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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS*

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Abstract

In this paper, we study the asymptotic behavior of the second order difference equation

$$(*)\qquad \Delta(r(n)\Delta x(n))+f(n, x(n))=0.$$

we obtain some sufficient conditions which ensure that all the solutions of (*) are bounded, and also obtain some conditions which guarantee that for every solution $x(n)$ of (*) satisfies $|x(n)|=O(R(n, n_0))$ as $n\to\infty$, where $R(n, s)=\sum_{k=s}^{n-1}\frac{1}{r(k)}$.

1. A discrete inequality

In the sequel we will require the following discrete inequality which extends the known discrete inequality obtained by Meng [5].

DEFINITION. A function $g(u)$ is said to belong to $\mathcal{F}$ if $g(u)$ is nondecreasing and continuous on $(0, \infty)$ and

$$g(u)/v\leqq g(u/v), \quad u\geqq 0,\ v\geqq 1.$$

Every where we mean that $\sum_{k=s}^{n}\alpha(k)=0$ if $n<s$.

LEMMA. *Let $x(n)$, $h_i(n)$, $i=1, 2, \cdots, m$ be real valued nonnegative functions defined on $N(n_0)=\{n_0, n_0+1, \cdots\}$, $n_0\in\{1, 2, \cdots\}$, $f(n)\geqq 1$ be nondecreasing on $N(n_0)$, $g_i(u)\in\mathcal{F}$, $i=1, 2, \cdots, m$. Suppose that the discrete inequality*

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$$(1)\qquad x(n)\leqq f(n)+\sum_{i=1}^{m}\Big(\sum_{s=n_0}^{n-1}h_i(s)g_i(x(s))\Big)$$

holds for all $n\in N(n_0)$. *Then we have*

$$(2)\qquad x(n)\leqq f(n)\prod_{i=1}^{m}E_i(n),\quad n\in N(n_0).$$

Where $G_k(u)=\int_{u_0}^{u}\frac{1}{g_k(s)}ds$, $0<u_0$, $u\geqq 0$,

$$E_k(n)=G_k^{-1}\Big[G_k(1)+\sum_{s=n_0}^{n-1}h_k(s)\prod_{i=1}^{k-1}E_i(s)\Big],\quad k=1,2,\cdots,m.$$

G_k^{-1} *is the inverse of* G_k. *Here it is supposed that*

$$\prod_{i=1}^{0}E_i(n)=1,\quad n\in N(n_0).$$

Proof. The proof is by Mathematical induction. We first suppose that here $m=1$. Since $f(n)$ is nondecreasing, $g_1\in\mathcal{F}$ and $f(n)\geqq 1$, we have from (1)

$$(3)\qquad \frac{x(n)}{f(n)}\leqq 1+\sum_{s=n_0}^{n-1}\frac{h_1(s)}{f(s)}g_1(x(s))$$

$$\leqq 1+\sum_{s=n_0}^{n-1}h_1(s)g_1\Big(\frac{x(s)}{f(s)}\Big).$$

Now using the discrete Bihari's inequality [4], we obtain

$$(4)\qquad x(n)\leqq f(n)E_1(n),\quad n\in N(n_0),$$

where

$$E_1(n)=G_1^{-1}\Big[G_1(1)+\sum_{s=n_0}^{n-1}h_1(s)\Big].$$

This proves the (2) is true for $m=1$. Now suppose that (2) is true for $m=k$. Then for $m=k+1$ we may rewrite the inequality (1) as

$$(5)\qquad x(n)\leqq f_1(n)+\sum_{i=1}^{k}\Big(\sum_{s=n_0}^{n-1}h_i(s)g_i(x(s))\Big),\quad n\in N(n_0),$$

where

$$f_1(n)=f(n)+\sum_{s=n_0}^{n-1}h_{k+1}(s)g_{k+1}(x(s)).$$

Obviously, here $f_1(n)$ satisfies the condition for $f(n)$, so by the inductive assumption we obtain from (5)

$$x(n)\leqq f(n)R(n)+R(n)\Big(\sum_{s=n_0}^{n-1}h_{k+1}(s)g_{k+1}(x(s))\Big)$$

where

$$R(n)=\prod_{i=1}^{k} E_i(n).$$

So we have

$$\frac{x(n)}{f(n)R(n)} \leqq 1+\sum_{s=n_0}^{n-1} h_{k+1}(s)R(s)g_{k+1}\left(\frac{x(s)}{f(s)R(s)}\right). \tag{6}$$

This inequality is of the form (3), and hence by the first step of our proof we get from (6)

$$\begin{aligned} x(n) &\leqq f(n)R(n)G_{k+1}^{-1}\left[G_{k+1}(1)+\sum_{s=n_0}^{n-1} h_{k+1}(s)R(s)\right] \\ &= f(n)\prod_{i=1}^{k+1} E_i(n), \quad n\in N(n_0). \end{aligned}$$

This proves that (2) is true for $m=k+1$. The proof is complete.

Example. The following example illustrates the Lemma. Let

$$x(n)\leqq f(n)+\sum_{i=1}^{m}\left(\sum_{s=n_0}^{n-1} h_i(s)[x(s)]^{r_i}\right)$$

where $0<r_i<1$, $i=1, 2, \cdots, m$. We note that $x^{r_i}\in\mathcal{F}$ and x, f, h_i ($i=1, 2, \cdots, m$) are defined as in Lemma. It is easy to observe

$$G_i(u)=\frac{1}{1-r_i}[u^{1-r_i}-u_0^{1-r_i}], \quad i=1, 2, \cdots, m, \ 0<r_i<1,$$

and

$$G_i^{-1}(u)=[(1-r_i)u+u_0^{1-r_i}]^{1/(1-r_i)}, \quad i=1, 2, \cdots, m, \quad 0<r_i<1.$$

Similarly, we obtain

$$E_1(n)=\left[1+(1-r_1)\sum_{s=n_0}^{n-1} h_1(s)\right]^{1/(1-r_1)},$$

$$E_i(n)=\left[1+(1-r_i)\sum_{s=n_0}^{n-1} h_i(s)\prod_{k=1}^{i-1} E_k(s)\right]^{1/(1-r_i)}, \quad i=2, 3, \cdots, m.$$

We conclude that

$$x(n)\leqq f(n)\prod_{i=1}^{m} E_i(n), \quad n\in N(n_0).$$

2. Boundedness conditions

We consider the following second order nonlinear difference equation

$$\Delta(r(n)\Delta x(n))+f(n, x(n))=0, \quad n\in N(n_0), \tag{7}$$

where $N(n_0)$ as in above Lemma, Δ is the forward difference operator, i.e.

$\Delta x(n)=x(n+1)-x(n)$. $r(n)$ is the real sequences and $f: N(n_0)\times R\to R$ (R is real line). With regard to the equation (7) we always assume that:

(H) $r(n)>0$ for all $n\in N(n_0)$; in addition we always have

$$|f(n, x)|\leqq \sum_{i=1}^{m} b_i(n)g_i(|x|)+b_{m+1}(n), \quad (n, x)\in N(n_0)\times R,$$

where $b_i(n)$, $i=1, 2, \cdots, m+1$ are known nonnegative real functions on $N(n_0)$, $g_i(u)\in\mathcal{F}$, $i=1, 2, \cdots, m$.

To simplity the notation in the sequel we will define

$$R(n, s)=\sum_{k=s}^{n-1}\frac{1}{r(k)}.$$

for any $s\in N(n_0)$ and all $n\in N(s+1)$. According to the limit of $R(n, n_0)$ as $n\to\infty$, there are two different case, namely:

$$\text{(a)}\quad \lim_{n\to\infty} R(n, n_0)<\infty, \qquad \text{(b)}\quad \lim_{n\to\infty} R(n, n_0)=\infty.$$

Recently some results concerning the oscillatory and nonoscillatory properties of solutions of nonlinear difference equations of second order have been established in [1] (see also references therein).

The purpose of this paper is to present theorems that give sufficient conditions for all solutions of (7) to be bounded. Also, we obtain a result on the growth of solutions of (7). The obtained results are discrete analogues of some known theorems for nonlinear differential equations due to Yang [2] and Hatvanti [3].

We first establish some results for the case (a) as follows.

THEOREM 1. *In addition to the hypothesses (H), suppose further that*

(i) $\sum_{s=n_0}^{n-1} b_j(s)R(n, s+1)$ *is bounded on* $N(n_0)$ *for* $1\leqq j\leqq m+1$;

(ii) $\lim_{n\to\infty} R(n, n_0)<\infty$.

Then every solution $x(n)$ *of* (7) *is bounded on* $N(n_0)$.

Proof. Let $x(n)$ be a solution of (7). From (7) by successive summations, we obtain

$$x(n)=x(n_0)+r(n_0)\Delta x(n_0)\sum_{k=n_0}^{n-1}\frac{1}{r(k)}-\sum_{k=n_0}^{n-1}\frac{1}{r(k)}\sum_{i=n_0}^{k-1} f(i, x(i)).$$

To exchange the order of the summation of the last term, we have

$$x(n)=x(n_0)+r(n_0)\Delta x(n_0)\sum_{k=n_0}^{n-1}\frac{1}{r(k)}-\sum_{k=n_0}^{n-1} f(k, x(k))R(n, k+1).$$

So

$$|x(n)|\leqq a(n)+\sum_{i=1}^{m}\sum_{k=n_0}^{n-1} R(n, k+1)b_i(k)g_i(|x_i(k)|),$$

where

$$a(n)=1+|x(n_0)|+r(n_0)|\Delta x(n_0)|R(n, n_0)+\sum_{k=n_0}^{n-1} b_{m+1}(k)R(n, k+1).$$

By our Lemma, we obtain from above inequality

$$|x(n)|\leqq a(n)\prod_{i=1}^{m} E_i(n), \quad n\in N(n_0). \tag{8}$$

Where

$$E_i(n)=G_i^{-1}\Big(G_i(1)+\sum_{s=n_0}^{n-1} b_i(s)R(n, s+1)\prod_{j=1}^{i-1} E_j(s)\Big), \quad i=1, 2, \cdots, m.$$

G_i, G_i^{-1} are defined as in Lemma. By the conditions (i) and (ii) we have $a(n)$ and $E_i(n)$ are bounded on $N(n_0)$. From (8) we have $x(n)$ is bounded on $N(n_0)$. The proof is complete.

COROLLARY 1. *In addition to (H), suppose that the following conditions are satisfied:*

(i) *there exists a constant $K>0$ such that $r(n)\geqq K$ for $n\in N(n_0)$;*
(ii) $\lim_{n\to\infty} R(n, n_0)=A<\infty$;
(iii) $\sum_{k=n_0}^{\infty} b_i(k)<\infty$, $i=1, 2, \cdots, m+1$.

Then every solution $x(n)$ of (7) is bounded on $N(n_0)$.

Proof. It is easily seen that the conditions (ii) and (iii) in Corollary 1 imply the condition (i) in Theorem 1.

Example 1. Consider the equation

$$\Delta(e^{cn}\Delta x)+\sum_{i=1}^{m} q_i(n)x^{p_i}=0, \quad n\in N(n_0). \tag{9}$$

Here $c>0$ and $p_i\in(0, 1]$ are constants, and $q_i(n)$ are real polynomials. Clearly, all of the conditions in Theorem 1 are satisfied for (9). Thus all solutions $x(n)$ of (9) are bounded on $N(n_0)$.

The following result is concerned with the case (b).

THEOREM 2. *Suppose the hypotheses (H) hold, suppose moreover that*

(i) $\sum_{k=n_0}^{\infty} b_{m+1}(k)<\infty$, $\sum_{k=n_0}^{\infty} b_i(k)R(k, n_0)<\infty$, $(i=1, 2, \cdots, m)$;
(ii) $\lim_{n\to\infty} R(n, n_0)=\infty$.

Then for every solution $x(n)$ of (7) satisfies

$$|x(n)|=O(R(n, n_0)), \quad |r(n)\Delta x(n)|=O(1), \quad \text{as } n\to\infty.$$

Proof. Let $x(n)$ be a solution of (7). From (7), by successive summations, we obtain

(10) $$r(n)\Delta x(n)=r(n_0)\Delta x(n_0)-\sum_{k=n_0}^{n-1} f(k,\ x(k)).$$

and

$$x(n)=x(n_0)+r(n_0)\Delta x(n_0)R(n,\ n_0)-\sum_{k=n_0}^{n-1}\frac{1}{r(k)}\sum_{i=n_0}^{k-1} f(i,\ x(i)),\quad n\in N(n_0).$$

And, so

$$|x(n)|\leqq C_1R(n,\ n_0)+\sum_{j=1}^{m}\sum_{k=n_0}^{n-1}\frac{1}{r(k)}\sum_{i=n_0}^{k-1} b_j(i)g_j(|x(i)|)+\sum_{k=n_0}^{n-1}\frac{1}{r(k)}\sum_{i=n_0}^{k-1} b_{m+1}(i).$$

For some constant $C_1>0$ and all $n\geqq n_1\in N(n_0)$. By the first condition in (i), there exists constant $C_2>0$ such that

$$\sum_{k=n_0}^{n-1}\frac{1}{r(k)}\sum_{i=n_0}^{k-1} b_{m+1}(i)\leqq C_2R(n,\ n_0).$$

Since $\lim_{n\to\infty}R(n,\ n_0)=\infty$, $R(n,\ n_0)\geqq 1$, for $n\geqq n_1$, $R(n,\ n_0)\geqq R(n,\ s)$, $s\geqq n_0$ we obtain

$$|x(n)|\leqq CR(n,\ n_0)+R(n,\ n_0)\sum_{j=1}^{m}\sum_{k=n_0}^{n-1} b_j(k)g_j(|x(k)|).$$

For some constant $C\geqq 1$, and so

(11) $$\frac{|x(n)|}{R(n,\ n_0)}\leqq C+\sum_{j=1}^{m}\Bigl(\sum_{k=n_0}^{n-1} b_j(k)R(k,\ n_0)g_j\Bigl(\frac{|x(k)|}{R(k,\ n_0)}\Bigr)\Bigr).$$

Applying the Lemma to (11), we have

(12) $$\frac{|x(n)|}{R(n,\ n_0)}\leqq C\prod_{i=1}^{m} V_i(n),\quad n\geqq n_1.$$

Where

$$V_i(n)=G_i^{-1}\Bigl(G_i(1)+\sum_{s=n_0}^{n-1} b_i(s)R(s,\ n_0)\Bigl(\prod_{j=1}^{i-1} V_j(s)\Bigr)\Bigr),\quad i=1,\ 2,\ \cdots,\ m.$$

Now, letting $n\to\infty$ in (12) and in view of the conditions (i) and (ii), then we obtain the desired relation $|x(n)|=O(R(n,\ n_0))$. Finaly, by (12) we derive from (10)

$$r(n)|\Delta x(n)|\leqq r(n_0)|\Delta x(n_0)|+\sum_{k=n_0}^{n-1} b_{m+1}(k)+\sum_{i=1}^{m} g_i(M)\Bigl(\sum_{k=n_0}^{n-1} b_i(k)R(k,\ n_0)\Bigr)<\infty,$$

where the number $M>0$ is an upper bound for $C\prod_{i=1}^{m} V_i(n)$ on $N(n_0)$. This complete the proof.

COROLLARY 2. *We consider the equation*

(13) $$x(n+2)-2x(n+1)+x(n)+f(n,\ x(n))=0,\quad n\in N(n_0).$$

If the hypotheses (H) *and the following conditions are satisfied*:

$$\sum_{k=n_0}^{\infty} b_{m+1}(k)<\infty, \quad \sum_{k=n_0}^{\infty} b_i(k)k<\infty, \quad i=1, 2, \cdots, m.$$

Then all solutions $x(n)$ *of* (13) *satisfy the relations*: $|x(n)|=O(n)$ *and* $|x(n+1)-x(n)|=O(1)$ *as* $n\to\infty$.

Remark. Theorem 2 generalizes Theorem 8 in [1].

Example 2. Consider the following equation

$$x(n+2)-2x(n+1)+x(n)+\sum_{i=1}^{m} c_i(n)x^{p_i}(n)=0, \quad n\in N(n_0). \tag{14}$$

Where p_i are numbers from (0, 1], and $c_i(n)$ $(i=1, 2, \cdots, m)$ are real sequences. If

$$\sum_{k=n_0}^{\infty} |c_i(k)|k<\infty, \quad i=1, 2, \cdots, m.$$

Then by Corollary 2, all of the solutions of the equation (14) obey the relations: $|x(n)|=O(n)$ and $|x(n+1)-x(n)|=O(1)$, as $n\to\infty$.

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