, E_2 has the following properties: (a) If the sequence $\{r_n\}$ satisfies (2.2), then $\{r_n\} \subset E_2$. (b) If a sequence $\{r_n\} \subset E_2$ $(r_n \to \infty)$, then (2.2) holds for $\{r_n\}$. Now we affirm that logarithmic measure of E_2, $\operatorname{lm} E_2 = \infty$. In fact, if $\operatorname{lm} E_2 = \delta < \infty$, then from the definition of E_2, we have

$$\varlimsup_{\substack{r\to\infty \\ r\in(1,\infty)-E_2}} \frac{\log\log M(r, g)}{\log r} = \alpha_1 < \alpha. \tag{2.3}$$

Now for a given $\{r'_n\} \subset (1, \infty)$, $r'_n \to \infty$, there exists a point $r''_n \in [r'_n, (\delta+1)r'_n] - E_2$. From

$$\frac{\log\log M(r''_n, g)}{\log r''_n} \geqq \frac{\log\log M(r'_n, g)}{\log[(\delta+1)r'_n]} = \frac{\log\log M(r'_n, g)}{\log r'_n + \log(\delta+1)},$$

we have

$$\begin{aligned}\varlimsup_{r'_n\to\infty} = \frac{\log\log M(r'_n, g)}{\log r'_n} &= \varlimsup_{r'_n\to\infty} \frac{\log\log M(r'_n, g)}{\log r'_n + \log(\delta+1)} \\ &\leqq \varlimsup_{r''_n\to\infty} \frac{\log\log M(r''_n, g)}{\log r''_n}\end{aligned}$$

$$\leqq \varlimsup_{\substack{r\to\infty \\ r\in(1,+\infty)-E_2}} \frac{\log\log M(r, g)}{\log r}.$$

Since $\{r_n'\}$ is arbitrary, we have $\alpha \leqq \alpha_1$. This is a contradiction, hence $\operatorname{lm} E_2=\infty$.

By $\sigma(g)=\alpha<\infty$ and Theorem D, we have

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log \mu(r)}=1, \tag{2.4}$$

where $\mu(r)$ is the maximum term of the power series of $g(z)$, $\mu(r)=|a_{\nu_g(r)}|r^{\nu_g(r)}$ By (2.4), for sufficiently large r,

$$\log M(r, g)\leqq 2\log\mu(r)\leqq 2\log^+|a_{\nu_g}|+2\nu_g(r)\cdot\log r.$$

From

$$\frac{\log\log M(r, g)}{\log r}\leqq\frac{\log\nu_g(r)}{\log r}+\frac{\log^+\log^+|a_{\nu_g}|+2\log 2+\log\log r}{\log r},$$

we have

$$\alpha=\lim_{\substack{r\to\infty \\ r\in E_2}}\frac{\log\log M(r, g)}{\log r}=\varliminf_{\substack{r\to\infty \\ r\in E_2}}\frac{\log\log M(r, g)}{\log r}$$

$$\leqq\varliminf_{\substack{r\to\infty \\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}\leqq\varlimsup_{\substack{r\to\infty \\ r\in(0,\infty)}}\frac{\log\nu_g(r)}{\log r}=\alpha,$$

i.e.

$$\lim_{\substack{r\to\infty \\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}=\alpha.$$

LEMMA 3. *Suppose that $g(z)$ is an entire function with $\sigma(g)=\infty$, then there is a set $E_2\subset(1, \infty)$ that has infinite logarithmic measure such that*

$$\lim_{\substack{r\to\infty \\ r\in E_2}}\frac{\log\nu_g(r)}{\log r}=\infty. \tag{2.5}$$

Proof. Using the same proof as in the upper half part of Lemma 2, we can prove Lemma 3.

LEMMA 4 (see [4]). *Suppose that $u(z)$ is a meromorphic function with $\sigma(u)=\beta<\infty$, $\varepsilon>0$ is a given constant. Then there exists a set $E_3\subset(1, \infty)$ that has finite logarithmic measure, such that*

$$\left|\frac{u^{(j)}(z)}{u(z)}\right|\leqq r^{j(\beta-1+\varepsilon)}\quad (j=1, \cdots, k) \tag{2.6}$$

hold for all z satisfying $|z|=r\notin[0, 1]\cup E_3$.

LEMMA 5. *Suppose that $u(z)$ is a meromorphic function with $\sigma(u)=\beta<\infty$, (m is integer), $\varepsilon>0$ is a given constant. Then there exists a set $E_3\subset(1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z|=r\notin[0, 1]\cup E_3$,* *we have*

$$|u(z)\cdot(u^{-1}(z))^{(m)}|\leqq r^{m(\beta-1+\varepsilon)}. \tag{2.7}$$

Proof. First we use the mathematical induction to prove

$$u\left(\frac{1}{u}\right)^{(m)}=\sum_{(j_1\cdots j_m)}\alpha_{(j_1\cdots j_m)}\left(\frac{u'}{u}\right)^{j_1}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}, \tag{2.8}$$

where $\alpha_{(j_1\cdots j_m)}$ is a constant, $j_1, \cdots, j_m$ satisfy $1\cdot j_1+2\cdot j_2+\cdots+m\cdot j_m=m$. For $m=1$, (2.8) holds obviously. For m, assume that (2.8) holds. So, we have for $m+1$,

$$\begin{aligned}\left(\frac{1}{u}\right)^{(m+1)}&=\left[\left(\frac{1}{u}\right)^{(m)}\right]'=\left[\frac{1}{u}\sum_{(j_1\cdots j_m)}\alpha_{(j_1\cdots j_m)}\left(\frac{u'}{u}\right)^{j_1}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}\right]'\\
&=-\frac{u'}{u^2}\sum_{(j_1\cdots j_m)}\alpha_{(j_1\cdots j_m)}\left(\frac{u'}{u}\right)^{j_1}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}+\frac{1}{u}\sum_{(j_1\cdots j_m)}\alpha_{(j_1\cdots j_m)}\\
&\quad\cdot\left\{\sum_{d=1}^{m}\left(\frac{u'}{u}\right)^{j_1}\cdots\left(\frac{u^{(d-1)}}{u}\right)^{j_{d-1}}\left[j_d\left(\frac{u^{(d)}}{u}\right)^{j_d-1}\left(\frac{u^{(d+1)}}{u}\right)\right.\right.\\
&\quad\left.\left.-j_d\left(\frac{u^{(d)}}{u}\right)^{j_d}\left(\frac{u'}{u}\right)\right]\cdot\left(\frac{u^{(d+1)}}{u}\right)^{j_{d+1}}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}\right\}\\
&=\frac{1}{u}\sum_{(j_1\cdots j_m)}\alpha_{(j_1\cdots j_m)}\left(\frac{u'}{u}\right)^{j_1+1}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}+\frac{1}{u}\sum_{(j_1\cdots j_m)}\alpha_{(j_1\cdots j_m)}\\
&\quad\cdot\left\{\sum_{d=1}^{m}\left[-j_d\left(\frac{u'}{u}\right)^{j_1+1}\left(\frac{u''}{u}\right)^{j_2}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}\right.\right.\\
&\quad\left.\left.+j_d\left(\frac{u'}{u}\right)^{j_1}\cdots\left(\frac{u^{(d)}}{u}\right)^{j_d-1}\left(\frac{u^{(d+1)}}{u}\right)^{j_{d+1}+1}\cdots\left(\frac{u^{(m)}}{u}\right)^{j_m}\right]\right\},\end{aligned}$$

where the indexs satisfy $1\cdot(j_1+1)+2\cdot j_2+\cdots+m\cdot j_m=m+1$, or $1\cdot j_1+\cdots+d\cdot(j_d-1)+(d+1)\cdot(j_{d+1}+1)+\cdots+m\cdot j_m=m+1$. Therefore (2.8) holds.

Now by (2.8) and Lemma 4, it is easy to see that Lemma 5 holds.

LEMMA 6. *Suppose that $b_0, \cdots, b_{k-1}, H\not\equiv 0$ are meromorphic functions, $\sigma(H)=\beta<\infty$, that there are a set $E_3\subset(1, +\infty)$ that has finite logarithmic measure and a constant number $C_1>0$, such that for $|z|=r\notin[0, 1]\cup E_3$,*

$$|b_j(z)|\leqq r^{C_1}\quad (j=0, \cdots, k-1) \tag{2.9}$$

hold. If an entire function $g(z)$ solves the equation

$$g^{(k)}+b_{k-1}g^{(k-1)}+\cdots+b_0g=H, \tag{2.10}$$

then $\sigma(g)<\infty$.

Proof. Assume that $\sigma(g)=\infty$, $\mu(r)$ denotes the maximum term of the power series of $g(z)$, $\nu_g(r)$ denotes the centralindex of $g(z)$. By Lemma 3, we know that there is a set $E_2\subset(1, \infty)$ that has infinite logarithmic measure such that

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log \nu_g(r/2)}{\log(r/2)} = \infty . \tag{2.11}$$

Since $\nu_g(r)$ is a step function in r, we can assume that t_j ($j=0, 1, \cdots, 0=t_0<t_1<t_2<\cdots$) are discontinuous points of $\nu_g(r)$. As $t\in(t_j, t_{j+1})$, we have $\mu(t)=|a_{\nu_g(t)}|\cdot t^{\nu_g(t)}$, where centralindex $\nu_g(t)=m$ is fixed constant. Hence

$$\mu'(t)=m|a_m|t^{m-1}=\mu\cdot\nu_g(t)/t$$

holds for $t\in(t_j, t_{j+1})$. Since $\mu(t)$ is a continuous function, we have for $r>2$

$$\begin{aligned}\log\mu(r)-\log\mu(1)&=\int_1^r[\mu'(t)/\mu(t)]\mathrm{d}t\\ &=\int_1^r[\nu_g(t)/t]\mathrm{d}t>\int_{r/2}^r(\nu_g(t)/t)\mathrm{d}t\geqq\nu_g(r/2)\cdot\log 2 .\end{aligned}$$

By Cauchy's inequality, it is easy to know that $\mu(r)\leqq M(r, g)$. So,

$$\nu_g(r/2)\cdot\log 2\leqq\log M(r, g)-\log\mu(1) . \tag{2.12}$$

For a given large α such that

$$\alpha>\max\{C_1, \beta\}+k , \tag{2.13}$$

by (2.11), (2.12), we obtain

$$\nu_g(r)\geqq\nu_g(r/2)\geqq(r/2)^\alpha=C_2r^\alpha, \tag{2.14}$$

$$M(r, g)\geqq C_3\cdot\exp\{C_1r^\alpha\} \tag{2.15}$$

for $r\in E_2$, $r\to\infty$, where C_2, C_3, C_4 are positive constants.

From the Wiman-Valiron theory (see [6], [8], [9]) we have basic formulas

$$\frac{g^{(j)}(z)}{g(z)}=\left(\frac{\nu_g(r)}{z}\right)^j(1+o(1)) \quad (j=1, \cdots, k) \tag{2.16}$$

where $|z|=r$, $|g(z)|=M(r, g)$, $r\notin E_4$, $\int_{E_4}\frac{\mathrm{d}r}{r}<\infty$.

By Lemma 1, we have

$$|H(z)|\leqq\exp\{r^{\beta+1/2}\} \tag{2.17}$$

for $|z|=r\in[1, +\infty)-E_1$, $\int_{E_1}\frac{\mathrm{d}r}{r}<\infty$.

Now, we take sufficiently large $|z|=r\in E_2-(E_1\cup E_3\cup E_4)$, $|g(z)|=M(r, g)$, logarithmic measure $\mathrm{lm}[E_2-(E_1\cup E_3\cup E_4)]=\infty$. (2.10) and (2.16) give

$$\left(\frac{\nu_g(r)}{z}\right)^k(1+o(1))+b_{k-1}\left(\frac{\nu_g(r)}{z}\right)^{k-1}(1+o(1))+\cdots+b_0=\frac{H(z)}{g(z)},$$

$$\frac{\nu_g(r)}{z^k}(1+o(1))=\frac{H(z)}{g(z)\nu_g^{k-1}(r)}-\frac{b_{k-1}}{z^{k-1}}(1+o(1))$$
$$-\frac{b_{k-2}}{z^{k-2}\nu_g(r)}(1+o(1))-\cdots-\frac{b_0}{\nu_g^{k-1}(r)} \tag{2.18}$$

By (2.13)-(2.15), (2.17), we have

$$\frac{|H(z)|}{|g(z)|}=\frac{|H(z)|}{M(r, g)}\leqq\frac{1}{C_3}\exp\{r^{\beta+(1/2)}-C_4r^{\alpha}\}\to 0, \tag{2.19}$$

$$\left|\frac{b_j(z)}{z^j\nu_g^{k-1-j}(r)}\right|\to 0\quad (j=0, \cdots, k-2) \tag{2.20}$$

hold for $|z|=r\in E_2-(E_1\cup E_3\cup E_4)$, $r\to\infty$. And (2.14), (2.19) and (2.20) give

$$\left|\frac{H(z)}{g(z)\cdot\nu_g^{k-1}(r)}-\frac{b_{k-1}}{z^{k-1}}(1+o(1))-\frac{b_{k-2}}{z^{k-2}\nu_g(r)}(1+o(1))-\cdots-\frac{b_0}{\nu_g^{k-1}(r)}\right|$$
$$=O\left(\frac{b_{k-1}}{z^{k-1}}\right)=O(r^{C_1-k+1}). \tag{2.21}$$

On the other hand, by (2.14), we have

$$\left|\frac{\nu_g(r)}{z^k}(1+o(1))\right|\geqq C_2r^{\alpha-k}>r^{C_1} \tag{2.22}$$

for $r\in E_2$, $r\to\infty$. And (2.21) contradicts (2.22) by (2.18). Therefore $\sigma(g)<\infty$.

LEMMA 7. *Suppose that b_{k-j} $(j=1, \cdots, k)$ are rational functions having a pole at ∞ of order $n_{k-j}\geqq 0$, $k\geqq 0$, $H(z)\not\equiv 0$ is a meromorphic function with $\sigma(H)=\beta$. If* (1.1) *has a meromorphic solution f, then*

(a) *If $1+\max_{1\leqq j\leqq k} n_{k-j}/j<\beta<\infty$, then $\sigma(f)=\beta$.*

(b) *If $\beta\leqq 1+\max_{1\leqq j\leqq k} n_{k-j}/j$, then $\beta\leqq\sigma(f)\leqq 1+\max_{1\leqq j\leqq k} n_{k-j}/j$.*

Proof. We have $\sigma(f)\geqq\beta$ from (1.1). By (1.1) and fact that b_{k-j} $(j=1, \cdots, k)$ have only finitely many poles, we know that if $|z|$ $(<\infty)$ is sufficiently large, then either f and H are both analytic at z, or f has a pole at z of order m_1 if and only if H has a pole at z of order m_1+k. So,

$$\bar{\lambda}(1/f)=\bar{\lambda}(1/H). \tag{2.23}$$

From

$$n(r, f)\leqq n(r, H)+O(1)\quad\text{and}\quad n(r, H)\leqq(k+1)n(r, f)+O(1),$$

it follows that

$$\lambda(1/f)=\lambda(1/H). \tag{2.24}$$

Set $f(z)=g(z)/(z^{m_2}\cdot u(z))=g(z)/u_1(z)$, where m_2 is a nonnegative integer, $g(z)$ is an entire function, $u(z)$ is a canonical product (or polynomial) formed with the nonzero poles $\{z_j : j=1, 2, \cdots\}$ $(|z_j|=r_j,\ 0<r_1\leqq r_2\leqq \cdots)$ of f, $u_1(z)=z^{m_2}u(z)$, then $\lambda(u_1)=\sigma(u_1)=\lambda(1/f)=\lambda(1/H)\leqq\beta$.

Now we suppose that $\sigma(f)=\alpha>\beta$. By $f=g/u_1$ and $\sigma(u_1)\leqq\beta$, we have $\sigma(g)=\sigma(f)=\alpha$. For any given ε $(0<2\varepsilon<\alpha-\beta)$, by Lemma 1, it follows that there is a set $E_1\subset(1, +\infty)$ that has finite logarithmic measure, such that

$$|1/u_1(z)|\leqq\exp\{r^{\beta+\varepsilon}\} \tag{2.25}$$

holds for $|z|=r\notin[0, 1]\cup E_1$, $r\to\infty$. From (2.24), (2.25) and the fact that the poles of f can only occur at poles of H except at most finitely many poles, it follows that

$$|H(z)|\leqq\exp\{r^{\beta+\varepsilon}\} \tag{2.26}$$

holds for $|z|=r\notin[0, 1]\cup E_1$, $r\to\infty$. By $f(z)=g(z)/u_1(z)$, we have for $n=1, \cdots, k$

$$\frac{f^{(n)}}{f}=\frac{g^{(n)}}{g}+C_n^1\cdot u_1\left(\frac{1}{u_1}\right)'\frac{g^{(n-1)}}{g}+\cdots+C_n^{(n-1)}\cdot u_1\left(\frac{1}{u_1}\right)^{(n-1)}\frac{g'}{g}+u_1\left(\frac{1}{u_1}\right)^{(n)} \tag{2.27}$$

where C_n^j $(j=1, \cdots, n)$ are the usual notation for the binomial coefficients. (2.27) and (1.1) give

$$\frac{g^{(k)}}{g}+d_{k-1}\frac{g^{(k-1)}}{g}+\cdots+d_1\frac{g'}{g}+d_0=\frac{H\cdot u_1}{g} \tag{2.28}$$

where

$$d_{k-j}=C_k^j\cdot u_1\left(\frac{1}{u_1}\right)^{(j)}+b_{k-1}\cdot C_{k-1}^{j-1}u_1\left(\frac{1}{u_1}\right)^{(j-1)}+\cdots+b_{k-j+1}\cdot C_{k-j+1}^1u_1\left(\frac{1}{u_1}\right)'+b_{k-j} \tag{2.29}$$

$$(j=1, \cdots, k)$$

By $\sigma(u_1)\leqq\beta$ and Lemma 6, there is a set $E_3\subset(1, \infty)$ that has finite logarithmic measure, such that for $|z|=r\notin[0, 1]\cup E_3$, for $j=1, \cdots, k$, we have

$$|u_1(z)(u_1^{-1}(z))^{(j)}|\leqq r^{j(\beta-1+\varepsilon)}. \tag{2.30}$$

(a) Suppose $1+\max_{1\leq j\leq k} n_{k-j}/j<\beta<\infty$. Now we prove $\sigma(f)=\alpha>\beta$ fails. From (2.30) and

$$n_{k-j}\leqq j(\beta-1)\quad (j=1, \cdots, k), \tag{2.31}$$

we have for $|z|=r\notin[0, 1]\cup E_3$, $r\to\infty$.

$$|b_{k-q}(z)u_1(z)(u_1^{-1}(z))^{j-q}|\leqq r^{n_{k-q}+(j-q)(\beta-1+\varepsilon)}<r^{j(\beta-1+\varepsilon)}\quad (q=1, \cdots, j). \tag{2.32}$$

(2.29) and (2.32) give for $|z|=r\notin[0, 1]\cup E_3$, $r\to\infty$

$$|d_{k-j}(z)|=O(r^{j(\beta-1+\varepsilon)})\quad (j=1, \cdots, k). \tag{2.33}$$

By Lemma 6 and (2.28), (2.33), we have $\sigma(g)=\alpha<\infty$. From Lemma 2 and $\sigma(g)<\infty$, there is a set $E_2\subset(1, \infty)$ that has infinite logarithmic measure such that

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log\log M(r, g)}{\log r} = \lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log \nu_g(r)}{\log r} = \alpha . \tag{2.34}$$

From the Wiman-Valiron theory, there is a set $E_4 \subset (1, \infty)$ that has finite logarithmic measure, such that for $|z| = r \notin E_4$, $|g(z)| = M(r, g)$, (2.16) holds. By (2.34), we have

$$M(r, g) \geqq \exp\{r^{\alpha-\varepsilon}\} \tag{2.35}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])$ and sufficiently large r. From (2.26), (2.35) and $|u_1(z)| \leqq \exp\{r^{\beta+\varepsilon}\} (|z| = r \to \infty)$, and $\beta + \varepsilon < \alpha - \varepsilon$, we get

$$\left| \frac{u_1(z) \cdot H(z)}{g(z)} \right| = \left| \frac{u_1(z) \cdot H(z)}{M(r, g)} \right| \leqq \exp\{2r^{\beta+\varepsilon} - r^{\alpha-\varepsilon}\} \to 0 \tag{2.36}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])$, $|g(z)| = M(r, g)$, $r \to \infty$. By (2.34),

$$\nu_g(r) = r^{\alpha + o(1)} \tag{2.37}$$

holds for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])$, $r \to \infty$. Since the logarithmic measure of $E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])$, $\operatorname{lm}[E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])] = \infty$, and by (2.16), (2.28), (2.33), (2.36), we obtain

$$\begin{aligned} &\left(\frac{\nu_g(r)}{z}\right)^k (1+o(1)) + O(r^{\beta-1+\varepsilon}) \left(\frac{\nu_g(r)}{z}\right)^{k-1} (1+o(1)) + \cdots + \\ &O(r^{(k-1)(\beta-1+\varepsilon)}) \frac{\nu_g(r)}{z} (1+o(1)) + O(r^{k(\beta-1+\varepsilon)}) = o(1) \end{aligned} \tag{2.38}$$

for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])$, $|g(z)| = M(r, g)$, $r \to \infty$. By (2.37), (2.38) and $0 < 2\varepsilon < \alpha - \beta$, for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4 \cup [0, 1])$, $|g(z)| = M(r, g)$, $r \to \infty$, it is easy to see that there is only one term $(\nu_g(r)/z)^k (1+o(1))$ with the degree $k(\alpha - 1)$ being the highest one among all terms of (2.38). This is impossible. Therefore, $\sigma(f) = \beta$.

(b) Suppose that $\beta \leqq 1 + \max_{1\leq j\leq k} n_{k-j}/j$. Now we prove $1 + \max_{1\leq j\leq k} n_{k-j}/j < \sigma(f) = \alpha$ fails. We set $1 + \max_{1\leq j\leq k} n_{k-j}/j = m < \alpha$, then

$$n_{k-j} \leqq j(m-1) \quad (j = 1, \cdots, k) \quad \text{and} \quad \beta \leqq m. \tag{2.39}$$

By $\beta \leqq m$ and (2.30), (2.39) we have for $|z| = r \notin E_3 \cup [0, 1]$, $r \to \infty$

$$\begin{aligned} |b_{k-q}(z) u_1(z) (u_1^{-1}(z))^{(j-q)}| &\leqq r^{n_{k-q} + (j-q)(\beta-1+\varepsilon)} \\ &\leqq r^{q(m-1) + (j-q)(\beta-1+\varepsilon)} \\ &< r^{j(m-1+\varepsilon)} \quad (q = 1, \cdots, j). \end{aligned} \tag{2.40}$$

(2.29) and (2.40) give for $|z| = r \notin E_3 \cup [0, 1]$, $r \to \infty$

$$|d_{k-j}(z)| = O(r^{j(m-1+\varepsilon)}) \quad (j = 1, \cdots, k). \tag{2.41}$$

By Lemma 6 and (2.28), (2.41), we have $\sigma(f)=\alpha<\infty$. Using the same reasoning as in (a), it is easy to know that (2.34)-(2.37) hold. Since $\text{lm}[E_2-(E_1\cup E_3\cup E_4\cup[0, 1])]=\infty$ and (2.16), (2.28), (2.36), (2.41), we have

$$(2.42)\qquad \left(\frac{\nu_g(r)}{z}\right)^k(1+o(1))+O(r^{m-1+\varepsilon})\left(\frac{\nu_g(r)}{z}\right)^{k-1}(1+o(1))+\cdots+$$
$$O(r^{(k-1)(m-1+\varepsilon)}\frac{\nu_g(r)}{z}(1+o(1))+O(r^{k(m-1+\varepsilon)})=o(1)$$

for $|z|=r\in E_2-(E_1\cup E_3\cup E_4\cup[0, 1])$, $|g(z)|=M(r, g)$, $r\to\infty$. From (2.37), (2.42) and $\alpha>m$, for $|z|=r\in E_2-(E_1\cup E_3\cup E_4\cup[0, 1])$, $|g(z)|=M(r, g)$, $r\to\infty$, it is easy to see that there is only one term $(\nu_g(r)/z)^k(1+o(1))$ with the degree $k(\alpha-1)$ being the highest one among all terms of (2.42). This is impossible. Therefore, $\beta\leqq\sigma(f)\leqq m$.

LEMMA 8. *Suppose that β is a positive integer and $\beta>1$, B_{k-j} $(j=1, \cdots, k)$ are rational functions having a pole at ∞ of order $n_{k-j}=j(\beta-1)$, $U\not\equiv 0$ is a meromorphic function with $\sigma(U)<\beta$. If the equation*

$$(2.43)\qquad y^{(k)}+B_{k-1}y^{(k-1)}+\cdots+B_0y=U$$

has a meromorphic solution y, then $\sigma(y)=\beta$ except at most one possible exceptional meromorphic solution y_0 with $\sigma(y_0)<\beta$.

If $y\not\equiv 0$ is a meromorphic solution of the equation

$$(2.44)\qquad y^{(k)}+B_{k-1}y^{(k-1)}+\cdots+B_0y=0$$

that is the corresponding homogeneous differential equation of (2.43), the $\sigma(y)=\beta$.

Proof. Set $\sigma(y)=\alpha$, then $\alpha\geqq\sigma(U)=d$ by (2.43). Now assume that $\sigma(y)=\alpha>d$. Set $y(z)=g(z)/u_1(z)$ where $g(z)$, $u_1(z)$ are functions defined in the same way as in Lemma 7. Using the same reasoning as in Lemma 7, we have $\sigma(u_1)\leqq d$ and

$$(2.45)\qquad |u_1(z)|\leqq\exp\{r^{d+\varepsilon}\}\quad(|z|=r\to\infty).$$

And there is a set $E_4\subset(1, \infty)$ has finite logarithmic measure such that (2.16) holds for $|z|=r\notin[0, 1]\cup E_4$, $|g(z)|=M(r, g)$. For any given ε $(0<2\varepsilon<\min\{\alpha-d, \beta-d\})$, there is a set $E_1\subset(1, +\infty)$ that has finite logarithmic measure such that for $|z|=r\notin E_1\cup[0, 1]$, $r\to\infty$

$$(2.46)\qquad |U(z)|\leqq\exp\{r^{d+\varepsilon}\}.$$

By Lemma 5 and the hypotheses, there is a set $E_3\subset(1, +\infty)$ that has finite logarithmic measure such that for $|z|=r\notin[0, 1]\cup E_3$,

$$(2.47)\qquad |u_1(z)(u_1^{-1}(z))^{(j)}|\leqq r^{j(d-1+\varepsilon)}\quad(j=1, \cdots, k).$$

Substituting $y=g/u_1$ into (2.43), we get

$$\frac{g^{(k)}}{g}+D_{k-1}\frac{g^{(k-1)}}{g}+\cdots+D_0=\frac{U\cdot u_1}{g} \tag{2.48}$$

where

$$D_{k-j}=C_k^j u_1(u_1^{-1})^{(j)}+B_{k-1}C_{k-1}^{j-1}u_1(u_1^{-1})^{(j-1)}+\cdots+B_{k-j+1}C_{k-j+1}^1 u_1(u_1^{-1})'+B_{k-j} \quad (j=1, \cdots, k). \tag{2.49}$$

Since (2.47), (2.49) and $B_{k-j}(z)=C_{k-j}\cdot z^{j(\beta-1)}(1+o(1))$ (($C_{k-j}\neq 0$ is constant) ($j=1, \cdots, k$), and $\beta>d+\varepsilon$, we have

$$D_{k-j}=C_{k-j}z^{j(\beta-1)}(1+o(1)) \quad (j=1, \cdots, k). \tag{2.50}$$

for $|z|=r\notin E_3\cup[0, 1]$, $r\to\infty$.

By (2.48), (2.50) and Lemma 6, we know that $\sigma(g)=\alpha<\infty$. By Lemma 2, there is a set $E_2\subset(1, +\infty)$ that has infinite logarithmic measure, such that (2.34) holds. Using the same reasoning as in Lemma 7, we have

$$\left|\frac{u_1(z)\cdot U(z)}{g(z)}\right|=\left|\frac{u_1(z)\cdot U(z)}{M(r, g)}\right|\leqq\exp\{2r^{d+\varepsilon}-r^{\alpha-\varepsilon}\}\to 0. \tag{2.51}$$

for $|z|=r\in E_2-([0, 1]\cup E_1\cup E_3\cup E_4)$, $|g(z)|=M(r, g)$, $r\to\infty$. By (2.48), (2.50), (2.51), and (2.16), we get

$$\left(\frac{\nu_g(r)}{z}\right)^k(1+o(1))+C_{k-1}z^{\beta-1}\left(\frac{\nu_g(r)}{z}\right)^{k-1}(1+o(1))+\cdots+ C_1z^{(k-1)(\beta-1)}\left(\frac{\nu_g(r)}{z}\right)(1+o(1))+C_0z^{k(\beta-1)}(1+o(1))=o(1) \tag{2.52}$$

for $|z|=r\in E_2-([0, 1]\cup E_1\cup E_3\cup E_4)$, $|g(z)|=M(r, g)$, $r\to\infty$. By (2.34), we have

$$\nu_g(r)=r^{\alpha+o(1)} \tag{2.53}$$

for $|z|=r\in E_2-([0, 1]\cup E_1\cup E_3\cup E_4)$, $|g(z)|=M(r, g)$, $r\to\infty$. By (2.53) and ε arbitrarily small, we see that the degrees of all terms of the left of (2.52) are respectively

$$k(\alpha-1),\ (k-j)(\alpha-1)+j(\beta-1) \quad (j=1, \cdots, k).$$

From the Wiman-Valiron theory, we get $\alpha=\beta$, i.e. $\sigma(y)=\sigma(g)=\beta$.

Using the same manner as above, we can prove that if $y(z)\not\equiv 0$ is a meromorphic solution of (2.44), then $\sigma(y)=\beta$.

If y_0 and y_1 ($y_1\not\equiv y_0$) are both meromorphic solutions of (2.43) with $\sigma(y_j)<\beta$ ($j=0, 1$), then $\sigma(y_1-y_0)<\beta$. But $y_1-y_0\not\equiv 0$ is a meromorphic solution of (2.44), we have $\sigma(y_1-y_0)=\beta$ by the proof given above. Therefore, (2.43) has at most one exceptional meromorphic solution y_0 with $\sigma(y_0)<\beta$.

LEMMA 9. *Suppose that b_{k-j} $(j=1, \cdots, k)$ are rational functions having a pole at ∞ of order $n_{k-j} \geqq 0$, $k \geqq 1$, $H(z)$ is a meromorphic function, $\sigma(H)=\beta<\infty$. If f is a meromorphic solution of* (1.1), *then*

$$\max\{\lambda(f), \lambda(1/f)\} \geqq \max\{\lambda(H), \lambda(1/H)\}. \tag{2.54}$$

Proof. Set $f=g/u_1$, where g and u_1 are functions defined in the same way as in Lemma 7. Using the same method as in the proof of Lemma 7, we know that (2.28), (2.33) hold. By Lemma 6, we have $\sigma(g)<\infty$, hence $\sigma(f)<\infty$.

Using the same reasoning as in Lemma 7 of [3] we can prove (2.54) holds.

LEMMA 10 (see [3]). *Suppose that b_{k-j} $(j=1, \cdots, k)$ are rational functions, $H \not\equiv 0$ is a meromorphic function with $\sigma(H)<\infty$. If f is a meromorphic solution of* (1.1) *such that $\sigma(H)<\sigma(f)<\infty$, then $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.*

3. Proof of Theorem 1

(a) By Lemma 7, we have $\sigma(f)=\beta$.

(b) By (2.23) and (2.24) in proof of Lemma 7, we have $\bar{\lambda}(1/f)=\bar{\lambda}(1/H)$, $\lambda(1/f)=\lambda(1/H)$. If $\lambda(H)>\lambda(1/H)$, we have $\lambda(f) \geqq \lambda(H)$ by Lemma 9.

(c) If $\beta>\max\{\lambda(H), \lambda(1/H)\}$, then set $H=Ue^p$, where $U=z^s(v_1/v_2)$ (s is an integer), v_1 and v_2 are canonical products (or polynomials) formed respectively with the nonzero zeros and nonzero poles of H, $\sigma(U)=\max\{\lambda(H), \lambda(1/H)\}<\beta$, p is a polynomial with $\deg p=\beta$.

Now set $f=ge^p$, then $f(z)$ and $g(z)$ have the same zeros and poles. From

$$f^{(m)}=\left\{g^{(m)}+mp'g^{(m-1)}+\sum_{j=2}^{m} C_m^j (p')^j+H_{j-1}(p')g^{(m-j)}\right\}e^p \tag{3.1}$$

where $m=2, 3, \cdots, k$, $H_{j-1}(p')$ are differential polynomials in p' and its derivatives of total degree $j-1$ with constant coefficients. It is easy to see that the derivatives of $H_{j-1}(p')$ as to z are of the same form $H_{j-1}(p')$. Substituting $f=ge^p$, $H=Ue^p$ into (1.1), we have by (3.1)

$$g^{(k)}+B_{k-1}g^{(k-1)}+\cdots+B_0 g=U \tag{3.2}$$

where

$$\begin{cases} B_{k-j}=b_{k-j}+(k-j+1)b_{k-j+1}p'+\sum_{n=2}^{j} b_{k-j+n}C_{k-j+n}^n (p')^n+H_{n-1}(p') \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=2, \cdots, k,\ b_k \equiv 1) \\ B_{k-1}=b_{k-1}+kp' \end{cases} \tag{3.3}$$

Since $\beta>1+\max_{1 \leq j \leq k} n_{k-j}/j$, the degree $j(\beta-1)$ of the term $b_k C_k^j (p')^j=C_k^j (p')^j$ $(n=j)$ is the highest one in the first equality of (3.3). Hence B_{k-j} $(j=2, \cdots, k)$ *must have a pole at ∞ of order $j(\beta-1)$. By* $\deg p'=\beta-1>n_{k-1}$, *the rational function B_{k-1}* has a pole at ∞ of order $1 \cdot (\beta-1)$. By Lemma 8, we see that all meromorphic solutions of (3.2) satisfy $\sigma(g)=\beta$ except at most one possible exceptional meromorphic solution g_0 with $\sigma(g_0)<\beta$. If $\sigma(g_0)<\beta$, then $\lambda(g_0)<\beta$. If $\sigma(g)=\beta$, by $\sigma(U)<\beta$ and Lemma 10, we have $\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta$. Therefore, all meromorphic solutions $f=ge^p$ of (1.1) satisfy $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta$ except at most one possible one $f_0=g_0e^p$ satisfying $\lambda(f_0)<\beta$.

4. Proof of Theorem 2

(a) By Lemma 7, we have $\beta\leqq\sigma(f)\leqq 1+\max_{1\leq j\leq k} n_{k-j}/j$.
(b) Using the same reasoning as in the proof of Theorem 1 (b), we have (b).
(c) If $\sigma(f)>\sigma(H)$, then by Lemma 10, we have $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

Acknowledgement. The author would like to thank the referees for valuable suggestions to improve my paper.

References

[1] S. Bank and I. Laine, On the oscillation theory of $f''+Af=0$ where A is entire, Trans. Amer. Math. Soc., 273 (1982), 351-361.
[2] R. P. Boas, Entire Functions, Academic Press INC., New York, 1954.
[3] Z.-X. Chen and S.-A. Gao, On the complex oscillation of non-homogeneous linear differential equations with meromorphic coefficients, Kodai Math. J., 15 (1992), 65-78.
[4] G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, J. London Math. Soc. (2), 37 (1988), 88-104.
[5] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[6] Y.-Z. He and X.-Z. Xiao, Algebroid Functions and Ordinary Differential Equations, Science Press, 1988 (in Chinese).
[7] R.-F. Li, C.-J. Dai and G.-D. Song, The Continuation of Theory of Functions of a Complex Variable, Higher Education Press, 1986 (in Chinese).
[8] G. Valiron, Lectures on the General Theory of Integral Functions, Clelsea, New York, 1949.
[9] G. Valiron, Functions Analytiques, Presses Universitaires de France, Paris, 1954.

Department of Mathematics
Jiangxi Normal University
Nanchang, 330027
Jiangxi Prov.
P.R. China