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KODAI MATH. J.
19 (1996), 117—136

ORTHOGONAL DECOMPOSITION RELATED TO MAGNETIC FIELD, AND GRUNSKY INEQUALITY

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1. Introduction

Let D be a bounded domain in $\boldsymbol{R}^3$ with C^ω smooth boundary surfaces Σ. Let $\sigma = adx + bdy + cdz$ be a C^∞ closed 1-form on $\bar{D}$ $(=D\cup\Sigma)$. By putting $\tilde{\sigma}=\sigma$ in $\bar{D}$ and $=0$ outside D, we consider the usual Weyl's orthogonal decomposition: $\tilde{\sigma} = *\omega + dF$ in $\boldsymbol{R}^3$, where ω is a L^2 closed 2-form in $\boldsymbol{R}^3$ and $dF \in \mathrm{Cl}[dC_0^\infty(\boldsymbol{R}^3)]$.

In §4 we shall show that ω is a harmonic 2-form in $\boldsymbol{R}^3\setminus\Sigma$ of the form $\omega = dp$ and that p and F are written into the following integral formulas:

$$p(x) = \Big(\frac{1}{4\pi}\int_\Sigma \frac{(a,b,c)\times \boldsymbol{n}_y}{\|x-y\|}dS_y\Big)\cdot dx \qquad \text{for } x\in\boldsymbol{R}^3,$$

$$F(x) = \frac{1}{4\pi}\int_\Sigma \frac{(a,b,c)\cdot \boldsymbol{n}_y}{\|x-y\|}dS_y - \frac{1}{4\pi}\int_D \frac{\operatorname{div}(a,b,c)}{\|x-y\|}dv_y \qquad \text{for } x\in\boldsymbol{R}^3,$$

where $\boldsymbol{n}_y$ is the unit outer normal vector of Σ at y, $dx=(dx, dy, dz)$, and $\cdot$ means the formal inner product.

In §2 we briefly recall the definition of surface current densities on Σ and their properties studied in [6]. In §3 we shall prove an approximation lemma concerning improper integrals. This lemma is not only useful to prove the above integral formulas but also to show the fact that ω is related to the magnetic field. Precisely, if we write $\omega = \alpha dy\wedge dz + \beta dz\wedge dx + \gamma dx\wedge dy$ and define $B=(\alpha, \beta, \gamma)$ in $\boldsymbol{R}^3\setminus\Sigma$, then B is a magnetic field induced by a surface current density JdS_x on Σ such that B is the strong limit of a sequence of usual magnetic fields $\{B_n\}_n$ in $\boldsymbol{R}^3$: $\lim_{n\to\infty}\int_{\boldsymbol{R}^3}\|B_n(x)-B(x)\|^2 dv_x = 0$. In §5 we shall show that this fact implies the existence of equilibrium current densities $\mathcal{J}dS_x$ on Σ. The notion of equilibrium current densities were introduced in [6] motivated by the electric solenoid.

In §6 the integral formulas in $\boldsymbol{R}^3$ stated above is extended into those in the complex z-plane. We then obtain a new proof of Grunsky inequality (cf. [4]), which implies a necessary and sufficient condition for the case when the inequality is reduced to equality. It gives us many examples of such cases.

Received October 25, 1994; revised May 8, 1995.

The main result (Theorem 4.1) in this paper is motivated by the elementary part of Okabe's fluctuation and dissipation principle in [3]. The author thanks Professors Y. Okabe and Y. Nakano for their conversation. He also appreciates the referee for his kind comments.

2. Surface current density

We shall use the notation: $x=(x, y, z)=(x_1, x_2, x_3)\in \boldsymbol{R}^3$. Let $J=(f_1, f_2, f_3)$ be a C^∞ vector field in $\boldsymbol{R}^3$ with compact support. If $\operatorname{div} J(x)=\sum_{i=1}^3 \partial f_i/\partial x_i=0$, then Jdv_x, where dv_x is a volume element of $\boldsymbol{R}^3$, is called *a volume current density in* $\boldsymbol{R}^3$. Let γ be a 1-cycle in $\boldsymbol{R}^3$. By taking a 2-chain Q in $\boldsymbol{R}^3$ such that $\partial Q=\gamma$, we set $J[\gamma]=\int_Q J(x)\cdot \boldsymbol{n}_x dS_x$, where $\boldsymbol{n}_x$ denotes the unit outer normal vector of Q at x. We call $J[\gamma]$ *the total current of* Jdv_x *through* $[\gamma]$. We consider the vector valued-integrals:

$$A(x)=\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{J(y)}{\|x-y\|}dv_y \qquad \text{for } x\in \boldsymbol{R}^3 \tag{2.1}$$

$$B(x)=\operatorname{rot} A(x)=\frac{1}{4\pi}\int_{\boldsymbol{R}^3} J(y)\times\frac{x-y}{\|x-y\|^3}dv_y \qquad \text{for } x\in \boldsymbol{R}^3. \tag{2.2}$$

Following Biot-Savart we call $A(x)$ *the vector potential for* Jdv_x, and $B(x)$ *the magnetic field induced by* Jdv_x.

Let $D\Subset \boldsymbol{R}^3$ be a domain bounded by C^ω smooth surfaces Σ. We denote by dS_x the surface area element of Σ, and put $D'=\boldsymbol{R}^3\setminus \bar{D}$. Let $J=(f_1, f_2, f_3)$ be a C^∞ vector field on Σ. If there exists a sequence of volume current densities $\{J_n dv_x\}_n$ in $\boldsymbol{R}^3$ which converges to JdS_x on Σ in the sense of distribution, then JdS_x is called *a surface current density on* Σ. Precisely speaking, $\{\operatorname{Supp} J_n\}_n$ is uniformly bounded and $\lim_{n\to\infty}\int_{\boldsymbol{R}^3}\phi J_n dv_x=\int_\Sigma \phi J dS_x$ for $\forall \phi\in C_0^\infty(\boldsymbol{R}^3)$. For a 1-cycle γ in $\boldsymbol{R}^3\setminus\Sigma$, we set $J[\gamma]=\lim_{n\to\infty}J_n[\gamma]$, which is called *the total current of* JdS_x *through* $[\gamma]$. We consider

$$A(x)=\frac{1}{4\pi}\int_\Sigma\frac{J(y)}{\|x-y\|}dS_y \qquad \text{for } x\in \boldsymbol{R}^3$$

$$B(x)=\operatorname{rot} A(x)=\frac{1}{4\pi}\int_\Sigma J(y)\times\frac{x-y}{\|x-y\|^3}dS_y \qquad \text{for } x\in \boldsymbol{R}^3\setminus\Sigma.$$

We say that $A(x)$ is *the vector potential for* JdS_x, and $B(x)$ *the magnetic field induced by* JdS_x.

We summarize some results in [7] which we use in this note:

PROPOSITION 2.1. *Let* $J=(f_1, f_2, f_3)$ *be a* C^∞ *vector field on* Σ *and let* $\eta=f_1dx+f_2dy+f_3dz$ *on* Σ. *We put* $\boldsymbol{n}_x\times J(x)=(g_1, g_2, g_3)$ *for* $x\in\Sigma$, *and* $\star\eta=g_1dx+g_2dy+g_3dz$ *on* Σ *(which is called the conjugate* 1*-form of* η *on* Σ*). Then* JdS_x *is a surface current density on* Σ, *if and only if* J *is tangential on*

Σ and $\star\eta$ is a closed 1-form on Σ.

When we regard Σ as a Riemann surface with conformal structure induced by the euclidean metric of $\boldsymbol{R}^3$, the above condition says that η is a co-closed differential on Σ, namely, $\star\eta$ is the conjugate differential of η on Σ such that $d\star\eta=0$ on Σ (which is inherited from condition $\operatorname{div} J_n=0$ $(n=1, 2, \cdots)$ in $\boldsymbol{R}^3$ that $J_n dv_x$ is a volume current density in $\boldsymbol{R}^3$).

PROPOSITION 2.2. *Let $JdS_x=(f_1, f_2, f_3)dS_x$ be a surface current density on Σ and, $B(x)=(\alpha, \beta, \gamma)$ the magnetic field in $\boldsymbol{R}^3\backslash\Sigma$ induced by JdS_x. We put $\eta=f_1dx+f_2dy+f_3dz$ on Σ and $\boldsymbol{\omega}=\alpha dy\wedge dz+\beta dz\wedge dx+\gamma dx\wedge dy$ in $\boldsymbol{R}^3\backslash\Sigma$. Then we have*

(1) *$\boldsymbol{\omega}$ is a harmonic 2-form in $\boldsymbol{R}^3\backslash\Sigma$ such that $\boldsymbol{\omega}(x)=O(1/\|x\|^2)$ at $x=\infty$.*

(2) *We simply write $D^+=D$ and $D^-=D'$. If we put $B(x)=B^\pm(x)$ for $x\in D^\pm$, then $B^\pm(x)$ are continuous up to Σ from $D^\pm$, respectively, and has the following gap: $B^+(x)-B^-(x)=\boldsymbol{n}_x\times J(x)$ for $x\in\Sigma$. In other words, if we put $\boldsymbol{\omega}(x)=\boldsymbol{\omega}^\pm(x)$ for $x\in D^\pm$, then $\boldsymbol{\omega}^\pm(x)$ are continuous up to Σ from $D^\pm$, respectively, in such a way that $*\boldsymbol{\omega}^+(x)-*\boldsymbol{\omega}^-(x)=\star\eta(x)$ on Σ.*

(3) *For a 1-cycle $\gamma\subset D\cup D'$, we have $J[\gamma]=\int_\gamma *\boldsymbol{\omega}=\int_{\gamma'}\star\eta$, where $\gamma'=Q\cap\Sigma$ and Q is a 2-chain in $\boldsymbol{R}^3$ such that $\partial Q=\gamma$.*

Given $x\in\boldsymbol{R}^2$ sufficiently close to Σ, we find a unique point $\xi=\xi(x)\in\Sigma$ such that

$$x-\xi=R(x)\boldsymbol{n}_\xi \quad \text{where } R(x)\in\boldsymbol{R}, \tag{2.3}$$

where $\boldsymbol{n}_\xi$ is the unit outer normal vector of Σ at ξ. Then $R(x)$ becomes a C^ω function in a neighborhood U of Σ in $\boldsymbol{R}^3$ such that $\boldsymbol{n}_x=\nabla R(x)=(\partial R/\partial x_1, \partial R/\partial x_2, \partial R/\partial x_3)$ on Σ and

$$U\cap D \text{ (resp. } \Sigma,\ U\cap D')=\{x\in U\,|\,R(x)<(\text{resp.} =, >)0\}, \tag{2.4}$$

For a given $\delta>0$ we set $U(\delta):=\{x\in U\,|\,-\delta<R(x)<\delta\}$. We fix an integer n_0 such that $U(1/n_0)\Subset U$, and put $\Gamma_n:=\{x\in U\,|\,-1/n\leqq R(x)\leqq -1/2n\}$ for $n\geqq n_0$. We take a sequence of C^∞ functions $\{\chi_n(R)\}_{n\geqq 1}$ on $(-\infty, \infty)$ such that

$$\begin{aligned} &0\leqq\chi_n(R)\leqq 1 \qquad \chi_n(R)=\begin{cases} 1 & \text{on } (-\infty, -1/n] \\ 0 & \text{on } [-1/2n, +\infty), \end{cases} \\ &0\leqq|\chi_n'(R)|\leqq nM, \quad |\chi''(R)|\leqq n^2M, \end{aligned} \tag{2.5}$$

where $M>0$ is a constant independent of n $(\geqq 1)$ and $R\in(-\infty, \infty)$. For $n\geqq n_0$, we can consider a function $\tilde{\chi}_n(x)$ in $\boldsymbol{R}^3$ defined by

$$\tilde{\chi}_n(x)=\begin{cases} 1 & \text{in } D\backslash U \\ \chi_n(R(x)) & \text{in } U \\ 0 & \text{in } D'\backslash U. \end{cases} \tag{2.6}$$

Thus, $\tilde{\chi}_n(x)\in C_0^\infty(\boldsymbol{R}^3)$. The functions $\chi_n'(R(x))$ and $\chi_n''(R(x))$ are of class C^∞ in U with support in Γ_n ($\Subset U$), so we extend them to $\boldsymbol{R}^3$ by putting 0 in $\boldsymbol{R}^3\backslash U$.

PROPOSITION 2.3. *Let $f\in C_0^\infty(R^3)$. Then we have*

(1) *$\chi_n'(R(x))f(x)dv_x\to -f(x)dS_x$ on Σ in the sense of distribution.*
(2) *$\{\chi_n''(R(x))f(x)dv_x\}_{n\geq n_0}$ is convergent on Σ in the sense of distribution, if and only if $f(x)=0$ on Σ. In this case, the limit is $(\partial f/\partial n_x)dS_x$ on Σ.*

Assertion (2) followed from the fact that, for $\forall\phi\in C_0^\infty$,

$$\lim_{n\to\infty}\int_{R^3}\chi_n''(R(x))\phi(x)dv_x=\int_\Sigma\left\{\frac{\partial\phi}{\partial n_x}+\phi H\right\}dS_x, \tag{2.7}$$

where $H(x)$ denotes the mean curvature of Σ at x (cf. Lemma 1.1 in [7]).

Now let D be a domain in $\boldsymbol{R}^3$ (which may be $\boldsymbol{R}^3$ itself). For $i=1, 2$ we consider the space $L_i^2(D)$ of all L^2 i-forms in D and their subspace:

$C_{i,0}^\infty(D)=$the set of C^∞ i-forms with compact support in D,

$Z_i^\infty(\bar{D})=$the set of all C^∞ closed i-forms on $\bar{D}$,

$B_i(D)=\mathrm{Cl}[dC_{i-1,0}^\infty(D)]$, $Z_i(D)=\mathrm{Cl}[Z_i^\infty(\bar{D})]$,

$H_i(D)=$the set of all L^2 harmonic i-forms in D.

Then Weyl's orthogonal decomposition theorems hold:

$$L_i^2(D)=*Z_{3-i}(D)\dotplus B_i(D), \quad Z_i(D)=H_i(D)\dotplus B_i(D).$$

In case D is a bounded domain in $\boldsymbol{R}^3$ with C^ω smooth boundary surfaces Σ, we define

$$H_{20}(D)=\{\omega\in H_2(D)\,|\,\omega \text{ is of class } C^\omega \text{ up to } \Sigma, \text{ and } \omega=0 \text{ along } \Sigma\},$$

where $\omega=0$ *along* Σ means that the normal component of ω vanishes on Σ. As an analogue to Ahlfors' theorem in [1], we have

PROPOSITION 2.4. *Let $\{\gamma_j\}_{j=1,\dots,q}$ be a 1-dimensional homology base of D. Then, for each i ($1\leqq i\leqq q$), there exists a unique $\omega_i\in H_{20}(D)$ such that $\int_{\gamma_j}*\omega_i=\delta_{ij}$ ($1\leqq\forall j\leqq q$).*

3. Approximation lemma

We shall show the following approximation

LEMMA 3.1. *Let $g(x)\in C^{\infty}(U)$ be given. For $n\geqq n_0$, we consider the C^{∞} functions $I_{1,n}(x)$ and $I_{2,n}(x)$ in $\boldsymbol{R}^3$ defined by*

$$I_{1,n}(x)=\int_U \frac{\chi_n'(R(y))g(y)}{\|y-x\|}dv_y\,,\quad I_{2,n}(x)=\int_U \frac{\chi_n''(R(y))g(y)}{\|y-x\|}dv_y\,.$$

We put

$$I_1(x)=-\int_{\Sigma}\frac{g(y)}{\|y-x\|}dS_y \qquad \textit{for } x\in\boldsymbol{R}^3 \tag{3.1}$$

$$I_2(x)=\int_{\Sigma}\left\{\frac{\partial}{\partial n_y}\left(\frac{g(y)}{\|y-x\|}\right)+\frac{g(y)H(y)}{\|y-x\|}\right\}dS_y \quad \textit{for } x\in\boldsymbol{R}^3\backslash\Sigma. \tag{3.2}$$

Then we have

(1) $\lim_{x\to\infty} I_{1,n}(x)=I_1(x)$ *uniformly in* $\boldsymbol{R}^3$.

(2) $\lim_{x\to\infty} I_{2,n}(x)=I_2(x)$ *uniformly on any compact set in* $\boldsymbol{R}^3\backslash\Sigma$.

(3) *Both* $\{I_{1,n}(x)\}_{n\geqq n_0}$ *and* $\{I_{2,n}(x)\}_{n\geqq n_0}$ *are uniformly bounded in* $\boldsymbol{R}^3$.

Proof. It is clear that $I_1(x)$ and $I_2(x)$ are continuous in $\boldsymbol{R}^3$ and $\boldsymbol{R}^3\backslash\Sigma$, respectively, and that $I_2(x)$ has the gap $4\pi g(x)$ for $x\in\Sigma$. (Thus the convergence of (2) is not uniform in U in general.) Since $\operatorname{Supp}\chi_n'(R(x))\to\Sigma$ $(n\to\infty)$, we see from (1) of Proposition 2.3 that $\lim_{n\to\infty}I_{1,n}(x)=I(x)$ pointwise in $\boldsymbol{R}^3\backslash\Sigma$. For each $n\geqq n_0$, the function $I_{1,n}(x)$ is of class C^{∞} in $\boldsymbol{R}^3$ such that, for $x\in\boldsymbol{R}^3$ and $i=1, 2, 3$,

$$\frac{\partial I_{1,n}}{\partial x_i}(x)=\int_U\frac{\chi_n''(R(y))\dfrac{\partial R}{\partial y_i}(y)g(y)}{\|y-x\|}dv_y+\int_U\frac{\chi_n'(R(y))\dfrac{\partial g(y)}{\partial y_i}}{\|y-x\|}dv_y\,.$$

Therefore, if (3) is true, then the family $\{(\partial I_{1,n}/\partial x_i)(x)\}_{n\geqq n_0}$ is uniformly bounded in $\boldsymbol{R}^3$. Hence, the family $\{I_{1,n}(x)\}_{n\geqq n_0}$ is bounded and equicontinuous on *any* compact set K in $\boldsymbol{R}^3$. It follows from Ascoli-Arzelà's theorem that the sequence $\{I_{1,n}(x)\}_{n\geqq n_0}$ uniformly converges to a function $g_1(x)$ on K. As K, we take a large closed ball $\bar{B}_0$ such that $B_0\supset U$. Since $I_{1,n}(x)$ is harmonic in $\boldsymbol{R}^3\backslash\Gamma_n$, it follows from the expression of $I_{1,n}(x)$ that there exists an $A_1>0$ such that $|I_{1,n}(x)|\leqq A_1/\|x\|$ for $\forall n\geqq n_0$ and $\forall x\in\boldsymbol{R}^3\backslash B_0$. Hence, $\{I_{1,n}(x)\}_{n\geqq n_0}$ uniformly converges to a function $g_1(x)$ in $\boldsymbol{R}^3$. Since $I_1(x)=g_1(x)$ in $\boldsymbol{R}^3\backslash\Sigma$ and since $I_1(x)$ and $g_1(x)$ are continuous in $\boldsymbol{R}^3$, we have (1). Following the proof of (2.7), we obtain (2). It rests to prove (3) for $k=1, 2$. The proof for $k=1$ is easy as follows: By simple calculation we find a constant $c>0$ such that

$$\int_{\Gamma_n}\frac{1}{\|x-y\|}dv_y\leqq\frac{c}{n} \quad \text{for } \forall x\in\boldsymbol{R}^3 \text{ and } \forall n\geqq n_0\,.$$

We put $M_1 := \sup\{|g(y)| \mid y \in U(1/n_0)\} < +\infty$. Since $|\chi_n'(R)| \leqq nM$ on $(-\infty, +\infty)$ by (2.5), it follows that $|I_{1,n}(x)| \leqq cMM_1$ for $\forall x \in \boldsymbol{R}^3$ and $\forall n \geqq n_0$. Thus, the case $k=1$ is proved. The proof for $k=2$ is rather delicate. The proof will be done by use of Morse's theorem concerning regular singular point as follows:

In this proof we take and fix $0<\delta^*<1/n_0$, so that $\Sigma \subset U(\delta^*) \Subset U(1/n_0)$. We simply put $I^*=(-\delta^*, +\delta^*)$. Each $I_{2,n}(x)$, $n \geqq n_0$ is a C^∞ function in $\boldsymbol{R}^3$ and harmonic in $\boldsymbol{R}^3 \backslash \Gamma_n$. By the expression of $I_{2,n}(x)$ and (2), we find a constant $A_2>0$ (independent of $n \geqq n_0$) such that $|I_{2,n}(x)| \leqq A_2/\|x\|$ outside a ball $B_0 \supset \bar{D}$. It follows from (2) that $\{I_{2,n}(x)\}_{n \geqq n_0}$ is uniformly bounded in $\boldsymbol{R}^3 \backslash U(\delta^*)$. Therefore, it suffices to prove the following

CLAIM. *There exist an integer n_1 $(\geqq n_0)$ and a constant $C>0$ such that*

$$|I_{2,n}(p+R^*\boldsymbol{n}_p)| \leqq C \quad \text{for } \forall (p, R^*) \in \Sigma \times I^* \text{ and } \forall n \geqq n_1.$$

1st *step*. Let $p \in \Sigma$ be given arbitrarily. By a euclidean motion, we may assume that p is the origin O in the (x, y, z)-space and the unit outer normal vector $\boldsymbol{n}_p$ is equal to $(0, 0, 1)$. We identify p with O in this proof. The tangent plane of Σ at O is thus

$$\zeta=\phi(\xi, \eta)=a\xi^2+2b\xi\eta+c\eta^2+\{\text{higher order terms of } \xi \text{ and } \eta\},$$

where the Taylor series $\{\ \}$ uniformly converges in a disk $D_1=\{\xi^2+\eta^2<\rho_1\}$ (for future use, we prefer notation (ξ, η, ζ) to (x, y, z)). We consider the following transformation $\mathcal{S}: (\xi, \eta, R) \mapsto y=(x, y, z)$ from a neighborhood W_1 of the origin $(0, 0, 0)$ in the (ξ, η, R)-space onto a neighborhood V_1 of the origin O in the (x, y, z)-space of the form

$$\mathcal{S}: y=(\xi, \eta, \phi(\xi, \eta))+R\boldsymbol{n}_\xi, \tag{3.3}$$

where $\boldsymbol{n}_\xi$ denotes the unit outer normal vector of Σ at $(\xi, \eta, \phi(\xi, \eta))$. So, R is equal to $R(y)$ defined by (2.3). Then we have, for $y \in V_1$ and $R^* \in I^*$,

$$\begin{aligned} l(y, R^*) &:= \|y-(p+R^*\boldsymbol{n}_p)\|^2 \\ &= \|(\xi, \eta, \phi(\xi, \eta))+R\boldsymbol{n}_\xi-(0, 0, R^*)\|^2 \\ &= (\xi-K_{\xi,\eta}\phi_\xi R)^2+(\eta-K_{\xi,\eta}\phi_\eta R)^2+(\phi(\xi, \eta)+K_{\xi,\eta}R-R^*)^2, \end{aligned} \tag{3.4}$$

where $K_{\xi,\eta}=1/\sqrt{1+\phi_\xi^2+\phi_\eta^2}$. It follows that for any (ξ, η) sufficiently close to $(0, 0)$, say $(\xi, \eta) \in D_1'=\{\xi^2+\eta^2<\rho_1'\}$ where $0<\rho_1'<\rho_1$, we have

$$\begin{aligned} l(y, R^*) = (R-R^*)^2+(1+AR)\xi^2+2BR\xi\eta+(1+CR)\eta^2 \\ +\{\text{higher order terms of } \xi \text{ and } \eta\}, \end{aligned}$$

where A, B, C are C^ω functions of ξ, η, R, R^*. We thus find an interval $I_1 := (-\delta_1, +\delta_1)$ such that

$$(1+AR)(1+CR)>|BR|^2+1/2 \quad \text{for } \forall(\xi, \eta)\in D_1', \ \forall R\in I_1, \text{ and } \forall R^*\in I^*.$$

First, regarding R and R^* as parameters, we apply Morse's theorem to obtain a C^2 transformation $\mathcal{M}_{R,R^*}$ from a neighborhood $\mathcal{D}_1'(R, R^*)$ of $(0, 0)$ in the (X, Y)-plane onto a neighborhood $D_1'(R, R^*)$ $(\subset D_1')$ of $(0, 0)$ in the (ξ, η)-plane such that

$$\begin{aligned} &\mathcal{M}_{R,R^*}:(X, Y)\mapsto(\xi, \eta)=(f(X, Y, R, R^*), g(X, Y, R, R^*)) \\ &l(y, R^*)=(R-R^*)^2+X^2+Y^2. \end{aligned} \tag{3.5}$$

By the construction of $\mathcal{M}_{R,R^*}$ under the form (3.4) of $l(y, R^*)$, the functions f and g may be chosen to be of class C^2 for $(R, R^*)\in I_1\times I^*$. By smoothness we can take a common neighborhood $\mathcal{D}_2\subset\mathcal{D}_1'(R, R^*)$ of $(0, 0)$ in the (X, Y)-plane for $\forall(R, R^*)\in I_1\times I^*$, so that

(3.6) $\quad f(X, Y, R, R^*)$ and $g(X, Y, R, R^*)$ are of class C^2 in $\mathcal{D}_2\times(I_1\times I^*)$.

Next, regarding $R^*\in I^*$ as parameter, we put $\mathcal{M}:(X, Y, R)\mapsto(\xi, \eta, R)=(f, g, R)$, and consider the C^2 transformation $\mathcal{T}:=\mathcal{S}\circ\mathcal{M}$ from a product neighborhood $\mathcal{V}_2:=\mathcal{D}_2\times I_1$ of the origin O in the (X, Y, R)-space onto a neighborhood V_2 $(\subset V_1)$ of the origin O in the (x, y, z)-space. We write

$$\begin{aligned} &\mathcal{T}:(X, Y, R)\in\mathcal{V}_2\mapsto y \\ &=(F(X, Y, R, R^*), G(X, Y, R, R^*), H(X, Y, R, R^*))\in V_2. \end{aligned}$$

By differentiability of (3.6) we can find an $L>1$ such that

$$\begin{aligned} &\text{Modules of } \left\{\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \cdots, \frac{\partial^2 H}{\partial Y\partial R}, \frac{\partial^2 H}{\partial R^2}\right\}\leq L \\ &\frac{1}{L}\leq J_{\mathcal{T}}(X, Y, R)=\frac{\partial(x, y, z)}{\partial(X, Y, R)}\leq L \end{aligned} \tag{3.7}$$

for $\forall(X, Y, R)\in\mathcal{V}_2$ and $\forall R^*\in I^*$. Note that $\mathcal{T}$ depends on $(p, R^*)\in\Sigma\times I^*$, so do $\mathcal{V}_2$ and L. Thus, it should better to write $\mathcal{V}_2=\mathcal{V}_2(p, R^*)$ and $L=L(p, R^*)$. However, since the surface Σ is of C^ω smooth, we see from the construction of the mapping $\mathcal{T}$ that there exists a small common product neighborhood $\mathcal{V}_0\subset\mathcal{V}_2(p, R^*)$ centered at $(0, 0, 0)$ in the (X, Y, R)-space and a large common $L_0>L(p, R^*)>0$ such that (3.5) and (3.7) are satisfied for $\forall(X, Y, R)\in\mathcal{V}_0$ and $\forall(p, R^*)\in\Sigma\times I^*$. We write

$$\mathcal{V}_0=\mathcal{D}_0\times I_0 \quad \text{where } D_0=\{X^2+Y^2<\rho_0\} \text{ and } I_0=(-\delta_0, +\delta_0).$$

As an integer n_1 in the claim, we take an n_1 $(\geqq n_0)$ such that $\Gamma_n\subset U(\delta_0)$ for $\forall n\geqq n_1$. We put $O_{p,R^*}:=\mathcal{T}(\mathcal{V}_0)$, where $\mathcal{T}$ is constructed above depending on $(p, R^*)\in\Sigma\times I^*$. Thus, O_{p,R^*} is a neighborhood of (p, R^*) in the (x, y, z)-space. From (3.7), we find a small common disk $E_\tau:=\{\xi^2+\eta^2<\tau^2\}$, where $\tau>0$, in the (ξ, η)-plane such that

(3.8) $$\mathcal{S}(E_\tau\times I_0)\subset O_{p,R^*} \quad \text{for } \forall(p, R^*)\in\Sigma\times I^*,$$

where $\mathcal{S}$ is defined by (3.3) depending on (p, R^*).

2^{nd} *step*. Let $(p, R^*)\in\Sigma\times I^*$ and $n\geqq n_1$ be given arbitrarily. We set

(3.9) $$\begin{aligned} I_{2,n}(p+R^*\boldsymbol{n}_p) &= \left\{\int_{O_{p,R^*}}+\int_{U\setminus O_{p,R^*}}\right\}\frac{\chi_n''(R(y))g(y)}{\|y-(p+R^*\boldsymbol{n}_p)\|}dv_y \\ &\equiv S_n(p, R^*)+T_n(p, R^*). \end{aligned}$$

We first show the uniform boundedness of the second terms $\{T_n(p, R^*)\}_{n\geqq n_1}$ in $\Sigma\times I^*$. For $R\in I_0$, we consider the level surface: $\Sigma(R)=\{y\in U\,|\,R(y)=R\}$ in the (x, y, z)-space, where $R(y)$ is defined by (2.3). For $y\in\Sigma(R)$ and $R\in I_0$, we set $dv_y=j(y)dS_y dR$, where dS_y denotes the surface area element of $\Sigma(R)$ at y. Thus, $j(y)$ becomes a C^ω function in $U(\delta_0)$ such that $j(y)=1$ on Σ. We put, for $\forall R\in I_0$,

$$F_{p,R^*}(R):=\int_{\Sigma(R)\setminus O_{p,R^*}}\frac{g(y)j(y)}{\|y-(p+R^*\boldsymbol{n}_p)\|}dS_y\,.$$

By (3.8) we have $\|y-(p+R^*\boldsymbol{n}_p)\|>\tau$ for $\forall y\in\Sigma(R)\setminus O_{p,R^*}$ and $\forall(p, R^*)\in\Sigma\times I^*$. Hence, the integrand is a bounded C^∞ function for $y\in\Sigma(R)\setminus O_{p,R^*}$ such that its boundedness is uniform for $(R, p, R^*)\in I_0\times\Sigma\times I^*$. Further, since $\Sigma(R)\setminus O_{p,R^*}$ varies C^2 smoothly with respect to $(R, p, R^*)\in I_0\times\Sigma\times I^*$, it follows that $F_{p,R^*}(R)$ varies smoothly with these variables. We thus find an $M_2>0$ such that

$$\left|\frac{\partial F_{p,R^*}(R)}{\partial R}\right|\leqq M_2 \quad \text{for } \forall(R, p, R^*)\in I_0\times\Sigma\times I^*.$$

Note that $\chi_n'(-1/n)=\chi_n'(-1/2n)=0$ and $\operatorname{Supp}\chi_n''\subset[-1/n, -1/(2n)]$. By the integration by parts, we have

$$T_n(p, R^*)=\int_{-1/n}^{-1/2n}\chi_n''(R)F_{p,R^*}(R)dR=-\int_{-1/n}^{-1/2n}\chi_n'(R)\frac{\partial F_{p,R^*}(R)}{\partial R}dR\,.$$

Since $|\chi_n'(R)|\leqq nM$ by (2.5), it follows that

$$|T_n(p, R^*)|\leqq C_1:=MM_2/2 \quad \text{for } \forall(p, R^*)\in\Sigma\times I^* \text{ and } \forall n\geqq n_1.$$

We next show the uniform boundedness of the first terms $\{S_n(p, R^*)\}_{n\geqq n_1}$ in $\Sigma\times I^*$. By the change of variables from $y=(x, y, z)$ to (X, Y, R) by $\mathcal{T}$ (depending on (p, R^*)), we have

$$S_n(p, R^*)=\int_{\mathcal{D}_0\times I_0}\frac{\chi_n''(R)\tilde{g}(X, Y, R)}{\sqrt{(R-R^*)^2+X^2+Y^2}}J_{\mathcal{T}}(X, Y, R)dXdYdR,$$

where $\tilde{g}=g\#\mathcal{T}$. We use the polar coordinates $(X, Y)=(r\cos\theta, r\sin\theta)$ in $\mathcal{D}_0$ and put $\tilde{G}(r, \theta, R):=\tilde{g}(X, Y, R)J_{\mathcal{T}}(X, Y, R)$. Note that $\tilde{G}$ depends on (p, R^*) $\in\Sigma\times I^*$. By (3.7) we find an $L_1>0$ such that

$$\text{Modules of } \left\{\tilde{G}(r,\ \theta,\ R),\ \frac{\partial \tilde{G}}{\partial r},\ \cdots,\ \frac{\partial^2 \tilde{G}}{\partial \theta \partial R},\ \frac{\partial^2 \tilde{G}}{\partial R^2}\right\} \leqq L_1$$

for $\forall(r,\ \theta) \in [0,\ \rho_0] \times [0,\ 2\pi]$ and $\forall(R,\ p,\ R^*) \in I_0 \times \Sigma \times I^*$. Since $\chi'(-1/n) = \chi'(-1/2n) = 0$ and $\operatorname{Supp} \chi_n''(R) \subset [-1/n,\ -1/(2n)]$, we use the integration by parts for R to obtain

$$S_n(p,\ R^*) = \int_0^{2\pi}\int_0^{\rho_0} r\left\{\int_{-1/n}^{-1/2n} \chi_n''(R) \frac{\tilde{G}(r,\ \theta,\ R)}{\sqrt{(R-R^*)^2 + r^2}} dR\right\} dr d\theta$$

$$= \int_0^{2\pi}\int_0^{\rho_0} r\left\{-\int_{-1/n}^{-1/2n} \chi_n'(R) \frac{\partial}{\partial R}\left(\frac{\tilde{G}(r,\ \theta,\ R)}{\sqrt{(R-R^*)^2 + r^2}}\right) dR\right\} dr d\theta .$$

We conveniently put $Z = Z(r,\ R,\ R^*) = 1/\sqrt{(R-R^*)^2 + r^2}$. It follows from $r(\partial Z/\partial R) = (R-R^*)\partial Z/\partial r$ that

$$S_n(p,\ R^*) = -\int_0^{2\pi}\int_{-1/n}^{-1/2n}\int_0^{\rho_0} \left\{r\chi_n'(R)\left(\frac{\partial Z}{\partial R}\tilde{G} + Z\frac{\partial \tilde{G}}{\partial R}\right)\right\} dr dR d\theta$$

$$= -\int_0^{2\pi}\int_{-1/n}^{-1/2n}\int_0^{\rho_0} \left\{(R-R^*)\chi_n'(R)\frac{\partial Z}{\partial r}\tilde{G}\right\} dr dR d\theta$$

$$-\int_0^{2\pi}\int_{-1/n}^{-1/2n}\int_0^{\rho_0} \left\{rZ\chi_n'(R)\frac{\partial \tilde{G}}{\partial R}\right\} dr dR d\theta$$

$$\equiv S_n^{(1)}(p,\ R^*) + S_n^{(2)}(p,\ R^*) .$$

Since $|rZ| \leqq 1$ and $|\chi_n'(R)| \leqq nM$, we have $|S_n^{(2)}(p,\ R^*)| \leqq 2\pi(1/2n)(nM)L_1\rho_0 = \pi M L_1 \rho_0$ for $\forall(r,\ R^*) \in \Sigma \times I^*$ and $\forall n \geqq n_1$. Using the integration by parts for r in $S_n^{(1)}(p,\ R^*)$, we have from $|(R-R^*)Z| \leqq 1$ and $|\chi_n'(R)| \leqq nM$,

$$|S_n^{(1)}(p,\ R^*)| = \left|\int_0^{2\pi}\int_{-1/n}^{-1/2n} (R-R^*)\chi_n'(R)\left\{\int_0^{\rho_0} \frac{\partial Z}{\partial r}\tilde{G} dr\right\} dR d\theta\right|$$

$$= \left|\int_0^{2\pi}\int_{-1/n}^{-1/2n} (R-R^*)\chi_n'(R)\left\{[Z\tilde{G}]_0^{\rho_0} - \int_0^{\rho_0} Z\frac{\partial \tilde{G}}{\partial r} dr\right\} dR d\theta\right|$$

$$\leqq 2\pi\frac{1}{2n}(nM)(2L_1 + \rho_0 L_1) = \pi M L_1(2+\rho_0)$$

for $\forall(r,\ R^*) \in \Sigma \times I^*$ and $\forall n \geqq n_1$. Hence, $|S_n(p,\ R^*)| \leqq C_2 := \pi M L_1(3+\rho_0)$ in $\Sigma \times I^*$ for $\forall n \geqq n_1$. It follows from (3.9) that $|I_{2,n}(p + R^* \boldsymbol{n}_p)| \leqq C := C_1 + C_2$ in $\Sigma \times I^*$ for $\forall n \geqq n_1$. Our claim is thus proved. □

COROLLARY 3.1. *Let $J dS_x$ be a surface current density on Σ and denote by $A(x)$ and $B(x)$ its vector potential in $\boldsymbol{R}^3$ and its magnetic field in $\boldsymbol{R}^3 \backslash \Sigma$. Then there exists a sequence of volume current densities $\{J_n dv_x\}_n$ with the following properties: If we denote by $A_n(x)$ and $B_n(x)$ the vector potential and the magnetic field for $J_n dv_x$ respectiveiy, then it holds*

(1) *$\{A_n(x)\}_n$ converges $A(x)$ uniformly in $\boldsymbol{R}^3$.*
(2) *$\{B_n(x)\}_n$ converges $B(x)$ uniformly on any compact set in $\boldsymbol{R}^3 \backslash \Sigma$.*

(3) $\{B_n(x)\}_n$ *is uniformly bounded in* $\boldsymbol{R}^3$.

(4) $\lim_{n\to\infty}\int_{\boldsymbol{R}^3}\|B_n(x)-B(x)\|^2 dv_x=0$.

Proof. In Corollary 1.1 in [7] we constructed a sequence of volume current densities $\{J_n dv_x\}_n$ converging the given JdS_x on Σ in the sense of distribution such that their $\{A(x)\}_n$ and $\{B(x)\}_n$ converge $A(x)$ and $B(x)$ uniformly on any compact set in $\boldsymbol{R}^3\backslash\Sigma$. In that proof, $J_n dv_x=(f_{1n}, f_{2n}, f_{3n})dv_x$ was of the form

$$f_{1n}(x)=\chi_n'(R(x))\Big(\tilde{g}_3(x)\frac{\partial R(x)}{\partial x_2}-\tilde{g}_2(x)\frac{\partial R(x)}{\partial x_3}\Big) \quad \text{etc.,}$$

where $\tilde{g}_2(x)$ and $\tilde{g}_3(x)$ are C^∞ functions in U $(\supset\Gamma_n)$ and are independent of n $(\geqq n_0)$. We shall show this $\{J_n dv_x\}_n$ satisfies (1)~(4) of Corollary 3.1. In fact, (2) is already proved in [7]. Applying (1) of Lemma 3.1 to definition (2.1) of $A_n(x)$, we have (1). Since $B_n(x)=\operatorname{rot} A_n(x)$, we see that each component of $B_n(x)$ is of the form

$$\int_{\boldsymbol{R}^3}\frac{\chi_n'(R(y))h(y)+\chi_n''(\boldsymbol{R}(y))k(y)}{\|x-y\|}dv_y, \tag{3.10}$$

where $h(y)$ and $k(y)$ are functions of class C^∞ in U and independent of n $(\geqq n_0)$. Hence, (3) of Lemma 3.1 implies (3). From (2) and definition (2.2) of $B_n(x)$ we can find an $M_1>0$ such that $\|B_n(x)\|\leqq M_1/\|x\|^2$ outside a ball $B_0\supset\bar{D}$ for $\forall n\geqq n_0$. This together with (3) implies (4). □

4. Main theorem

Given a C^∞ 1-form $\sigma=\sum_{i=1}^3 f_i dx_i$ in a domain $U\subset\boldsymbol{R}^3$, we put $\|\sigma\|(x)=(\sum_{i=1}^3 f_i(x)^2)^{1/2}\geqq 0$, $\Delta\sigma=\sum_{i=1}^3(\Delta f_i)dx_i$, and $\delta=*d*$, where Δ is Laplacian and the operator $*$ is determined by $\sigma\wedge*\sigma=\|\sigma\|^2(x)dv_x$ in U. When $\sigma\in C_{1,0}^\infty(\boldsymbol{R}^3)$, we put

$$\mathfrak{N}\sigma(x) \text{ or } \frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{\sigma(y)}{\|x-y\|}dv_y:=\frac{1}{4\pi}\sum_{i=1}^3\Big(\int_{\boldsymbol{R}^3}\frac{f_i(y)}{\|x-y\|}dv_y\Big)dx_i.$$

This as well as $\Delta\sigma$ is a 1-form. We analogously define the corresponding ones for C^∞ i-form σ_i $(i=0, 1, 2, 3)$. By the symmetry of the Newton kernel $1/\|x-y\|$ with respect to x and y in $\boldsymbol{R}^3$, we easily obtain, for $\sigma_i\in C_{i,0}^\infty(\boldsymbol{R}^3)$,

$$d\mathfrak{N}\sigma_i=\mathfrak{N}d\sigma_i,\quad *\mathfrak{N}\sigma_i=\mathfrak{N}*\sigma_i,\quad \delta\mathfrak{N}\sigma_i=\mathfrak{N}\delta\sigma_i.$$

Further we have (see, for example, [5])

$$\Delta\sigma_i=(-1)^i(\delta d-d\delta)\sigma_i \quad\text{and}\quad \Delta\mathfrak{N}\sigma_i=-\sigma_i \text{ (Poisson's equation)}.$$

We use the following Maxwell's theorem in the time independent case (see [7]):

PROPOSITION 4.1. *Let* $\eta\in *Z_{20}^{\infty}(\boldsymbol{R}^3)$ $(=*[Z_2(\boldsymbol{R}^3)\cap C_{2,0}^{\infty}(\boldsymbol{R}^3)])$. *If we put* $p(x)=\mathfrak{N}\eta(x)$ *and* $\omega(x)=dp(x)$ *in* $\boldsymbol{R}^3$, *then* $\delta\omega=\eta$ *holds in* $\boldsymbol{R}^3$.

We shall show the following main theorem which gives a new interpretation of Weyl's orthogonal decomposition theorem related to magnetic fields induced by surface current densities on Σ:

THEOREM 4.1. *Let* $\sigma=adx+bdy+cdz$ *be a* C^∞ *closed* 1-*form on* $\bar{D}$. *We put* $\boldsymbol{a}(x)=(a, b, c)$ *for* $x\in\bar{D}$. *Then we have*

(1) $JdS_x:=\boldsymbol{a}(x)\times\boldsymbol{n}_x dS_x$ *is a surface current density on* Σ.
We denote by $B(x)=(\alpha, \beta, \gamma)$ *in* $D\cup D'$ *the magnetic field induced by* JdS_x, *and put* $\boldsymbol{\omega}=\alpha dy\wedge dz+\beta dz\wedge dx+\gamma dx\wedge dy$ *in* $D\cup D'$.

(2) *If we put* $\tilde{\sigma}=\sigma$ *in* D *and* $=0$ *in* D', *then it holds*

$$\tilde{\sigma}=*\boldsymbol{\omega}\dotplus dF \quad in\ D\cup D', \tag{4.1}$$

where

$$F(x)=\frac{1}{4\pi}\int_{\Sigma}\frac{\boldsymbol{a}(y)\cdot\boldsymbol{n}_y}{\|x-y\|}dS_y-\frac{1}{4\pi}\int_{D}\frac{\operatorname{div}\boldsymbol{a}(y)}{\|x-y\|}dv_y \quad for\ x\in\boldsymbol{R}^3.$$

(3) *Formula* (4.1) *is the Weyl's orthogonal decomposition of* $\tilde{\sigma}$ *in* $\Gamma_1^2(\boldsymbol{R}^3)$, *that is,* $\boldsymbol{\omega}\in Z_2(\boldsymbol{R}^3)$ *and* $dF\in B_1(\boldsymbol{R}^3)$. *In our case,* $F\in C(\boldsymbol{R}^3)\cap C^\infty(\boldsymbol{R}^3\backslash\Sigma)$ *and* $\boldsymbol{\omega}\in H_2(\boldsymbol{R}^3\backslash\Sigma)$ *such that* $F(x)=O(1/\|x\|^2)$ *and* $\boldsymbol{\omega}(x)=O(1/\|x\|^3)$ *at* $x=\infty$.

Proof. Although (1) is clear from Proposition 2.1, we verify it again for the proof of (2) and (3). Using the function $\tilde{\chi}_n(x)$ in $\boldsymbol{R}^3$ defined by (2.6) for $n\geqq n_0$, we consider $\tilde{\chi}_n\sigma\in C_{1,0}^{\infty}(\boldsymbol{R}^3)$ with support in D. If we put

$$\begin{aligned} \eta_n(x)&=*d(\tilde{\chi}_n\sigma)=f_{1n}dx+f_{2n}dy+f_{3n}dz && \text{in } \boldsymbol{R}^3 \\ J_n dv_x&=(f_{1n}, f_{2n}, f_{3n})dv_x && \text{in } \boldsymbol{R}^3, \end{aligned} \tag{4.2}$$

then $J_n dv_x$ is a volume current density in $\boldsymbol{R}^3$. Since σ is closed on $\bar{D}$, we get

$$f_{1n}(x)=\chi_n'(R(x))\left\{\frac{\partial R}{\partial y}c-\frac{\partial R}{\partial z}b\right\} \quad \text{etc.} \tag{4.3}$$

It follows from (1) of Proposition 2.3 and $\nabla R(x)=\boldsymbol{n}_x$ on Σ that $J_n dv_x\to JdS_x$ $(n\to\infty)$ on Σ in the sense of distribution. Thus (1) is proved. Denoting by $B_n=(\alpha_n, \beta_n, \gamma_n)$ the magnetic field in $\boldsymbol{R}^3$ induced by $J_n dv_x$, we have $B_n(x)\to B(x)$ $(n\to\infty)$ pointwise in $D\cup D'$. We put $\boldsymbol{\omega}_n(x)=\alpha_n dy\wedge dz+\beta_n dz\wedge dx+\gamma_n dx\wedge dy$ in $\boldsymbol{R}^3$, so that

$$\boldsymbol{\omega}_n(x)=d\left(\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{\eta_n(y)}{\|x-y\|}dv_y\right) \quad \text{for } x\in\boldsymbol{R}^3, \tag{4.4}$$

and $\boldsymbol{\omega}_n(x)\to\boldsymbol{\omega}(x)$ $(n\to\infty)$ pointwise.

We here note that $\delta(*\tilde{\chi}_n\sigma)=\eta_n$ in $\boldsymbol{R}^3$. By Proposition 4.1, we have $\delta\omega_n=\eta_n$ in $\boldsymbol{R}^3$. Since $d\omega_n=0$ in $\boldsymbol{R}^3$, we have the orthogonal decomposition: $*\tilde{\chi}_n\sigma=\omega_n\dot{+}(*\tilde{\chi}_n\sigma-\omega_n)$ in $\boldsymbol{R}^3$. Since $\Delta=\delta d-d\delta$ for 2-forms, it follows from (4.2) and Poisson's equation that, for any fixed $x\in\boldsymbol{R}^3$,

$$\omega_n(x)=d\delta\Big(\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{*\tilde{\chi}_n\sigma}{\|x-y\|}dv_y\Big) \tag{4.5}$$

$$=(-\Delta+\delta d)\Big(\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{*\tilde{\chi}_n\sigma}{\|x-y\|}dv_y\Big)$$

$$=*\tilde{\chi}_n\sigma(x)+*dF_n(x)$$

where

$$F_n(x)=\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{*d(*\tilde{\chi}_n\sigma)}{\|x-y\|}dv_y$$

$$=\frac{1}{4\pi}\int_D\frac{\chi_n'(R(y))\nabla R(y)\cdot\boldsymbol{a}(y)+\tilde{\chi}_n(y)\operatorname{div}\boldsymbol{a}(y)}{\|x-y\|}dv_y.$$

Consequently,

$$\tilde{\chi}_n\sigma=*\omega_n(x)\dot{+}d(-F_n)\quad\text{in }\boldsymbol{R}^3. \tag{4.6}$$

By its expression, $F_n(x)$ is of class C^∞ in $\boldsymbol{R}^3$ and harmonic in $\boldsymbol{R}^3\backslash\bar{D}$. Moreover, since $\tilde{\chi}_n(x)=0$ on Σ, we have

$$\lim_{x\to\infty}\frac{F_n(x)}{\|x\|}=\frac{1}{4\pi}\int_D\{\chi_n'(R(y))\nabla R(y)\cdot\boldsymbol{a}(y)+\tilde{\chi}_n(y)\operatorname{div}\boldsymbol{a}(y)\}\,dv_y$$

$$=\frac{1}{4\pi}\int_D d[\tilde{\chi}_n(y)*\sigma]=\frac{1}{4\pi}\int_{\partial D}\tilde{\chi}_n(y)*\sigma=0,$$

so that $F_n(x)=O(1/\|x\|^2)$ at $x=\infty$. Since $\omega_n\in Z_2^\infty(\boldsymbol{R}^3)$ and $dF_n\in B_1(\boldsymbol{R}^3)$, formula (4.6) for each $n\geqq n_0$ is the Weyl's orthogonal decomposition of $\tilde{\chi}_n\sigma$ in $\Gamma_1^2(\boldsymbol{R}^3)$. By (1) of Lemma 3.1, $F_n(x)\to -F(x)$ $(n\to\infty)$ uniformly in $\boldsymbol{R}^3$. Therefore, there exists an $M_1>0$ (independent of $n\geqq n_0$) such that $|F_n(x)|$, $|F(x)|\leqq M_1/\|x\|^2$ outside a ball $B_0\supset\bar{D}$. By (4.6) we may assume that $\|\omega_n\|(x)$, $\|\omega\|(x)\leqq M_1/\|x\|^3$ outside B_0. From (4.2), (4.3) and (4.4), each component α_n, β_n or γ_n of $\omega_n(x)$ is of the same form as (3.10). Hence, (3) of Lemma 3.1 implies that $\{\|\omega_n\|(x)\}_{n\geqq n_0}$ is uniformly bounded in $\boldsymbol{R}^3$. It follows that $\lim_{n\to\infty}\|\omega_n-\omega\|_{\boldsymbol{R}^3}^2=0$, and hence $\lim_{n\to\infty}\|dF_n+dF\|_{\boldsymbol{R}^3}^2=0$. In particular, $\omega\in Z_2(\boldsymbol{R}^3)$ and $dF\in B_1(\boldsymbol{R}^3)$. Letting $n\to\infty$ in (4.6), we get (2) and (3) of Theorem 4.1. □

COROLLARY 4.1. *Let JdS_x be a surface current density on Σ and, $B(x)$ the magnetic field induced by JdS_x. We use the same notations ω, η, $\star\eta$ as in Proposition 2.2. Assume that $\star\eta$ on Σ is extended to a C^∞ closed 1-form σ on $\bar{D}$. If we put $\tilde{\sigma}:=\sigma$ in D and $=0$ in D', then $*\omega$ is identical with the projection of $\tilde{\sigma}\in L_1^2(\boldsymbol{R}^3)$ to $*Z_2(\boldsymbol{R}^3)$ in the Weyl's orthogonal decomposition.*

In fact, we put $\star\eta=g_1dx+g_2dy+g_3dz$ on Σ and $\sigma=adx+bdy+cdz$ on $\bar{D}$, then $JdS_x=(g_1, g_2, g_3)\times \boldsymbol{n}_x dS_x=(a, b, c)\times \boldsymbol{n}_x dS_x$ for $x\in\Sigma$. Applying Theorem 4.1 to this σ, we have the corollary. □

5. Equilibrium surface density on Σ

If a surface current density JdS_x on Σ induces a magnetic field $B_J(x)$ in $D\cup D'$ such that $B_J(x)$ vanishes identically in D', we said in [6] that JdS_x is *an equilibrium current density on* Σ. In this case, (2) of Proposition 2.2 is reduced to $B_J^+(x)=\boldsymbol{n}_x\times J(x)$ and $\omega^+(x)=\star\eta(c)$ on Σ, which is called *Fleming's law*. In [7] we proved the following existence

THEOREM 5.1. *Let $\{\gamma_j\}_{j=1,\dots,q}$ be a base of the 1-dimensional homology group of D. Then there exist q equilibrium current densities $\{J_i dS_x\}_{i=1,\dots,q}$ on Σ such that $J_i[\gamma_j]=\delta_{ij}$ $(1\leqq\forall j\leqq q)$.*

We give another proof of this theorem by use of Theorem 4.1.

Proof. For each $i=1,\cdots,q$, we consider the 2-form $\omega_i=\alpha_i dy\wedge dz+\beta_i dz\wedge dx+\gamma_i dx\wedge dx\in H_{20}(D)$ defined in Proposition 2.4. As a C^∞ closed 1-form σ on $\bar{D}$ in Theorem 4.1, we can take $\sigma=*\omega_i$ on $\bar{D}$. We denote by $J_i dS_x$, B_i, Ω_i and $F_i(x)$ things obtained through $*\tilde{\omega}_i$ which correspond to JdS_x, B, ω and $F(x)$ obtained through $\tilde{\sigma}$ in Theorem 4.1. Therefore,

$$*\tilde{\omega}_i=*\Omega_i+dF_i \quad \text{in } D\cup D', \qquad J_i dS_x=((\alpha_i, \beta_i, \gamma_i)\times\boldsymbol{n}_x)dS_x \quad \text{on } \Sigma.$$

Since $(\alpha_i, \beta_i, \gamma_i)\perp\boldsymbol{n}_x$ on Σ and $\operatorname{div}(\alpha_i, \beta_i, \gamma_i)=0$ in D, we have $F_i(x)=0$ in $\boldsymbol{R}^3$, so that $*\tilde{\omega}_i=*\Omega_i$ in $D\cup D'$, that is, $\Omega_i=\omega_i$ in D and $\Omega_i=0$ in D', which is equivalent to $B_i(x)=(\alpha_i, \beta_i, \gamma_i)$ in D and $=0$ in D'. Hence $J_i dS_x$ is an equilibrium current density on Σ. By (3) of Proposition 2.2, we have $J_i[\gamma_j]=\int_{\gamma_j}*\omega_i=\delta_{ij}$. □

Let $u(x)$ be a harmonic function on $\bar{D}$. Applying Theorem 4.1 for $\sigma=du$, we see that $JdS_x:=(\nabla u(x)\times\boldsymbol{n}_x)dS_x$ is a surface current density on Σ and that

$$\widetilde{du}=*\omega+d\left(\frac{1}{4\pi}\int_\Sigma\frac{\partial u/\partial n_y}{\|x-y\|}dS_y\right) \quad \text{in } \boldsymbol{R}^3, \tag{5.1}$$

where $\omega\in Z_2(\boldsymbol{R}^3)$ with the following property: If we set $\omega(x)=\alpha dy\wedge dz+\beta dz\wedge dx+\gamma dx\wedge dy$ in $D\cup D'$, then (α, β, γ) is the magnetic field induced by JdS_x. On the other hand, it is well known (cf. [2]) that, if we put

$$c=1,\ 1/2,\ 0 \quad \text{on } D,\ \Sigma,\ D', \text{ respectively,} \tag{5.2}$$

then it holds

(5.3)
$$cu(x)=\frac{1}{4\pi}\int_{\Sigma}\frac{\partial u/\partial n_y}{\|x-y\|}dS_y-\frac{1}{4\pi}\int_{\Sigma}u(y)\frac{\partial}{\partial n_y}\frac{1}{\|x-y\|}dS_y$$
$$\equiv p_1(x)-p_2(x)$$

for $x\in \boldsymbol{R}^3$. We thus obtain

COROLLARY 5.1. *Under notations* (5.1) *and* (5.3), *we have*

(5.4)
$$\omega(x)=*d\left(-\frac{1}{4\pi}\int_{\Sigma}u(y)\frac{\partial}{\partial n_y}\frac{1}{\|x-y\|}dS_y\right) \quad \text{in } D\cup D'$$
$$\|du\|_D^2=\|dp_1\|_{\boldsymbol{R}^3}^2+\|dp_2\|_{\boldsymbol{R}^3}^2 .$$

The former formula physically means that the gradient of the double layer potential with density $u(x)dS_x$ on Σ is equal to the magnetic field induced by the surface current density $(\boldsymbol{n}_x\times\nabla u(x))dS_x$ on Σ. The latter says $dp_1\perp dp_2$ in $\boldsymbol{R}^3$ (not in D!).

COROLLARY 5.2. *Let $V(x)=(a, b, c)$ be a C^ω vector field on $\bar{D}$ such that* $\operatorname{div} V(x)=\operatorname{rot} V(x)=0$ *in D. Then there exists a surface current density JdS_x on Σ whose magnetic field restricted to D is equal to $V(x)$, if and only if $\int_{\Sigma_i}V(x)\cdot\boldsymbol{n}_x dS_x=0$ for each component Σ_i $(i=1, \cdots, m)$ of Σ.*

Proof. Let $V(x)=(a, b, c)$ be given as above. We put $\omega=ady\wedge dz+bdz\wedge dx+cdx\wedge dy$ on $\bar{D}$, so that $*\omega\in H_1(\bar{D})$. First, assume that $\int_{\Sigma_i}V(x)\cdot\boldsymbol{n}_x dS_x=0$ $(i=1, \cdots, m)$. By Proposition 2.4 we find $\omega_0=\alpha dy\wedge dz+\beta dz\wedge dx+\gamma dx\wedge dy\in H_{20}(D)$ such that $\int_{\gamma_j}*\omega_0=\int_{\gamma_j}*\omega$ $(1\leqq\forall j\leqq q)$. By the same reasoning as in the proof of Theorem 5.1, we see that $J_0dS_x:=((\alpha, \beta, \gamma)\times\boldsymbol{n}_x)dS_x$ is an equilibrium current density on Σ which induces the magnetic field (α, β, γ) in D and 0 in D'. We can find a harmonic function $h(x)$ on $\bar{D}$ such that $*\omega-*\omega_0=dh$. Since $\int_{\Sigma i}\frac{\partial h}{\partial n_x}dS_x=0$ $(i=1, \cdots, m)$, it follows from Fredholm theory of integral equations that there exists a C^ω function ϕ on Σ such that

$$h(x)=\frac{1}{4\pi}\int_{\Sigma}\phi(y)\frac{\partial}{\partial n_y}\frac{1}{\|x-y\|}dS_y \quad \text{for } x\in D.$$

We here solve the Dirichlet problem on $\bar{D}$ with boundary values $\phi(x)$ on Σ and denote by $u(x)$ its solution on $\bar{D}$. By (5.1), $J_1dS_x=(\boldsymbol{n}_x\times\nabla u(x))dS_x$ is a surface current density on Σ which induces the magnetic field $\nabla h(x)$ in D. It follows that the surface current density $JdS_x:=J_0dS_x+J_1dS_x$ on Σ induces the magnetic field $B_J(x)$ whose restriction to D is identical with $V(x)$.

Next, assume that there exists JdS_x on Σ which induces the magnetic field $B_J=(\alpha, \beta, \gamma)$ in $\boldsymbol{R}^3\backslash\Sigma$ such that $B_J=V$ in D. If we put $\omega_J=\alpha dy\wedge dz+\beta dz\wedge dx$

$+dx\wedge dy$ in $\boldsymbol{R}^3\backslash\Sigma$, then $\omega_J\in Z_2(\boldsymbol{R}^3)$ by Corollary 4.1. We draw a closed smooth surface Σ_i' in D homologous to Σ_i $(i=1, \cdots, m)$. Since div $V=0$ on $\bar{D}$, it follows that

$$\int_{\Sigma_i} V(x)\cdot \boldsymbol{n}_x dS_x=\int_{\Sigma_i'} V(x)\cdot \boldsymbol{n}_x dS_x=\int_{\Sigma_i'} \omega_J=0. \qquad \square$$

6. Grunsky inequality

In this section we consider the kernel $\log 1/|z-\zeta|$ in the complex plane $\boldsymbol{C}$ instead of $1/\|x-y\|$ in $\boldsymbol{R}^3$ in the previous section. Let D be a bounded domain in $\boldsymbol{C}$ with a C^∞ boundary smooth contour L. We recall the remarkable contrast between the properties of the single and double layer potentials as

PROPOSITION 6.1. *For $f_1, f_2\in C^1(L)$, we denote by v_1 and v_2 the single and double layer potentials with density $f_1 ds_z$ and $f_2 ds_z$ on L, respectively*:

$$v_1(z)=\frac{1}{2\pi}\int_L f_1(\zeta)\log\frac{1}{|z-\zeta|}ds_\zeta \qquad \textit{for } z\in\boldsymbol{C}$$

$$v_2(z)=\frac{1}{2\pi}{}_L f_2(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_\zeta \qquad \textit{for } z\in\boldsymbol{C}\backslash L,$$

where ds_ζ is the arc length element of L at ζ. We conveniently put $D^+=D$, $D^-=\boldsymbol{C}\backslash\bar{D}$, $\partial D^\pm=L^\pm$ (where $L^+=L$ and $L^-=-L$). If we write $v_i(z)=v_i^\pm(z)$ $(i=1, 2)$ for $z\in D^\pm$, then we have

(1) *Both $v_i^\pm(z)$, $i=1, 2$, are harmonic functions in $D^\pm$ and continuous up to $L^\pm$, in such a way that, for $z^\pm\in L^\pm$ over $z\in L$,*

$$\begin{cases} v_1^+(z^+)=v_1^-(z^-) \\ \dfrac{\partial v_1^+}{\partial n_z}(z^+)-\dfrac{\partial v_1^-}{\partial n_z}(z^-)=f_1(z) \end{cases} \qquad \begin{cases} v_2^+(z^+)-v_2^-(z^-)=-f_2(z) \\ \dfrac{\partial v_2^+}{\partial n_z}(z^+)=\dfrac{\partial v_2^-}{\partial n_z}(z^-), \end{cases}$$

where both $\boldsymbol{n}_z$ denote the same unit outer normal vector of L at z.

(2) *$v_1(z)=O(\log 1/|z|)$ and $v_2(z)=O(1/|z|)$ at $z=\infty$. Moreover, three conditions $v_1(z)=O(1/|z|)$ at $z=\infty$, $\int_L f_1(z)ds_z=0$, and $\int_L \frac{\partial v_1}{\partial n_z}dS_z=0$ are equivalent.*

Let $u(z)$ be a harmonic function in D and of class C^1 up to the boundary L. By use of notation c of (5.2), it is well known (cf. [2]) that

$$\begin{aligned} (6.1) \qquad cu(z)&=\frac{1}{2\pi}\int_L\frac{\partial u}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_\zeta-\frac{1}{2\pi}\int_L u(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_\zeta \\ &\equiv q_1(z)-q_2(z). \end{aligned}$$

for $z\in\boldsymbol{C}$. Formula (5.4) changes to the following one:

(6.2) $$\|du\|_D^2=\|dq_1\|_C^2+\|dq_2\|_C^2.$$

Proof. Since $\int_L \partial u/\partial n_z ds_z=0$, (2) of Proposition 6.1 implies $\|dq_i\|_C^2<\infty$ for $i=1, 2$. If we put $q_i(z)=q_i^\pm(z)$ for $x\in D^\pm$, then it also implies $\lim_{R\to\infty}\int_{|\zeta|=R} q_1^-(z)\cdot\frac{\partial q_2^-}{\partial n_\zeta}ds_\zeta=0$. It follows from (1) of Proposition 6.1 that

$$(dq_1, dq_2)_C=(dq_1, dq_2)_D+(dq_1, dq_2)_{D'}$$
$$=\int_L q_1^+(z)\frac{\partial q_2^+}{\partial n_\zeta}(z)ds_\zeta-\int_L q_1^-(z)\frac{\partial q_2^-}{\partial n_\zeta}(z)ds_\zeta=0.$$

This together with (6.1) proves (6.2). □

Proposition 6.1 implies

(6.3) $$\|dq_1\|_C^2=\frac{1}{2\pi}\int_L\int_L \frac{\partial u}{\partial n_z}\frac{\partial u}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_z ds_\zeta\equiv I_L(u),$$

which is called *the energy of* $(\partial u/\partial n_z)ds_z$ *on* L in the potential theory. Hence,

(6.4) $$\|du\|_D^2=I_L(u)+\|dq_2\|_C^2.$$

We consider the case when D is the unit disk D_0 of center the origin and L is the unit circle $L_0=\{e^{i\theta}|0\leqq\theta\leqq 2\pi\}$. Let $u(z)$ be a harmonic function $u(z)$ in D_0 and of class C^1 up to L_0. Then we have

LEMMA 6.1. $$I_{L_0}(u)=\frac{1}{2}\|du\|_{D_0}^2.$$

Proof. For any fixed $z\in L_0$, we have from Stokes' formula

$$\frac{1}{2\pi}\int_{L_0}\frac{\partial u}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_\zeta$$
$$=\frac{1}{2\pi}\left(\pi u(z)+\int_{L_0}u(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_\zeta\right)$$
$$=\frac{1}{2\pi}\left(\pi u(z)-\int_0^{2\pi}u(\zeta)\frac{d\theta}{2}\right)=\frac{1}{2}(u(z)-u(0)).$$

It follows that $I_{L_0}(u)=\frac{1}{2}\int_{L_0}(u(z)-u(0))\frac{\partial u}{\partial n_z}ds_z=\frac{1}{2}\|du\|_{D_0}^2$. □

We similarly verify that (6.4) and Lemma 6.1 are true for the unbounded domain D and the exterior $E_0=\{|z|>1\}$ of D_0 as follows: Let D be a unbounded domain with C^∞ smooth boundary contours L. We determine the orientation of L by $\partial D=L$. Let $U(w)$ be a harmonic function on $D\cup\{\infty\}$ which is of class C^1 up to L. Then we have

$$\|dU\|_D^2=\frac{1}{2\pi}\int_L\int_L\frac{\partial U}{\partial n_w}\frac{\partial U}{\partial n_\xi}\log\frac{1}{|w-\xi|}ds_w ds_\xi+\|dP_2\|_C^2,$$

where $P_2(w)$ is the double layer potential with density $U(w)ds_w$ on L. Let $V(z)$ be a harmonic function in $E_0\cup\{\infty\}$ which is of class C^1 up to the unit circle L_0 (where $\partial E_0=-L_0$), we have

$$\frac{1}{2\pi}\int_{L_0}\int_{L_0}\frac{\partial V}{\partial n_z}\frac{\partial V}{\partial n_\zeta}\log\frac{1}{|z-\zeta|}ds_z ds_\zeta=\frac{1}{2}\|dV\|_{E_0}^2.$$

We write these two formulas into the following simple forms:

$$\text{(6.5)}\qquad \|dU\|_D^2=I_L(U)+\|dP_2\|_C^2,\quad I_{L_0}(V)=\frac{1}{2}\|dV\|_{E_0}^2.$$

We shall show that these imply the following Grunsky inequality. We consider a univalent function $g(z)$ in E_0 such that $g(z)=z+c_0+c_1/z+c_2/z^2+\cdots$ at $z=\infty$, and denote by $\mathcal{G}$ the set of all such univalent functions $g(z)$ in E_0.

THEOREM 6.1 (see [4]). *Let $g(z)\in\mathcal{G}$. If we set*

$$\log\frac{g(z)-g(\zeta)}{z-\zeta}=-\sum_{k,l=1}^{\infty}\frac{b_{kl}}{z^k z^l}\quad \textit{for } (z,\zeta)\in E_0\times E_0,$$

then we have

$$\text{(6.6)}\qquad \sum_{n=1}^{\infty}\frac{|\lambda_n|^2}{n}\geqq\lim_{N\to\infty}\left|\sum_{k,l=1}^{N}b_{kl}\lambda_k\lambda_l\right|$$

for any complex numbers $\{\lambda_n\}_{n=1,2,\ldots}$. We call $\{b_{k,l}\}_{k,l}$ the Grunsky coefficients of $g(z)$.

Proof. It suffices to prove the case when $g(z)$ is univalent on $\bar{E}_0$. We put $D=g(E_0)$ and $L=g(-L_0)$ so that $\partial D=L$. For $N\geqq 1$ we consider the following functions:

$$\text{(6.7)}\qquad V_N(z)=2\Re\left\{\sum_{n=1}^{N}\frac{\bar{\lambda}_n}{nz^n}\right\}\ \text{ on } \bar{E}_0,\qquad U_N(w)=V_N(g^{-1}(w))\ \text{ on } \bar{D}.$$

Thus, $V_N(z)$ and $U_N(w)$ are harmonic functions on $\bar{E}_0\cup\{\infty\}$ and $\bar{D}\cup\{\infty\}$, respectively. Since $(\partial/\partial n_z)ds_z$ and the Dirichlet integral are invariant under the conformal mapping $w=g(z)$, we have

$$\|dV_N\|_{E_0}^2=\|dU_N\|_D^2,$$

$$I_L(U_N)=\frac{1}{2\pi}\int_{L_0}\int_{L_0}\frac{\partial V_N}{\partial n_z}\frac{\partial V_N}{\partial n_\zeta}\log\left|\frac{1}{g(z)-g(\zeta)}\right|ds_z ds_\zeta.$$

We denote by $P_{N_2}(w)$ the double layer potential with density $U_N(w)ds_w$ on L. Applying equations (6.5) for $U=U_N$, $V=V_N$ and $P_2=P_{N_2}$, we have

$$\begin{aligned}\frac{1}{2}\|dV_N\|^2_{E_0}&=-\frac{1}{2}\|dV_N\|^2_{E_0}+\|dU_N\|^2_D\\&=-I_{L_0}(V_N)+I_L(U_N)+\|dP_{N_2}\|^2_C\\&=\frac{1}{2\pi}\int_{L_0}\int_{L_0}\frac{\partial V_N}{\partial n_z}\frac{\partial V_N}{\partial n_\zeta}\log\left|\frac{z-\zeta}{g(z)-g(\zeta)}\right|ds_z ds_\zeta+\|dP_{N_2}\|^2_C\\&=\frac{1}{2\pi}\Re\left\{\int_{L_0}\int_{L_0}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{b_{kl}}{z^k\zeta^l}\frac{\partial V_N}{\partial n_z}\frac{\partial V_N}{\partial n_\zeta}ds_z ds_\zeta\right\}+\|dP_{N_2}\|^2_C.\end{aligned}$$

Since

$$\frac{\partial V_N}{\partial z}=-\sum_{n=1}^{N}\frac{\bar{\lambda}_n}{z^{n+1}},\quad \|dV_N\|^2_{E_0}=4\left\|\frac{\partial V_N}{\partial z}\right\|^2_{E_0},$$

$$\frac{\partial V_N}{\partial n_z}ds_z=\frac{1}{i}\left(\frac{\partial V_N}{\partial z}dz-\frac{\partial V_N}{\partial \bar{z}}d\bar{z}\right),$$

it follows that

$$\begin{aligned}(6.8)\qquad 2\pi\sum_{n=1}^{N}\frac{|\lambda_n|^2}{n}&=\frac{1}{2\pi}\Re\left\{\sum_{k,l=1}^{N}b_{kl}\left(\int_{L_0}\frac{\partial V_N}{\partial \bar{z}}\frac{d\bar{z}}{z^k}\right)\left(\int_{L_0}\frac{\partial V_N}{\partial \bar{\zeta}}\frac{d\bar{\zeta}}{\zeta^l}\right)\right\}+\|dP_{N_2}\|^2_C\\&=2\pi\,\Re\left\{\sum_{k,l=1}^{N}b_{kl}\lambda_k\lambda_l\right\}+\|dP_{N_2}\|^2_C\\&\geqq 2\pi\,\Re\left\{\sum_{k,l=1}^{N}b_{kl}\lambda_k\lambda_l\right\}.\end{aligned}$$

Since $\{\lambda_n\}_n$ is arbitrary, we can replace $\{\ \}$ by $|\ |$ in the last inequality. By letting $n\to\infty$, we obtain Theorem 6.1. □

In [4], when Grunsky inequality is reduced to equality is studied in the case that at most a finite number of $\{\lambda\}_n$ do not vanish. We shall give a necessary and sufficient condition for this problem under the conditions that

(i) *$g(z)\in\mathcal{G}$ is holomorphically extended up to L_0 except for a finite point set $\{P_i\}$.*

(ii) $\sum_{n=1}^{\infty}|\lambda_n|<\infty$.

We set $D=g(E_0)$, $L=g(-L_0)$ and $K=\boldsymbol{C}\setminus D$. By (i), the set K is compact in $\boldsymbol{C}$ and its boundary $\partial E=-L$ is a piecewise real analytic smooth curve with a finite number of edge points $\{Q_i\}=\{g(P_i)\}$. It may happen that the interior K° of K is empty: $K^\circ=\emptyset$. In this case, as a point set, L is a piecewise real analytic smooth arc $\mathcal{L}$. We write

$$(6.9)\qquad L=\mathcal{L}^+ + \mathcal{L}^- \quad\text{and}\quad \mathcal{L}^+=-\mathcal{L}^-.$$

Precisely, for $w\in\mathcal{L}$ (except for two end points), we find two points $w^\pm\in\mathcal{L}^\pm$ over w. We denote by $\{b_{kl}\}_{k,l}$ the Grunsky coefficients of $g(z)$. By Grunsky inequality we have $1/k+1/l\geqq|b_{kl}|$ for all $k,l\geqq 1$. This together with (ii) imply $\sum_{k,l=1}^{\infty}|b_{kl}\lambda_k\lambda_l|<\infty$. We put $\Theta=1/2\,\mathrm{Arg}\{\sum_{k,l=1}^{\infty}b_{kl}\lambda_k\lambda_l\}$ and consider the following functions:

(6.10) $$V(z)=2\Re\left\{\sum_{n=1}^{\infty}\frac{\bar{\lambda}_n e^{i\theta}}{nz^n}\right\} \text{ in } E_0, \qquad U(w)=V(g^{-1}(w)) \text{ in } D.$$

By (ii), $V(z)$ is of class C^1 up to L_0 and $U(w)$ is continuous up to L and of class C^1 up to L except for the edge points $\{Q_i\}$. Under these situations we shall prove

COROLLARY 6.1. *Assume that $g(z)\in\mathcal{G}$ and $\{\lambda_n\}_n$ satisfies conditions* (i) *and* (ii). *Then Grunsky inequality* (6.6) *for $g(z)$ and $\{\lambda_n\}_n$ is reduced equality, if and only if*

(6.11) $$K^{\circ}=\emptyset \quad \textit{and} \quad U(w^+)=U(w^-) \quad \textit{for } w\in\mathcal{L}.$$

Proof. We denote by $P_2(w)$ the double layer potential with density $U(w)ds_w$ on L. In the proof of Theorem 6.1 we can use the function $V(z)$ of (6.10) instead of $V_N(z)$ of (6.7) to obtain the following formula corresponding to (6.8):

$$2\pi\sum_{n=1}^{\infty}\frac{|\lambda_n|^2}{n}=2\pi\left|\sum_{k,l=1}^{\infty}b_{kl}\lambda_k\lambda_l\right|+\|dP_2\|_C^2.$$

It follows that equality holds in (6.6) if and only in $\|dP_2\|_C^2=0$, or equivalently,

(6.12) $$P_2(w)=\text{const. } a,\ 0 \quad \text{on } K^{\circ},\ D, \text{ respectively.}$$

Note that this formula is true even when $K^{\circ}=\emptyset$. It thus suffices for Corollary 6.1 to prove that (6.11)$\Leftrightarrow$(6.12). We first assume (6.11). Since $U(w^+)=U(w^-)$ for $\forall w\in\mathcal{L}$, it follows from (6.9) that

$$P_2(w)=\frac{1}{2\pi}\int_L U(\xi)\frac{\partial}{\partial n_\xi}\log\frac{1}{|w-\xi|}ds_\xi=0 \quad \text{for } \forall w\in D.$$

Thus ($\Rightarrow$) is proved. For the converse we may assume some $\lambda_n\neq 0$ ($n\geqq 1$), so that $U(w)$ is non-constant in D by (6.10). If $K^{\circ}\neq\emptyset$, formula (6.12) and (1) of Proposition 6.1 imply $U(w)=a$ on ∂K $(=-L)$. Consequently, $U(w)$ is the constant a on D, which is a contradiction. We thus have $K^{\circ}=\emptyset$, and (6.9). Therefore,

$$P_2(w)=\int_{\mathcal{L}^+}U(\xi^+)\frac{\partial}{\partial n_\xi}\log\frac{1}{|\xi-w|}ds_\xi+\int_{\mathcal{L}^-}U(\xi^-)\frac{\partial}{\partial n_\xi}\log\frac{1}{|\xi-w|}ds_\xi$$

for $w\in\boldsymbol{C}$. Let $w_0\in\mathcal{L}\setminus\{\text{two edge points}\}$. We find a small disk $\mathcal{V}$ in $\boldsymbol{C}$ centered at w_0, and denote by $\mathcal{V}^+(w_0)$ and $\mathcal{V}^-(w_0)$ the left and right half sides of $\mathcal{V}$ along $\mathcal{L}^+$, respectively. From (6.12) and (1) of Proposition 6.1, we have,

$$U(w_0^+)-U(w_0^-)=\lim_{\substack{w\to w_0^+\\ w\in\mathcal{V}^+(w_0)}}\frac{\partial P_2}{\partial n_{w_0}}(w)-\lim_{\substack{w\to w_0^-\\ w\in\mathcal{V}^-(w_0)}}\frac{\partial P_2}{\partial n_{w_0}}(w)=0.$$

Thus ($\Leftarrow$) is proved. □

Examples. By the above consideration we can construct many exemples $g(z)\in\mathcal{G}$ and $\{\lambda_n\}_n$ for which equality holds in (6.6): First consider a piecewise C^ω smooth arc $\mathcal{L}$ in the w-plane with a finite number of edge points $\{Q_i\}$. We put $D=\boldsymbol{C}\backslash\mathcal{L}$, so that $D\cup\{\infty\}$ is simply connected and $\partial D=\mathcal{L}^{+}+\mathcal{L}^{-}$ such that there exist $w^{\pm}\in\mathcal{L}^{\pm}$ for $w\in\mathcal{L}$ (except for two end points). We have a unique $g(z)\in\mathcal{G}$ which transforms E_0 onto D. So, $g(z)$ satisfy condition (i). Next let $\phi(w)$ be a C^∞ real-valued function in a neighborhood of $\mathcal{L}$ in the w-plane such that $\phi(w)$ is a constant c_i near each Q_i. We construct the harmonic function $U(w)$ in $D\cup\{\infty\}$ with boundary values $\phi(w)$ at $w^{\pm}\in\mathcal{L}^{\pm}$. We set $V(z)=U(g(z))$ in E_0 and consider the Taylor series: $V(z)=2\Re\{\sum_{n=0}^{\infty}a_n/z^n\}$ in E_0. If we set $\lambda_n=n\bar{a}_n$ $(n=1, 2, \cdots)$, then equality holds in (6.6) for these $g(z)$ and $\{\lambda_n\}_n$:

$$\sum_{n=1}^{\infty}\frac{|\lambda_n|^2}{n}=\left|\sum_{k,l=1}^{\infty}b_{kl}\lambda_k\lambda_l\right|. \tag{6.13}$$

In fact, it is clear that

$$V(z)=2\Re\left\{\sum_{n=1}^{\infty}\frac{\bar{\lambda}_n}{nz^n}\right\}\quad\text{in } E_0,\qquad U(w)=V(g^{-}(w))\quad\text{in } D.$$

Since $U(z)$ is of class C^3 up to the boundary L_0, it follows that $+\infty>\|\partial^3U/\partial z^3\|_{E_0}^2=\pi\sum_{n=1}^{\infty}n^2(n+1)^2(n+2)|a_n|^2$, so that $\sum_{n=1}^{\infty}|\lambda_n|<\infty$. Consequently, the same argument as (6.8) is available for this $V(z)$ instead of $V_N(z)$, and we obtain

$$2\pi\sum_{n=1}^{\infty}\frac{|\lambda_n|^2}{n}=2\pi\,\Re\left\{\sum_{k,l=1}^{\infty}b_{kl}\lambda_k\lambda_l\right\}+\|dP_2\|_C^2,$$

where $P_2(w)$ is the double layer potential with density $U(w)ds_w$ for $w\in\partial D$. Since $\partial D=\mathcal{L}^{+}+\mathcal{L}^{-}$ and $U(w^{+})=U(w^{-})=\phi(w)$ for $w\in\mathcal{L}$, we have $P_2(w)=0$ in D, and $\|dP_2\|_C^2=0$. This and Grunsky inequality imply (6.13). □

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