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A SHORT ANALYTIC PROOF OF CLOSEDNESS OF LOGARITHMIC FORMS

Dedicated to Professor Nobuyuki Suita on the occasion of his 60th birthday

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§ 1. Introduction

Deligne [D, (3.2.14)] proved the d-closedness of logarithmic forms on a smooth complex quasi-projective variety by showing the degeneracy of some spectral sequence. Actually, his proof works for a Zariski open subspace of a compact Kähler manifold. The logarithmic forms play important roles in various aspects of analytic-algebraic geometry, including the value distribution of holomorphic mappings (see, e.g., [D], [I], [N1], [N2], [N3] and [N4]), and their closedness is fundamental. Therefore it may be of worth, and hence our purpose of this note to give its short proof based only on the classical harmonic integral theory [K].

Let D be an effective reduced divisor on a compact Kähler manifold M, and $\Omega_M^p(\log D)$ the sheaf of germs of logarithmic p-forms along D over M (see § 2 for the definition).

THEOREM. *Let $\omega \in H^0(M, \Omega_M^p(\log D))$ be a global section of $\Omega_M^p(\log D)$. Then $d\omega=0$ in the complement $M \backslash D$ of D.*

§ 2. Definitions and Lemmas

In the present note we denote by M an m-dimensional compact Kähler manifold with structure sheaf $\mathcal{O}_M$ of germs of holomorphic functions. Let Ω_M^p denote the sheaf of germs of holomorphic p-forms over M. Let D be an effective reduced divisor on M. Let $x \in M$ and take irreducible $\sigma_j \in \mathcal{O}_{M,x}$, $1 \leqq j \leqq k$, so that $\{\sigma_j=0\}$ define the local irreducible components of D at x. Then we define the sheaf $\Omega_M^1(\log D)$ of germs of logarithmic 1 forms along D by

$$\Omega_{M,x}^1(\log D)=\sum_{j=1}^{k} \mathcal{O}_{M,x} \frac{d\sigma_j}{\sigma_j}+\Omega_{M,x}^1 .$$

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Moreover, we set

$$\Omega_{M,x}^{p}(\log D)=\Omega_{M,x}^{1}(\log D)\wedge \cdots \wedge \Omega_{M,x}^{1}(\log D) \quad (p\text{-times}).$$

Let $\pi: \tilde{M}\to M$ be a desingularization of D by Hironaka such that $\tilde{M}$ is Kähler and the divisor $\tilde{D}=\pi^{-1}(D)$ has only simple normal crossings. Then it easily follows from the extension of holomorphic functions over analytic subsets of codimension $\geqq 2$ that the pull-back

$$\pi^*: H^0(M, \Omega_M^p(\log D)) \longrightarrow H^0(\tilde{M}, \Omega_{\tilde{M}}^p(\log \tilde{D}))$$

is an isomorphism. Henceforth we may assume that

D has only simple normal croossings.

Let $D=\sum_{i=1}^{l} D_i$ be the decomposition into irreducible components, and $\iota_{D_i}: D_i\to M$ be the inclusions. Take $x\in D_i$. Then there is a holomorphic local coordinate system $(x^1, \cdots, x^m)$ in a neighborhood U of x such that $\{x^1=0\}=D_i\cap U$. For $\omega\in H^0(M, \Omega_M^p(\log D))$ we write

$$\omega=\frac{dx^1}{x^1}\wedge\eta+\omega' \qquad \text{in } U,$$

so that $\eta\in H^0(U, \Omega_M^{p-1}(\log \sum_{j\neq i} D_j))$ and $\omega'\in H^0(U, \Omega_M^p)$ do not contain dx^1. Put

$$\operatorname{Res}_{D_i}(\omega)=\iota_{D_i}^*(\eta) \qquad \text{in } D_i\cap U.$$

Then $\operatorname{Res}_{D_i}(\omega)$ is well-defined globally, and then

$$\operatorname{Res}_{D_i}(\omega)\in H^0(D_i, \Omega_{D_i}^{p-1}(\log \sum_{j\neq i} D_j\cap D_i)). \tag{2.1}$$

We also consider $\operatorname{Res}_{D_i}(\omega)$ as a current of type $(p, 1)$ on M, and set

$$\operatorname{Res}(\omega)=\sum_i \operatorname{Res}_{D_i}(\omega),$$

which is called the Poincaré residue. We denote by $[\omega]$ the current of type $(p, 0)$ defined by ω. By Stokes's theorem we have

$$\operatorname{Res}(\omega)=\frac{1}{2\pi i}\bar{\partial}[\omega] \quad \text{(in the sense of currents)}. \tag{2.2}$$

Take $j\neq i$. If $D_j\cap D_i=\emptyset$, then we define

$$\operatorname{Res}_{D_j\cdot D_i}(\omega)=\operatorname{Res}_{D_j}(\operatorname{Res}_{D_i}(\omega))=0.$$

Otherwise, we define

$$\operatorname{Res}_{D_j\cdot D_i}(\omega)=\operatorname{Res}_{D_j\cap D_i}(\operatorname{Res}_{D_i}(\omega)).$$

Let $x\in D_j\cap D_i$ and take a holomorphic local coordinate system $(x^1, \cdots, x^m)$ such that $\{x^1=0\}=D_i$ and $\{x^2=0\}=D_j$, locally. Then we write

$$\omega=\frac{dx^1}{x^1}\wedge\eta_1+\frac{dx^2}{x^2}\wedge\eta_2+\frac{dx^1}{x^1}\wedge\frac{dx^2}{x^2}\wedge\eta_3\,,$$

where η_ν, $1\leqq\nu\leqq 3$, do not contain dx^1 nor dx^2. Then

$$\mathrm{Res}_{D_j\cdot D_i}(\omega)=\iota^*_{D_j\cap D_i}\eta_3 \qquad \text{around } x.$$

Thus we have

$$\text{(2.3)}\qquad \begin{cases} \mathrm{Res}_{D_j\cdot D_i}(\omega)\in H^0(D_j\cap D_i,\ \Omega^{p-2}_{D_j\cap D_i}(\log \sum_{k\neq j,i} D_k\cap D_j\cap D_i))\,, \\ \mathrm{Res}_{D_j\cdot D_i}(\omega)+\mathrm{Res}_{D_i\cdot D_j}(\omega)=0\,. \end{cases}$$

We recall a theorem from the theory of harmonic integrals. We fix a Kähler metric on M. As usual, we denote by H the harmonic projection, by G the Green operator, by δ (resp. ϑ, $\bar\vartheta$), the adjoint of d (resp. ∂, $\bar\partial$), and by Λ the adjoint of the multiplication operator by the Kähler form (see [W]).

(2.4) LEMMA ([K, Theorem 1.1.1]). *Let T be a current of type $(p, q+1)$ on M such that $HT=dT=0$. Then the current*

$$\Theta=\frac{1}{i}(d\Lambda+i\delta)GT$$

is of type (p, q) and satisfies $d\Theta=T$, so that $d\Theta=0$ in $M\backslash\mathrm{supp}\,T$, where $\mathrm{supp}\,T$ *denotes the support of T.*

§ 3. Proof of Theorem

We prove Theorem by induction on p, where M may be arbitrary. We keep the notation in § 2.

(i) The case of $p=1$. Put $T=\bar\partial[\omega]$. Then by (2.2) and (2.1) we have

$$T=2\pi i\,\mathrm{Res}\,(\omega)=\sum_i a_i D_i\,,\qquad a_i\in \boldsymbol{C}\,.$$

Therefore, $HT=dT=0$ and $\mathrm{supp}\,T\subset D$. It follows from Lemma (2.4) that

$$\Theta=\frac{1}{i}(d\Lambda+i\delta)GT$$

is of type (1, 0) and satisfies

$$\text{(3.1)}\qquad d\Theta=T\,,\qquad \text{so that } d\Theta=0 \text{ in } M\backslash D\,.$$

We put

$$S=[\omega]-\Theta\,,$$

and claim that S is harmonic; i.e., $\Delta S=0$, where Δ stands for the Laplace operator. Take an arbitrary differential form α of type $(m-1, m)$ on M. Then we have

$$\begin{aligned}\langle \Delta S, \alpha\rangle &= \int_M S\wedge \Delta\alpha = 2\int_M S\wedge(\bar\partial\bar\vartheta\alpha+\bar\vartheta\bar\partial\alpha)\\ &= 2\int_M S\wedge\bar\partial\bar\vartheta\alpha = 2\Big(\int_M \omega\wedge\bar\partial\bar\vartheta\alpha - \langle\Theta, \bar\partial\bar\vartheta\alpha\rangle\Big)\\ &= 2\Big(\int_M \omega\wedge\bar\partial\bar\vartheta\alpha - \langle\Theta, d\bar\vartheta\alpha\rangle\Big) = 2(-\langle T, \bar\vartheta\alpha\rangle + \langle d\Theta, \bar\vartheta\alpha\rangle)\\ &= -2\langle T-d\Theta, \bar\vartheta\alpha\rangle = 0 \qquad \text{by (3.1)}.\end{aligned}$$

Thus S is smooth everywhere and $dS=0$. Combining this with (3.1), we deduce that $d\omega=0$ in $M\backslash D$.

(ii) Assume that Theorem holds for $p-1$ $(p\geqq 2)$ for an arbitrary compact Kähler manifold. Take $\omega\in H^0(M, \Omega_M^p(\log D))$, and put

$$T=\bar\partial[\omega].$$

To use Lemma (2.4), we show that

$$HT=dT=0. \tag{3.2}$$

The first one is trivial. We show the second. Let α be an arbitrary $(2n-p-1)$-form on M. Then

$$\begin{aligned}\langle dT, \alpha\rangle &= (-1)^p\langle T, d\alpha\rangle\\ &= (-1)^p 2\pi i\langle \mathrm{Res}(\omega), d\alpha\rangle\\ &= 2\pi i(-1)^p\sum_i\int_{D_i}\mathrm{Res}_{D_i}(\omega)\wedge d\iota_{D_i}^*\alpha.\end{aligned}$$

It follows from (2.1) and the induction hypothesis that

$$d\,\mathrm{Res}_{D_i}(\omega)=0 \qquad \text{in } D_i\backslash\sum_{j\neq i}D_j.$$

Therefore we get

$$\begin{aligned}\langle dT, \alpha\rangle &= 2\pi i(-1)^p\sum_i(-1)^{p-1}\int_{D_i}d(\mathrm{Res}_{D_i}(\omega)\wedge\iota_{D_i}^*\alpha)\\ &= (2\pi i)^2\sum_i\sum_{j\neq i}\int_{D_j\cap D_i}\mathrm{Res}_{D_j\cdot D_i}(\omega)\wedge\iota_{D_j\cap D_i}^*\alpha\\ &= -4\pi^2\sum_{j\neq i}\langle\mathrm{Res}_{D_j\cdot D_i}(\omega), \alpha\rangle = 0 \qquad \text{by (2.3)}.\end{aligned}$$

Applying Lemma (2.4), we see that

$$\Theta=\frac{1}{i}(d\Lambda+i\delta)GT$$

is of type $(p, 0)$ and satisfies $d\Theta=T$. In the same way as in (i) we infer that $S=[\omega]-\Theta$ is harmonic, and so closed. Thus $d\omega=0$ in $M\backslash D$.

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