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2-TYPE SURFACES IN A HYPERSPHERE

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0. Introduction

Submanifolds of finite type were introduced by B. Y. Chen in [4]. This class of submanifolds is extremely large, including minimal submanifolds of the Euclidean space and minimal submanifolds of hyperspheres. In the last decade many researchers dealed with the study of finite type submanifolds.

In the present paper we study finite type surfaces in a hypersphere. We will review the definition of finite type submanifolds. Let M^n be an n-dimensional submanifold of the unit hypersphere S^m of the $(m+1)$-dimensional Euclidean space E^{m+1} equipped with the induced metric. Denote by Δ the Laplacian operator of M^n with sign convention such that $\Delta=-d^2/dt^2$ on the real line E. This operator can be extended in a natural way to E^{m+1}-valued maps on M^n. The submanifold M^n is said to be of finite k-type if the position vector x admits the spectral decomposition

$$x=x_0+x_{i_1}+\cdots+x_{i_k},$$

where x_0 is a constant vector and x_{i_j} $(j=1, \cdots, k)$ are non-constant E^{m+1}-valued maps on M^n such that

$$\Delta x_{i_j}=\lambda_{i_j}x_{i_j}, \quad \lambda_{i_j}\in R, \quad \lambda_{i_1}<\cdots<\lambda_{i_k}.$$

If the constant vector x_0 is the center of the hypersphere S^m then M^n is said to be *mass-symmetric* in S^m.

In terms of finite type terminology the well known Takahashi's Theorem [14] asserts that the submanifold M^n of S^m is of 1-type if and only if it is minimal in S^m or in a hypersphere of S^m. Moreover minimal submanifolds of S^m are mass-symmetric in S^m. From this point of view mass-symmetric 2-type submanifolds in S^m are the simplest submanifolds next to minimal submanifolds in S^m.

In [8] it was proved that a surface in S^3 is of 2-type if and only if it is a portion of a Riemannian product of two circles of different radii. Moreover in [9] it was proved that a hypersurface of S^m is of 2-type if and only if the scalar curvature and the mean curvature are constants.

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The following interesting problem has been posed in [5] by B.Y. Chen: *Do there exist mass-symmetric 2-type surfaces which lie fully in an even-dimensional hypersphere?*

In [1], M. Barros and B.Y. Chen proved that a compact stationary mass-symmetric 2-type surface in S^m is flat and lies fully in S^5 or in S^7. Furthermore they proved that there exist no mass-symmetric 2-type surfaces which lie fully in S^4. After that O.J. Garay [7] showed that a mass-symmetric 2-type Chen surface in S^m, that is a surface with zero allied mean curvature vector ([3], p. 203), is flat and lies fully in S^3 or in S^5 or in S^7 unless it is pseudoumbilical.

In [12] Y. Miyata based on a powerfull result of R. Bryant [2] treated mass-symmetric 2-type surfaces $M^2(c)$ in S^m with constant Gauss curvature c and proved, among others, that: Let $M^2(c)$ be a mass-symmetric 2-type surface in S^m with $0<\lambda_1<\lambda_2$ then, (i) there exists no such surface with $c<0$ and (ii) if $c\geqq 0$ and $M^2(c)$ lies fully in S^m then m is odd. Finally M. Kotani [11] showed that if M is a compact mass-symmetric 2-type surface of genus zero which lies fully in S^m then m is odd.

In this paper we will discuss this problem. The paper is organized as follows. In the first paragraph we prove some basic lemmas that will be needed in the proofs of the main results. In the second paragraph we give some local results. Specifically we prove that: If M is a mass-symmetric 2-type surface which lies fully in S^m and $\dim N_1\leqq 2$, where N_1 stands for the first normal space of M in S^m, then m is odd unless M is pseudoumbilical. In the third paragraph we prove global results for mass-symmetric 2-type surfaces in S^m. In particular we prove that: If M is a complete mass-symmetric 2-type surface in S^m with non-negative Gaussian curvature then M is flat and m is odd unless M is pseudoumbilical. Finally in the last paragraph we prove that there exist no mass-symmetric 2-type surfaces which lie fully in S^6.

1. Basic lemmas

Let M be a surface of the unit hypersphere S^m of E^{m+1} centered at the origin O. Denote by H, A, D the mean curvature vector, the Weingarten map and the normal connection of M in S^m respectively. Moreover $\bar{H}$, $\bar{A}$ and $\bar{D}$ denote the corresponding quantities for M in E^{m+1}. We choose a local orthonormal frame field $\{e_1, e_2, \cdots, e_{m+1}\}$ on M such that e_1, e_2 are tangent to M, $e_3=x$ and $H=|H|e_4$, where x is the position vector field with respect to the center O. Denote by ω_1, ω_2 the dual frame of e_1, e_2. The connection form ω_{AB}, $A, B=1, 2, \cdots, m+1$ is given by $\omega_{AB}(X)=\langle\bar{\nabla}_X e_A, e_B\rangle$, where $\bar{\nabla}$ stands for the usual Riemannian connection of E^{m+1}. By Cartan's Lemma we have

$$\omega_{i\alpha}=\sum_{j=1}^{2} h_{ij}^{\alpha}\omega_j, \quad \alpha\geqq 3, \ i=1, 2,$$

where h_{ij}^{α} are the coefficients of the second fundamental form.

The surface M is said to be pseudoumbilical at the point $p\in M$ if the Weingarten map A_4 is proportional to the identity map. Moreover M is said to be pseudoumbilical in $U\subset M$ or pseudoumbilical if M is pseudoumbilical everywhere on U or everywhere on M, respectively.

The mean curvature vector $\bar{H}$ is given by

$$\bar{H}=H-e_3,\quad 2H=(\mathrm{tr}\,A_4)e_4.$$

Henceforth we assume that M is mass-symmetric 2-type in S^m. Then we have the spectral decomposition

$$x=x_1+x_2,\quad \Delta x_1=\lambda_1 x_1,\quad \Delta x_2=\lambda_2 x_2,\quad \lambda_1<\lambda_2.$$

Because of the main result in [8] and Theorem 2 in [1] we may assume that $m\geqq 5$. Moreover from our choice of the frame we have $\mathrm{tr}\,A_\alpha=0$ for $\alpha\geqq 5$ and so we may set $h_{11}^\alpha=-h_{22}^\alpha=\rho_\alpha$ and $h_{12}^\alpha=h_{21}^\alpha=\mu_\alpha$. Thus we have

$$A_3=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\quad A_4=\begin{pmatrix}h_{11}^4 & h_{12}^4\\ h_{12}^4 & h_{22}^4\end{pmatrix}\ \text{and}\ A_\alpha=\begin{pmatrix}\rho_\alpha & \mu_\alpha\\ \mu_\alpha & -\rho_\alpha\end{pmatrix}\ \text{for}\ \alpha\geqq 5.$$

Using Chen's formula ([4], Lemma 4.1, p. 271) and the above expression for x we easily obtain the following necessary conditions for mass-symmetric and 2-type surface in S^m $(m\geqq 5)$.

$$(1.1)\qquad (\mathrm{tr}\,A_4)^2=2(\lambda_1+\lambda_2)-\lambda_1\lambda_2-4=(\lambda_2-2)(2-\lambda_1),$$

$$(1.2)\qquad \sum_{i=1}^{2} A_{D_{e_i}e_4}e_i=0$$

$$(1.3)\qquad \mathrm{tr}\,A_4^2+|De_4|^2=\lambda_2+\lambda_2-2$$

$$(1.4)\qquad \mathrm{tr}(A_4A_\alpha)-\mathrm{tr}\,\nabla\omega_{4\alpha}+\langle De_4, De_\alpha\rangle=0,\quad \text{for any } \alpha\geqq 5,$$

where

$$\langle De_4, De_\alpha\rangle=\sum_{i=1}^{2}\langle D_{e_i}e_4, D_{e_i}e_\alpha\rangle\quad \text{for any } \alpha\geqq 4.$$

and

$$\mathrm{tr}\,\nabla\omega_{4\alpha}=\sum_i(\nabla_{e_i}\omega_{4\alpha})e_i=\sum_i(e_i(\omega_{4\alpha}(e_i))-\omega_{4\alpha}(\nabla_{e_i}e_i)).$$

LEMMA 1.1. *Let M be a mass-symmetric 2-type surface in S^m $(m\geqq 5)$. The Weingarten map A_4 is a Codazzi tensor, that is A_4 satisfies equation $(\nabla_{e_1}A_4)e_2=(\nabla_{e_2}A_4)e_1$.*

Proof. Since $D_{e_i}e_4=\sum_{\alpha\geqq 5}\omega_{4\alpha}(e_i)e_\alpha$, from (1.2) we have

$$(1.5)\qquad \sum_{\alpha\geqq 5}(\omega_{4\alpha}(e_1)\rho_\alpha+\omega_{4\alpha}(e_2)\mu_\alpha)=0\ \text{and}\ \sum_{\alpha\geqq 5}(\omega_{4\alpha}(e_1)\rho_\alpha-\omega_{4\alpha}(e_2)\mu_\alpha)=0.$$

The last equations imply $A_{D_{e_1}e_4}e_2=A_{D_{e_2}e_4}e_1$ and so from the Codazzi equation

$$(\nabla_{e_1}A_4)e_2-(\nabla_{e_2}A_4)e_1=A_{D_{e_1}e_4}e_2-A_{D_{e_2}e_4}e_1$$

we conclude the desired result.

Let $N_1(p)$ be the first normal space of M in S^m at the point $p\in M$. Considering the linear map $\xi\to A_\xi$ from the normal bundle $(T_pM)^\perp$ of M in S^m into the bundle whose fiber at $p\in M$ is the space of symmetric linear transformations of T_pM we conclude that $\dim N_1(p)\leqq 3$. It is obvious that e_4 belongs to $N_1(p)$ for all $p\in M$.

Consider, now, the linear map $L_p: T_pM\to(T_pM)^\perp$ defined by $L_p(X)=D_Xe_4$. We set $d_p=\dim(L_p(T_pM))$. It is obvious that $d_p\leqq 2$ everywhere on M. In the following let

$$M_0=\{p\in M: d_p=0\},\quad M_1=\{p\in M: d_p=1\}\quad\text{and}\quad M_2=\{p\in M: d_p=2\}.$$

It is clear that M_2 is an open subset of M.

LEMMA 1.2. *Let* $\mathrm{Int}(M_0)\neq\emptyset$. *Every component* V_0 *of* $\mathrm{Int}(M_0)$ *lies fully in a totally geodesic* S^3 *of* S^m.

Proof. The vectors $D_{e_1}e_4$ and $D_{e_2}e_4$ vanish identically on V_0. Since M is 2-type from equation (1.1) we conclude that $\mathrm{tr}\,A_4=\mathrm{const.}\neq 0$. So V_0 has parallel mean curvature vector in S^m. According to a result due to B.Y. Chen and S.T. Yau (see for example [3], p. 106), V_0 lies in a sphere S^3 of S^m. Then, the main result in [8] implies that V_0 is an open portion of a Riemannian product of two circles of different radii in a great sphere S^3 of S^m.

LEMMA 1.3. *Let* $\mathrm{Int}(M_1)\neq\emptyset$. *On every component* V_1 *of* $\mathrm{Int}(M_1)$ *we may choose* $e_5, e_6, \cdots, e_\alpha$ *so that* $A_5=0$ *and* $A_\alpha=0$ *for any* $\alpha\geqq 8$. *Moreover* $D_{e_1}e_4=\omega_{45}(e_1)e_5$, $D_{e_2}e_4=\omega_{45}(e_2)e_5$ *and* $\omega_{45}^2(e_1)+\omega_{45}^2(e_2)\neq 0$ *everywhere on* V_1.

Proof. The vectors $D_{e_1}e_4$, $D_{e_2}e_4$ are linearly dependent everywhere on V_1 and at least one of them is nonzero. Denote by e_5 its common unit direction. Then we have

$$D_{e_1}e_4=\omega_{45}(e_1)e_5\quad\text{and}\quad D_{e_2}e_4=\omega_{45}(e_2)e_5$$

and $\omega_{45}^2(e_1)+\omega_{45}^2(e_2)\neq 0$, everywhere on V_1. In this case from equation (1.2) we find

$$\omega_{45}(e_1)\rho_5+\omega_{45}(e_2)\mu_5=0\quad\text{and}\quad\omega_{45}(e_1)\mu_5-\omega_{45}(e_2)\rho_5=0$$

from which we obtain $\mu_5=\rho_5=0$. Thus $A_5=0$, that is e_5 nowhere belongs to the first normal space. Then we choose e_6, e_7 so that $e_6, e_7\in N_1$ and thus $e_\alpha\notin N_1$ for any $\alpha\geqq 8$.

LEMMA 1.4. *Let $M_2 \neq \emptyset$. On every component V_2 of M_2 we may choose e_5, e_6 so that*

$$D_{e_1}e_4=\omega_{45}(e_1)e_5\,,\quad D_{e_2}e_4=\omega_{45}(e_2)e_5+\omega_{46}(e_2)e_6 \quad \text{and} \quad \omega_{45}(e_1)\cdot\omega_{46}(e_2)\neq 0$$

everywhere on V_2. Moreover we may suppose that V_2 consists in open neighbourhoods V_{21}, V_{22} such that one of the following holds

$$A_5\neq 0\,,\ A_6\neq 0 \qquad \text{everywhere on } V_{21}$$

$$A_5=0 \text{ and } A_6=0 \quad \text{everywhere on } V_{22}.$$

Proof. Because the vectors $D_{e_1}e_4$, $D_{e_2}e_4$ are linearly independent everywhere on V_2, by Gram-Schmidt orthogonalization process the first part of Lemma is established. For the proof of the second part it is necessary to prove that there is no open set of V_2 where $A_5\neq 0$ and $A_6=0$ or $A_5=0$ and $A_6\neq 0$. In fact if such an open set there exists, then from equation (1.2) we have

$$\rho_5\omega_{45}(e_1)+\mu_5\omega_{45}(e_2)=0\,,\quad \mu_5\omega_{45}(e_1)-\rho_5\omega_{45}(e_2)=0$$

or

$$\omega_{46}(e_2)\mu_6=0\,,\quad \omega_{46}(e_2)\rho_6=0$$

which are impossible.

Remark 1.5. In the following we refer quite often to neighbourhoods such as V_0, V_1, V_2, V_{21} and V_{22} which are defined in the above Lemmas.

In order to state another lemma which is useful in the proof of global results, we restrict to a nowhere pseudoumbilical neighbourhood U of M. In this situation we may choose e_1, e_2 as the eigenvectors of A_4 with corresponding eigenvalues k_1, k_2 $(k_1>k_2)$. In that case we have, $k_1+k_2=\operatorname{tr} A_4=\text{const.}\neq 0$, as it follows from equation (1.1) and because M is of 2-type.

LEMMA 1.6. *Let M be a mass-symmetric 2-type surface in S^m. In a nowhere pseudoumbilical neighbourhood U the curvature form ω_{12} and the Gaussian curvature K are given by*

$$\omega_{12}=\frac{1}{2}\,\frac{e_2(k_1-k_2)}{k_1-k_2}\,\omega_1-\frac{1}{2}\,\frac{e_1(k_1-k_2)}{k_1-k_2}\,\omega_2 \tag{1.6}$$

$$K=-\frac{1}{2}\Delta\,\log(k_1-k_2)\,. \tag{1.7}$$

Proof. From our choice of the frame we have $\omega_{i4}=k_i\omega_i$, $k_1+k_2=\text{const.}$ and $\omega_{1\alpha}=\rho_\alpha\omega_1+\mu_\alpha\omega_2$, $\omega_{2\alpha}=\mu_\alpha\omega_1-\rho_\alpha\omega_2$. Because of (1.6) we find

$$\sum_{\alpha\geq 5}\omega_{1\alpha}\wedge\omega_{\alpha 4}=0 \quad\text{and}\quad \sum_{\alpha\geq 5}\omega_{2\alpha}\wedge\omega_{\alpha 4}=0$$

and thus

$$d\omega_{14}=\omega_{12}\wedge\omega_{24}, \quad d\omega_{24}=\omega_{21}\wedge\omega_{14}.$$

The last equations imply (1.6). By using (1.6) and the Gauss equation

$$K=e_2(\omega_{12}(e_1))-e_1(\omega_{12}(e_2))-(\omega_{12}(e_1))^2-(\omega_{12}(e_2))^2$$

we conclude (1.7).

2. Local results

Let M be a mass-symmetric 2-type surface in S^m, $m\geqq 5$. In this section we prove some local results.

PROPOSITION 2.1. *There is no neighbourhood of the form V_0 which lies fully in S^m.*

Proof. This is a immediate consequence of Lemma 1.2.

PROPOSITION 2.2. *Let M be a mass-symmetric 2-type surface in S^m, $m\geqq 5$. If M has a neighbourhood of the form V_1 then M is nowhere pseudoumbilical in an open set of V_1 and V_1 is flat.*

Proof. By Lemma 1.3 we may set $\omega_{45}=\beta_1\omega_1+\beta_2\omega_2$, where $\beta_1^2+\beta_2^2\neq 0$. By exterior differentiation of $\omega_{46}=0$ and $\omega_{47}=0$ we get respectively

$$-\beta_2\omega_{56}(e_1)+\beta_1\omega_{56}(e_2)=(k_1-k_2)\mu_6$$

and

$$-\beta_2\omega_{57}(e_1)+\beta_1\omega_{57}(e_2)=(k_1-k_2)\mu_7.$$

Moreover (1.4) for $\alpha=6$ and $\alpha=7$ becomes respectively

$$\beta_1\omega_{56}(e_1)+\beta_2\omega_{56}(e_2)=(k_1-k_2)\rho_6$$

and

$$\beta_1\omega_{57}(e_1)+\beta_2\omega_{57}(e_2)=(k_1-k_2)\rho_7.$$

Hence we find

$$\omega_{56}(e_1)=\frac{k_1-k_2}{\beta_1^2+\beta_2^2}(\beta_1\rho_6-\beta_2\mu_6)$$

$$\omega_{56}(e_2)=\frac{k_1-k_2}{\beta_1^2+\beta_2^2}(\beta_1\mu_6+\beta_2\rho_6)$$

$$\omega_{57}(e_1)=\frac{k_1-k_2}{\beta_1^2+\beta_2^2}(\beta_1\rho_7-\beta_2\mu_7)$$

$$\omega_{57}(e_2)=\frac{k_1-k_2}{\beta_1^2+\beta_2^2}(\beta_1\mu_7+\beta_2\rho_7).$$

Differentiation of $\omega_{15}=\omega_{25}=0$ gives

$$k_1\beta_2+\rho_6\omega_{65}(e_2)-\mu_6\omega_{65}(e_1)+\rho_7\omega_{75}(e_2)-\mu_7\omega_{75}(e_1)=0$$

$$-\beta_1k_2+\mu_6\omega_{65}(e_2)+\rho_6\omega_{65}(e_1)+\mu_7\omega_{75}(e_2)+\rho_7\omega_{75}(e_1)=0\,.$$

Substituting $\omega_{65}(e_1)$, $\omega_{65}(e_2)$, $\omega_{75}(e_1)$, $\omega_{75}(e_2)$ into the last two equations we get respectively

$$k_1\beta_2-\beta_2(\rho_6^2+\mu_6^2+\rho_7^2+\mu_7^2)\frac{k_1-k_2}{\beta_1^2+\beta_2^2}=0$$

$$k_2\beta_1+\beta_1(\rho_6^2+\mu_6^2+\rho_7^2+\mu_7^2)\frac{k_1-k_2}{\beta_1^2+\beta_2^2}=0\,.$$

Because k_1+k_2 is a nonzero constant from the last equations we infer $\beta_1\beta_2=0$. Moreover from these equations we conclude that M is nowhere pseudoumbilical in an open set of V_1. Without loss of generality we can suppose that $\beta_2=0$; then $\omega_{45}=\beta_1\omega_1$ and by exterior differentiation we find

$$\omega_{12}(e_1)=\frac{e_2(\beta_1)}{\beta_1}\,.$$

Using (1.6) we get

$$e_2\Big(\frac{\beta_1^2}{k_1-k_2}\Big)=0\,.$$

Furthermore (1.4) for $\alpha=5$ and taking account that $A_5=0$, becomes

$$e_1(\beta_1)+\beta_1\omega_{12}(e_2)=0$$

from which by using (1.6) we obtain

$$e_1\Big(\frac{\beta_1^2}{k_1-k_2}\Big)=0\,.$$

Hence there is a constant d such that $\beta_1^2=d(k_1-k_2)$. Now, using (1.3) and (1.1), we easily, deduce that k_1, k_2, β_1 are constant. The flatness of V_1 follows now from Lemma 1.6.

PROPOSITION 2.3. *Let M be a mass-symmetric 2-type surface in S^m, $m\geqq 6$. If M has a neighbourhood of form V_{21} then this lies fully in a totally geodesic sphere S^5 of S^m.*

Proof. In V_{21} by using Lemma 1.4 we set $\omega_{45}=\beta_1\omega_1+\beta_2\omega_2$ and $\omega_{46}=\gamma_2\omega_2$ where $\beta_1\gamma_2\neq 0$, everywhere in V_{21}. Moreover $\mu_5^2+\rho_5^2\neq 0$, $\mu_6^2+\rho_6^2\neq 0$ and $A_\alpha=0$ for any $\alpha\geqq 7$. Since $\omega_{1\alpha}=\omega_{2\alpha}=0$ for any $\alpha\geqq 7$, (1.2) implies

$$\beta_1\rho_5+\beta_2\mu_5+\gamma_2\mu_6=0 \tag{2.1}$$

$$\beta_1\mu_5-\beta_2\rho_5-\gamma_2\rho_6=0\,. \tag{2.2}$$

Because $\omega_{4\alpha}=0$ for any $\alpha\geqq 7$, relation (1.4) implies

(2.3) $$\beta_1\omega_{\alpha 5}(e_1)+\beta_2\omega_{\alpha 5}(e_1)+\gamma_1\omega_{\alpha 6}(e_2)+\gamma_2\omega_{\alpha 6}(e_2)=0, \quad \alpha\geqq 7.$$

On the other hand by exterior differentiation of $\omega_{4\alpha}=\omega_{1\alpha}=\omega_{2\alpha}=0$, $\alpha\geqq 7$ one finds

(2.4) $$\beta_1\omega_{\alpha 5}(e_2)-\beta_2\omega_{\alpha 5}(e_1)-\gamma_2\omega_{\alpha 6}(e_1)=0,$$

(2.5) $$\rho_5\omega_{\alpha 5}(e_1)+\mu_5\omega_{\alpha 5}(e_2)+\rho_6\omega_{\alpha 6}(e_1)+\mu_6\omega_{\alpha 6}(e_2)=0,$$

(2.6) $$-\mu_5\omega_{\alpha 5}(e_1)+\rho_5\omega_{\alpha 5}(e_2)-\mu_6\omega_{\alpha 6}(e_1)+\rho_6\omega_{\alpha 6}(e_2)=0.$$

We consider (2.3), (2.4), (2.5) and (2.6) as a linear system with respect to $\omega_{\alpha 5}(e_1)$, $\omega_{\alpha 5}(e_2)$, $\omega_{\alpha 6}(e_1)$, $\omega_{\alpha 6}(e_2)$. Denote by T the determinant of the system. We claim that T is everywhere positive on V_{21}. In fact by computation we find

$$T=(\beta_1^2+\beta_2^2)(\rho_6^2+\mu_6^2)+\gamma_2^2(\rho_5^2+\mu_5^2)+2\beta_1\gamma_2(\mu_5\rho_6-\rho_5\mu_6)-2\beta_2\gamma_2(\rho_5\rho_6+\mu_5\mu_6),$$

which by using (2.1) and (2.2) becomes

$$T=(\beta_1^2+\beta_2^2)(\rho_6^2+\mu_6^2)+\gamma_2^2(\rho_5^2+\mu_5^2)+2\gamma_2^2(\mu_6^2+\rho_6^2),$$

which proves the claim. So, the system yields $\omega_{5\alpha}=\omega_{6\alpha}=0$, for any $\alpha\geqq 7$. But $\omega_{4\alpha}=0$, for any $\alpha\geqq 7$. It follows that the first normal space which is spanned by e_4, e_5, e_6 is parallel in the normal bundle of M in S^m. So, the codimension of V_{21} is reduced to 3 by Erbacher's Theorem [6] and V_{21} lies in a totally geodesic sphere S^5 of S^m.

Since in the following we shall apply at times the result of Y. Miyata [12] the following lemma is necessary.

LEMMA 2.4. *Let M be a mass-symmetric 2-type surface in S^m Then* $0<\lambda_1<2<\lambda_2$.

Proof. Taking account of (1.1) one finds

$$k_1^2+k_2^2=\frac{(k_1+k_2)^2}{2}+\frac{(k_1-k_2)^2}{2}=\lambda_1+\lambda_2-\frac{\lambda_1\lambda_2}{2}-2+\frac{(k_1-k_2)^2}{2}$$

and because of (1.3) we have

$$\frac{(k_1-k_2)^2}{2}+|De_4|^2=\frac{\lambda_1\lambda_2}{2},$$

which proves the assertion, by using (1.1).

Now we are able to prove the following.

THEOREM 2.5. *Let M be a mass-symmetric 2-type surface in S^m, $m\geqq 5$. If* $\dim N_1\leqq 2$ *and M lies fully in S^m then m is odd unless M is pseudoumbilical.*

Proof. If $m=5$ there is nothing to prove. So, we may suppose $m\geqq 6$. Because of Propositions 2.1 and 2.3 M cannot has neighbourhoods of form V_0,

V_{21} which are fully in S^m with $m \geqq 6$. So, there is neighbourhood of form V_1 or of form V_{22} which lies fully in S^m with $m \geqq 6$. In the first case V_1 must be flat by Proposition 2.2. But then a result due to Y. Miyata ([12], Theorem and its Corollary) implies that m is odd.

From now on we suppose that M has a neighbourhood of form V_{22} nowhere pseudoumbilical which lies fully in S^m with $m \geqq 6$. Moreover, because of $\dim N_1 \leqq 2$ and $e_4 \in N_1$ we may assume that $\rho_8 = \mu_8 = 0$ everywhere on V_{22}. We set $\omega_{45} = \beta_1 \omega_1 + \beta_2 \omega_2$ and $\omega_{46} = \gamma_2 \omega_2$, where $\beta_1 \gamma_2 \neq 0$ by Lemma 1.4. Taking exterior differentiation of $\omega_{15} = \omega_{25} = 0$, we get

$$\rho_7 \omega_{75}(e_1) + \mu_7 \omega_{75}(e_2) = \beta_1 k_2$$

$$-\mu_7 \omega_{75}(e_1) + \rho_7 \omega_{75}(e_2) = -k_1 \beta_2 .$$

Similarly, from $\omega_{16} = \omega_{26} = 0$ we obtain

$$\rho_7 \omega_{76}(e_1) + \mu_7 \omega_{76}(e_2) = 0$$

$$-\mu_7 \omega_{76}(e_1) + \rho_7 \omega_{76}(e_2) = -k_1 \gamma_2 .$$

We claim that $\rho_7^2 + \mu_7^2 \neq 0$ on an open subset of V_{22}. Assume in the contrary that $\rho_7^2 + \mu_7^2 = 0$ in an open subset $V \subset V_{22}$. Then it follows from the above relations that $k_1 \cdot k_2 = 0$ and since $k_1 + k_2 = \text{const.}$ we would have from Lemma 1.6 that $K = 0$. On the other hand from Gauss equation we find $K = 1$ which is impossible. Therefore we get

$$\omega_{75}(e_1) = \frac{1}{\rho_7^2 + \mu_7^2} (k_2 \beta_1 \rho_7 + k_1 \beta_2 \mu_7)$$

$$\omega_{75}(e_2) = \frac{1}{\rho_7^2 + \mu_7^2} (-k_1 \beta_2 \rho_7 + k_2 \beta_1 \mu_7)$$

$$\omega_{76}(e_1) = \frac{1}{\rho_7^2 + \mu_7^2} k_1 \gamma_2 \mu_7$$

$$\omega_{76}(e_2) = -\frac{1}{\rho_7^2 + \mu_7^2} k_1 \gamma_2 \rho_7 .$$

Differentiating $\omega_{47} = 0$ we get

$$(k_2 - k_1) \mu_7 + \beta_1 \omega_{57}(e_2) - \beta_2 \omega_{57}(e_1) - \gamma_2 \omega_{67}(e_1) = 0 .$$

Moreover (1.4) for $\alpha = 7$ becomes

$$(k_1 - k_2) \rho_7 + \beta_1 \omega_{75}(e_1) + \beta_2 \omega_{75}(e_2) + \gamma_2 \omega_{76}(e_2) = 0 .$$

Substituting $\omega_{75}(e_1)$, $\omega_{75}(e_2)$, $\omega_{76}(e_1)$ and $\omega_{76}(e_2)$ into the last equations we obtain

$$(2.7) \qquad (k_1 - k_2) \mu_7 + \frac{1}{\rho_7^2 + \mu_7^2} (-k_1 \beta_1 \beta_2 \rho_7 + k_2 \beta_1^2 \mu_7 - k_2 \beta_1 \beta_2 \rho_7 - k_1 \beta_2^2 \mu_7 - k_1 \gamma_2^2 \mu_7) = 0$$

(2.8) $$(k_1-k_2)\rho_7+\frac{1}{\rho_7^2+\mu_7^2}(k_2\beta_1^2\rho_7+k_1\beta_1\beta_2\mu_7-k_1\beta_2^2\rho_7-k_2\beta_1\beta_2\mu_7-k_1\gamma_2^2\mu_7)=0.$$

Eliminating the terms $(k_1-k_2)\mu_7$ and $(k_1-k_2)\rho_7$ from (2.7) and (2.8) we find $\beta_1\beta_2=0$ and thus $\beta_2=0$. Hence (2.7) and (2.9) give

(2.9) $$k_1-k_2+\frac{k_2\beta_1^2-k_1\gamma_2^2}{\rho_7^2+\mu_7^2}=0.$$

Differentiating $\omega_{45}=\beta_1\omega_1$ we obtain

$$-e_2(\beta_1)+\beta_1\omega_{12}(e_1)=\gamma_2\omega_{56}(e_1)$$

and using (1.6) we get

$$e_2\Big(\frac{\beta_1^2}{k_1-k_2}\Big)=-\frac{2\beta_1\gamma_2}{k_1-k_2}\omega_{56}(e_1).$$

On the other hand (1.4) for $\alpha=6$ on account of $\beta_2=0$ and (1.6) becomes

$$e_2\Big(\frac{\gamma_2^2}{k_1-k_2}\Big)=\frac{2\beta_1\gamma_2}{k_1-k_2}\omega_{65}(e_1).$$

Therefore

$$e_2\Big(\frac{\beta_1^2-\gamma_2^2}{k_1-k_2}\Big)=0.$$

By a similar argument on $\omega_{46}=\gamma_2\omega_2$, (1.4) for $\alpha=5$ and (1.6) we also have

$$e_1\Big(\frac{\beta_1^2-\gamma_2^2}{k_1-k_2}\Big)=0.$$

Consequently,

(2.10) $$\beta_1^2-\gamma_2^2=d(k_1-k_2)$$

where d is a constant. Eliminating β_1 and γ_2 from (1.3), (2.9) and (2.10) we get

$$2(\rho_7^2+\mu_7^2)+k_1^2+k_2^2=\lambda_1+\lambda_2-2-d(k_1+k_2)$$

which implies that the Gaussian curvature of V_{22} is constant, since k_1+k_2=const. Using again the result of Y. Miyata [12] we infer that m is odd and the proof is completed.

COROLLARY 2.6. *Let M be a mass-symmetric 2-type surface in S^m. If M has flat normal connection and lies fully in S^m then m is odd unless M is pseudoumbilical.*

Proof. It is obvious that the flatness of the normal connection implies that $\dim N_1\leqq 2$.

3. Global results

In this section we give two global results.

THEOREM 3.1. *Let M be a complete mass-symmetric 2-type surface in S^m. If M has non-negative Gaussian curvature and lies fully in S^m then it is either pseudoumbilical or flat. In the last case m is odd.*

Proof. Since M is complete with Gaussian curvature $K \geqq 0$ by a theorem of A. Huber [10] we know that M is either compact or parabolic. At first we assume that M is compact. If K is not identically zero then M is homeomorphic to S^2. Therefore by a result of M. Kotani [11] m is odd. If $K=0$ identically, then the conclusion follows from the result of Y. Miyata [12].

Now assume that M is non-compact and parabolic. We know from Lemma 1.1 that A_4 is a Codazzi tensor with constant trace, by using a result of B. Wegner [13, Satz 1] we get

$$-\frac{1}{2}\Delta \operatorname{tr} A_4^2 = K(k_1-k_2)^2 + |\nabla A_4|^2. \tag{3.1}$$

Since $K \geqq 0$ the above equation implies that $\operatorname{tr} A_4^2$ is a subharmonic function on M. On the other hand from Gauss equation we obtain

$$\operatorname{tr} A_4^2 \leqq 4(|H|^2+1).$$

Therefore $\operatorname{tr} A_4^2$ is bounded from above. Consequently $\operatorname{tr} A_4^2$ is a constant since M is parabolic and thus k_1, k_2 are constants. It follows then from (3.1) that $K(k_1-k_2)^2=0$. So M is either pseudoumbilical or nowhere pseudoumbilical. In the last case using Lemma 1.1 we infer that M is flat and applying once more the result of Y. Miyata [12] we conclude that m is odd.

THEOREM 3.2. *Let M be a compact mass-symmetric 2-type surface in S^m with non-positive Gauss curvature K. If M is nowhere pseudoumbilical and lies fully in S^m then it is flat and m is odd.*

Proof. From our assumption and Lemma 1.1 we have $\Delta \log(k_1-k_2) \geqq 0$ on M. By the compactness of M and the maximum principle we obtain $k_1-k_2=$ const. Since $k_1+k_2=$const. we get that k_1, k_2 are constant on M. Appealing again to Lemma 1.1 we deduce $K=0$ on M and so on.

4. A non-existence theorem

In this paragraph we prove the following non-existence result.

THEOREM 4.1. *There exist no mass-symmetric 2-type surfaces which lie fully in S^6.*

Proof. By the proof of Theorem 2.5 it is enough to consider the case when there is a neighbourhood of the form V_{22} where M is pseudoumbilical. By Lemma 1.4 we may suppose that

$$A_4=\begin{pmatrix} k_1 & 0 \\ 0 & k_1 \end{pmatrix}, \quad A_5=A_6=0, \quad A_7=\begin{pmatrix} \rho_7 & 0 \\ 0 & -\rho_7 \end{pmatrix}$$

and $\omega_{45}=\beta_1\omega_1+\beta_2\omega_2$, $\omega_{46}=\gamma_2\omega_2$ where $\beta_1\gamma_2\neq 0$ everywhere on V_{22} and k_1 is a nonzero constant. Differentiating $\omega_{15}=0$, $\omega_{25}=0$, $\omega_{16}=0$, $\omega_{26}=0$ we get respectively

(4.1) $$\rho_7\omega_{75}(e_2)=-k_1\beta_2$$

(4.2) $$\rho_7\omega_{75}(e_1)=k_1\beta_1$$

(4.3) $$\rho_7\omega_{76}(e_2)=-k_1\gamma_2$$

(4.4) $$\rho_7\omega_{76}(e_1)=0\,.$$

It is obvious that ρ_7 is nonzere everywhere on V_{22}.

Taking exterior differentiation of $\omega_{47}=0$ we obtain

(4.5) $$\beta_1\omega_{75}(e_2)-\beta_2\omega_{75}(e_1)-\gamma_2\omega_{76}(e_1)=0\,.$$

Furthermore (1.4) for $\alpha=7$ becomes

(4.6) $$\beta_1\omega_{75}(e_1)+\beta_2\omega_{75}(e_2)+\gamma_2\omega_{76}(e_2)=0\,.$$

Substituting $\omega_{75}(e_1)$, $\omega_{75}(e_2)$, $\omega_{76}(e_1)$, $\omega_{76}(e_2)$ from (4.1), (4.2), (4.3) and (4.4) into (4.5) and (4.6) we get respectively

$$\beta_1\beta_2=0 \quad \text{and} \quad \beta_1^2-\beta_2^2=0\,.$$

From the first of the above equations we conclude that $\beta_2=0$ and so $\beta_1^2=\gamma_2^2$. Then (1.3) implies that β_1 and γ_2 are nonzero constants.

Taking, now, exterior differentiation of $\omega_{45}=\beta_1\omega_1$ and $\omega_{46}=\gamma_2\omega_2$ we get $\omega_{56}=\pm\omega_{12}$ and so, using $d\omega_{12}=-K\omega_1\wedge\omega_2$, we obtain

$$K=\pm\beta_1^2\Big(\frac{k_1^2}{\rho_7^2}-1\Big)\,.$$

On the other hand

$$K=1+k_1^2-\rho_7^2\,.$$

Comparing these equations we deduce that ρ_7 is a constant and moreover K is a nonzero constant. So we have a mass-symmetric 2-type surface with a constant Gauss curvature which lies fully in S^6 a contradiction of Y. Miyata [12] result. This completes the proof.

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