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PICARD CONSTANTS OF FOUR-SHEETED ALGEBROID SURFACES, II

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§1. Introduction

Under the same title we have reported a paper, in which the Picard constant of several four-sheeted algebroid surfaces with $P(y)=7$ has been decided. In this paper we shall continue the same work for four-sheeted algebroid surfaces with $P(y)=6$. Again the discriminant of surfaces with $P(y)=6$ plays a very important role in this paper.

In the first place we decide several four-sheeted algebroid surfaces with $P(y)=6$. We classify into representative surfaces by a linear transformation $\alpha y+\beta$. Next we compute their discriminants of representative surfaces. This process need a little bit hard work. Finally we get theorems, which decide the Picard constant.

§2. Surfaces with $P(y)=6$

Let us consider the four-sheeted algebroid surface defined by

$$F(z, y)\equiv y^4-S_1y^3+S_2y^2-S_3y+S_4=0.$$

By Rémoundos' theorem we consider the following equations:

$$\begin{array}{ccccccc} & & \text{(i)} & & \text{(ii)} & & \text{(iii)} \\ \begin{pmatrix} F(z, 0) \\ F(z, a_1) \\ F(z, a_2) \\ F(z, a_3) \\ F(z, a_4) \end{pmatrix} & = & \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \end{pmatrix}, & = & \begin{pmatrix} c_1 \\ c_2 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix}, & = & \begin{pmatrix} c_1 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \\ \beta_4 e^{H_4} \end{pmatrix}, \end{array}$$

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$$\text{(iv)} \quad = \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ c_2 \\ c_3 \\ \beta_2 e^{H_2} \end{pmatrix}, \quad \text{(v)} \quad = \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ c_2 \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix}, \quad \text{(vi)} \quad = \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \\ \beta_4 e^{H_4} \end{pmatrix},$$

where c_1, c_2, c_3 are non-zero constants, $\beta_1, \beta_2, \beta_3, \beta_4$ non-zero constants and H_1, H_2, H_3, H_4 are non-constant entire functions satisfying $H_j(0)=0$.

CASE (i). Then $S_4=c_1$ and

$$\begin{cases} a_1^4-S_1a_1^3+S_2a_1^2-S_3a_1+c_1=c_2, \\ a_2^4-S_1a_2^3+S_2a_2^2-S_3a_2+c_1=c_3, \\ a_3^4-S_1a_3^3+S_2a_3^2-S_3a_3+c_1=\beta_1e^{H_1}, \\ a_4^4-S_1a_4^3+S_2a_4^2-S_3a_4+c_1=\beta_2e^{H_2}. \end{cases}$$

Let us put

$$x_1=\frac{c_1}{a_1a_2a_3}, \quad x_2=\frac{c_2}{a_1(a_1-a_2)(a_3-a_1)}, \quad x_3=\frac{c_3}{a_2(a_1-a_2)(a_2-a_3)}$$

and

$$x_0=\frac{\beta_1}{a_3(a_2-a_3)(a_3-a_1)}.$$

Then from the first three equations

$$\begin{cases} S_1=x_1+x_2+x_3+a_1+a_2+a_3+x_0e^{H_1}, \\ S_2=(a_1+a_2+a_3)x_1+(a_2+a_3)x_2+(a_1+a_3)x_3+a_1a_2+a_2a_3+a_1a_3+(a_1+a_2)x_0e^{H_1}, \\ S_3=(a_1a_2+a_2a_3+a_1a_3)x_1+a_2a_3x_2+a_1a_3x_3+a_1a_2a_3+a_1a_2x_0e^{H_1}. \end{cases}$$

Put these into $F(z, a_4)=\beta_2e^{H_2}$. Making use of Borel's unicity theorem, we have $H_1=H_2(=H)$ and

$$x_0=\frac{\beta_2}{a_4(a_2-a_4)(a_4-a_1)},$$

$$\frac{x_1}{a_4}+\frac{x_2}{a_4-a_1}+\frac{x_3}{a_4-a_2}=1.$$

We impose the following condition: $F(z, a)=\beta e^L$ does not hold excepting $a=a_3, a_4$. Now

$$F(z, \alpha)=(\alpha-a_3)[-x_1(\alpha-a_1)(\alpha-a_2)-x_2\alpha(\alpha-a_2)$$
$$-x_3\alpha(\alpha-a_1)+\alpha(\alpha-a_1)(\alpha-a_2)]$$
$$-x_0e^H\alpha(\alpha-a_1)(\alpha-a_2).$$

Therefore we have three cases:

(a) $\alpha(\alpha-a_1)(\alpha-a_2)-x_1(\alpha-a_1)(\alpha-a_2)-x_2\alpha(\alpha-a_2)-x_3\alpha(\alpha-a_1)=(\alpha-a_4)^3$,

(b) $=(\alpha-a_3)(\alpha-a_4)^2$,

(c) $=(\alpha-a_3)^2(\alpha-a_4)$.

CASE (a). Then

$$3a_4=a_1+a_2+x_1+x_2+x_3,$$
$$3a_4^2=a_1a_2+(a_1+a_2)x_1+a_2x_2+a_1x_3$$

and

$$a_4^3=x_1a_1a_2.$$

Thus

$$c_1=a_3a_4^3,\quad c_2=(a_3-a_1)(a_4-a_1)^3,\quad c_3=(a_3-a_2)(a_4-a_2)^3.$$

Hence

$$\begin{cases} S_1=3a_4+a_3+x_0e^H, \\ S_2=3a_4(a_4+a_3)+(a_1+a_2)x_0e^H, \\ S_3=a_4^2(a_4+3a_3)+a_1a_2x_0e^H, \\ S_4=c_1=a_3a_4^3. \end{cases}$$

This surface is denoted by R_1.

CASE (b). Then

$$2a_4+a_3=a_1+a_2+x_1+x_2+x_3,$$
$$a_4^2+2a_3a_4=a_1a_2+(a_1+a_2)x_1+a_2x_2+a_1x_3,$$
$$a_3a_4^2=x_1a_1a_2=c_1/a_3.$$

Hence

$$c_1=a_3^2a_4^2,\quad c_2=(a_3-a_1)^2(a_4-a_1)^2,\quad c_3=(a_3-a_2)^2(a_4-a_2)^2.$$

Thus

$$\begin{cases} S_1=2a_3+2a_4+x_0e^H, \\ S_2=a_3^2+4a_3a_4+a_4^2+(a_1+a_2)x_0e^H, \\ S_3=2a_3a_4(a_3+a_4)+a_1a_2x_0e^H, \\ S_4=c_1=a_3^2a_4^2 \end{cases}$$

This surface is denoted by R_2.

CASE (c). Then

$$2a_3+a_4=a_1+a_2+x_1+x_2+x_3,$$

$$a_3^2+2a_3a_4=a_1a_2+(a_1+a_2)x_1+a_2x_2+a_1x_3$$

and

$$a_3^2a_4=x_1a_1a_2=c_1/a_3.$$

Thus

$$c_1=a_3^3a_4, \quad c_2=(a_3-a_1)^3(a_4-a_1), \quad c_3=(a_3-a_2)^3(a_4-a_2).$$

Hence

$$\begin{cases} S_1=3a_3+a_4+x_0e^H, \\ S_2=3a_3(a_3+a_4)+(a_1+a_2)x_0e^H, \\ S_3=3a_3^2a_4+a_3^3+a_1a_2x_0e^H, \\ S_4=c_1=a_3^3a_4. \end{cases}$$

This surface is denoted by R_3.

CASE (iv). Then $S_4=\beta_1e^{H_1}$ and

$$\begin{cases} a_1^4-S_1a_1^3+S_2a_1^2-S_3a_1+\beta_1e^{H_1}=c_1 \\ a_2^4-S_1a_2^3+S_2a_2^2-S_3a_2+\beta_1e^{H_1}=c_2 \\ a_3^4-S_1a_3^3+S_2a_3^2-S_3a_3+\beta_1e^{H_1}=c_3 \\ a_4^4-S_1a_4^3+S_2a_4^2-S_3a_4+\beta_1e^{H_1}=\beta_2e^{H_2}. \end{cases}$$

Let us put

$$x=\frac{c_1}{a_1(a_1-a_2)(a_1-a_3)}, \quad y=\frac{c_2}{a_2(a_1-a_2)(a_2-a_3)}, \quad z=\frac{c_3}{a_3(a_1-a_3)(a_2-a_3)}$$

and

$$x_0=\frac{\beta_1}{a_1a_2a_3}.$$

Then

$$\begin{cases} S_1=x_0e^{H_1}-(x-y+z-a_1-a_2-a_3), \\ S_2=(a_1+a_2+a_3)x_0e^{H_1}-\{(a_2+a_3)x-(a_1+a_3)y+(a_1+a_2)z \\ \qquad\qquad -a_1a_2-a_1a_3-a_2a_3\}, \\ S_3=(a_1a_2+a_1a_3+a_2a_3)x_0e^{H_1}-\{a_2a_3x-a_1a_3y+a_1a_2z-a_1a_2a_3\}. \end{cases}$$

We substitute these into $F(z, a_4)=\beta_2 e^{H_2}$. Then Borel's unicity theorem implies that

$$H_1=H_2(=H),$$

$$\beta_2=x_0(a_1-a_4)(a_2-a_4)(a_3-a_4),$$

$$\frac{x}{a_4-a_1}-\frac{y}{a_4-a_2}+\frac{z}{a_4-a_3}=-1.$$

We now impose the following condition: $F(z, \alpha)$ does not have any lacunary value of the second kind excepting at most 0 and a_4. Let us put

$$F(z, \alpha)=-(\alpha-a_1)(\alpha-a_2)(\alpha-a_3)x_0e^H+\alpha P(\alpha),$$

where

$$\begin{aligned}P(\alpha)=&\alpha^3+\alpha^2(x-y+z-a_1-a_2-a_3)\\&+\alpha\{-(a_2+a_3)x+(a_1+a_3)y-(a_1+a_2)z+a_1a_2+a_1a_3+a_2a_3\}\\&+a_2a_3x-a_1a_3y+a_1a_2z-a_1a_2a_3.\end{aligned}$$

Then there are three possible cases:

(a) $P(\alpha)=\alpha^2(\alpha-a_4)$, (b) $P(\alpha)=\alpha(\alpha-a_4)^2$, (c) $P(\alpha)=(\alpha-a_4)^3$.

We now consider Case (a). Then

$$\begin{cases}x-y+z-a_1-a_2-a_3=-a_4,\\(a_2+a_3)x-(a_1+a_3)y+(a_1+a_2)z=a_1a_2+a_1a_3+a_2a_3,\\a_2a_3x-a_1a_3y+a_1a_2z=a_1a_2a_3.\end{cases}$$

Hence

$$\begin{cases}S_1=x_0e^H+a_4,\\S_2=(a_1+a_2+a_3)x_0e^H,\\S_3=(a_1a_2+a_1a_3+a_2a_3)x_0e^H,\\S_4=a_1a_2a_3x_0e^H.\end{cases}$$

This surface is denoted by R_{16}.

Next we consider Case (b). Then

$$\begin{cases}x-y+z-a_1-a_2-a_3=-2a_4,\\-(a_2+a_3)x+(a_1+a_3)y-(a_1+a_2)z+a_1a_2+a_1a_3+a_2a_3=a_4^2,\\a_2a_3x-a_1a_3y+a_1a_2z-a_1a_2a_3=0.\end{cases}$$

Hence

$$\begin{cases} S_1 = x_0 e^H + 2a_4, \\ S_2 = (a_1 + a_2 + a_3) x_0 e^H + a_4^2, \\ S_3 = (a_1 a_2 + a_1 a_3 + a_2 a_3) x_0 e^H, \\ S_4 = a_1 a_2 a_3 x_0 e^H. \end{cases}$$

This surface is denoted by R_{17}.

We finally consider Case (c). Then

$$\begin{cases} x - y + z - a_1 - a_2 - a_3 = -3a_4, \\ -(a_2 + a_3)x + (a_1 + a_3)y - (a_1 + a_2)z + a_1 a_2 + a_1 a_3 + a_2 a_3 = 3a_4^2, \\ a_2 a_3 x - a_1 a_3 y + a_1 a_2 z - a_1 a_2 a_3 = -a_4^3. \end{cases}$$

Hence

$$\begin{cases} S_1 = x_0 e^H + 3a_4, \\ S_2 = (a_1 + a_2 + a_3) x_0 e^H + 3a_4^2, \\ S_3 = (a_1 a_2 + a_1 a_3 + a_2 a_3) x_0 e^H + a_4^3, \\ S_4 = a_1 a_2 a_3 x_0 e^H. \end{cases}$$

This surface is denoted by R_{18}.

Let $F(z, y)$ be $y^4 - S_1 y^3 + S_2 y^2 - S_3 y + S_4 = 0$. Let us put $y = \alpha Y + \beta$. Then

$$\begin{aligned} \alpha^4 G(z, Y) &\equiv F(z, \alpha Y + \beta) \\ &\equiv \alpha^4 (Y^4 - T_1 Y^3 + T_2 Y^2 - T_3 Y + T_4). \end{aligned}$$

$$T_1 = \frac{1}{\alpha}(S_1 - 4\beta),$$

$$T_2 = \frac{1}{\alpha^2}(S_2 - 3S_1\beta + 6\beta^2),$$

$$T_3 = \frac{1}{\alpha^3}(S_3 - 2S_2\beta + 3S_1\beta^2 - 4\beta^3)$$

and

$$T_4 = \frac{1}{\alpha_4}(S_4 - S_3\beta + S_2\beta^2 - S_1\beta^3 + \beta^4).$$

Now we put

$$\begin{cases} \alpha A_1+\beta=0, & \beta=a_4 \\ \alpha A_2=a_1-a_4, \\ \alpha A_3=a_2-a_4, \\ \alpha A_4=a_3-a_4. \end{cases}$$

Then we have $R_1\sim R_{16}$, $R_2\sim R_{17}$, $R_3\sim R_{18}$.

If we put

$$\begin{cases} \alpha A_1+\beta=0, & \beta=a_1 \\ \alpha A_2=a_2-a_1, \\ \alpha A_3=a_4-a_1, \\ a A_4=a_3-a_1, \end{cases}$$

then we have $R_1\sim R_3$.

§ 3. Surfaces with $P(y)=6$ (continued.)

We now consider Case (iii). Then $S_4=c_1$ and

$$\begin{cases} a_1^4-S_1a_1^3+S_2a_1^2-S_3a_1+c_1=\beta_1e^{H_1}, \\ a_2^4-S_1a_2^3+S_2a_2^2-S_3a_2+c_1=\beta_2e^{H_2}, \\ a_3^4-S_1a_3^3+S_2a_3^2-S_3a_3+c_1=\beta_3e^{H_3}, \\ a_4^4-S_1a_4^3+S_2a_4^2-S_3a_4+c_1=\beta_4e^{H_4}. \end{cases}$$

From the first three equations we have

$$S_1=x_1e^{H_1}+x_2e^{H_2}+x_3e^{H_3}+y_1+a_1+a_2+a_3,$$

$$S_2=(a_2+a_3)x_1e^{H_1}+(a_1+a_3)x_2e^{H_2}+(a_1+a_2)x_3e^{H_3}$$
$$+(a_1+a_2+a_3)y_1+a_1a_2+a_1a_3+a_2a_3,$$

$$S_3=a_2a_3x_1e^{H_1}+a_1a_3x_2e^{H_2}+a_1a_2x_3e^{H_3}+(a_1a_2+a_2a_3+a_1a_3)y_1$$
$$+a_1a_2a_3,$$

where

$$x_1=\frac{-\beta_1}{a_1(a_1-a_2)(a_1-a_3)},\quad x_2=\frac{\beta_2}{a_2(a_1-a_2)(a_2-a_3)},\quad x_3=\frac{-\beta_3}{a_3(a_1-a_3)(a_2-a_3)}$$

and

$$y_1=\frac{c_1}{a_1a_2a_3}.$$

Substituting these into the fourth equation $F(z, a_4)=\beta_4 e^{H_4}$, we can make use of Borel's unicity theorem. Then we have

$$H_1=H_2=H_3=H_4(\equiv H)$$

$$a_4(a_4-a_2)(a_4-a_3)x_1+a_4(a_4-a_1)(a_4-a_3)x_2+a_4(a_4-a_1)(a_4-a_2)x_3=-\beta_4$$

and

$$y_1(a_4-a_1)(a_4-a_2)(a_4-a_3)=a_4(a_4-a_1)(a_4-a_2)(a_4-a_3).$$

Hence $y_1=a_4$, that is, $c_1=a_1a_2a_3a_4$. Therefore

$$S_1=(x_1+x_2+x_3)e^H+a_1+a_2+a_3+a_4,$$

$$S_2=\{(a_2+a_3)x_1+(a_1+a_3)x_2+(a_1+a_2)x_3\}e^H$$
$$+a_1a_2+a_1a_3+a_2a_3+a_1a_4+a_2a_4+a_3a_4,$$

$$S_3=\{a_2a_3x_1+a_1a_3x_2+a_1a_2x_3\}e^H+a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4.$$

Now we impose the following condition:

$$F(z, \alpha)=(\alpha-a_1)(\alpha-a_2)(\alpha-a_3)(\alpha-a_4)+\alpha P(\alpha)e^H$$

does not reduce to a non-zero constant excepting for $\alpha=0$, where

$$-P(\alpha)=(\alpha-a_2)(\alpha-a_3)x_1+(\alpha-a_1)(\alpha-a_3)x_2+(\alpha-a_1)(\alpha-a_2)x_3.$$

Then

$$-P(\alpha)=\alpha^2(x_1+x_2+x_3)-\alpha\{(a_2+a_3)x_1+(a_1+a_3)x_2+(a_1+a_2)x_3\}$$
$$+a_2a_3x_1+a_1a_3x_2+a_1a_2x_3.$$

By our condition we have three possibilities:
(a) $P(\alpha)=k\alpha^2$, (b) $P(\alpha)=k\alpha$, (c) $P(\alpha)=k$ with a non-zero nonstant k.

CASE (a). Then $k=-(x_1+x_2+x_3)$ and

$$(a_2+a_3)x_1+(a_1+a_3)x_2+(a_1+a_2)x_3=0,$$
$$a_2a_3x_1+a_1a_3x_2+a_1a_2x_3=0.$$

In this case we can easily prove that

$$\frac{\beta_1}{a_1^3}=\frac{\beta_2}{a_2^3}=\frac{\beta_3}{a_3^3}$$

and

$$k=\frac{\beta_1}{a_1^3}(\equiv -x_0).$$

Hence we have

$$\begin{cases} S_1=x_0e^H+a_1+a_2+a_3+a_4\,, \\ S_2=a_1a_2+a_2a_3+a_1a_3+a_1a_4+a_2a_4+a_3a_4\,, \\ S_3=a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4\,, \\ S_4=c_1=a_1a_2a_3a_4\,. \end{cases}$$

This surface is denoted by R_{13}.

CASE (b). Then $x_1+x_2+x_3=0$ and

$$a_2a_3x_1+a_1a_3x_2+a_1a_2x_3=0\,,$$

$$k=(a_2+a_3)x_1+(a_1+a_3)x_2+(a_1+a_2)x_3\,.$$

It is easy to prove that

$$k=\frac{\beta_1}{a_1^2}=\frac{\beta_2}{a_2^2}=\frac{\beta_3}{a_3^2}(\equiv x_0)\,.$$

Then

$$\begin{cases} S_1=a_1+a_2+a_3+a_4\,, \\ S_2=x_0e^H+a_1a_2+a_2a_3+a_1a_3+a_1a_4+a_2a_4+a_3a_4\,, \\ S_3=a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4\,, \\ S_4=c_1=a_1a_2a_3a_4\,. \end{cases}$$

This surface is denoted by R_{14}.

CASE (c). Then $x_1+x_1+x_3=0$ and

$$(a_2+a_3)x_1+(a_1+a_3)x_2+(a_1+a_2)x_3=0\,,$$

$$k=-(a_2a_3x_1+a_1a_3x_2+a_1a_2x_3)\,.$$

It is easy to prove that

$$k=\frac{\beta_1}{a_1}=\frac{\beta_2}{a_2}=\frac{\beta_3}{a_3}(\equiv x_0)\,.$$

Then

$$\begin{cases} S_1=a_1+a_2+a_3+a_4\,, \\ S_2=a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4\,, \\ S_3=x_0e^H+a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4\,, \\ S_4=c_1=a_1a_2a_3a_4\,. \end{cases}$$

This surface is denoted by R_{15}.

CASE (vi). Then $S_4=\beta_1e^{H_1}$ and

$$\begin{cases} a_1^4 - S_1 a_1^3 + S_2 a_1^2 - S_3 a_1 + \beta_1 e^{H_1} = c_1, \\ a_2^4 - S_1 a_2^3 + S_2 a_2^2 - S_3 a_2 + \beta_1 e^{H_1} = \beta_2 e^{H_2}, \\ a_3^4 - S_1 a_3^3 + S_2 a_3^2 - S_3 a_3 + \beta_1 e^{H_1} = \beta_3 e^{H_3}, \\ a_4^4 - S_1 a_4^3 + S_2 a_4^2 - S_3 a_4 + \beta_1 e^{H_1} = \beta_4 e^{H_4}. \end{cases}$$

Let us put

$$x = \frac{\beta_1}{a_2 a_3 a_4}, \quad y = \frac{\beta_2}{a_2(a_2 - a_3)(a_2 - a_4)}, \quad z = \frac{\beta_3}{a_3(a_2 - a_3)(a_3 - a_4)},$$

$$u = \frac{\beta_4}{a_4(a_2 - a_4)(a_3 - a_4)}.$$

Then from the last three equations we have

$$\begin{cases} S_1 = x e^{H_1} - y e^{H_2} + z e^{H_3} - u e^{H_4} + a_2 + a_3 + a_4, \\ S_2 = (a_2 + a_3 + a_4) x e^{H_1} - (a_3 + a_4) y e^{H_2} + (a_2 + a_4) z e^{H_3} \\ \qquad - (a_2 + a_3) u e^{H_4} + a_2 a_3 + a_2 a_4 + a_3 a_4, \\ S_3 = (a_2 a_3 + a_2 a_4 + a_3 a_4) x e^{H_1} - a_3 a_4 y e^{H_2} + a_2 a_4 z e^{H_3} - a_2 a_3 u e^{H_4} + a_2 a_3 a_4, \\ S_4 = a_2 a_3 a_4 x e^{H_1}. \end{cases}$$

By the first equation we have

$$H_1 = H_2 = H_3 = H_4 (\equiv H),$$

$$(a_1 - a_2)(a_1 - a_3)(a_1 - a_4) x - a_1(a_1 - a_3)(a_1 - a_4) y$$

$$+ a_1(a_1 - a_2)(a_1 - a_4) z - a_1(a_1 - a_2)(a_1 - a_3) u = 0$$

and

$$c_1 = a_1(a_1 - a_2)(a_1 - a_3)(a_1 - a_4).$$

Now we impose the following condition:

$$F(z, \alpha) = \alpha(\alpha - a_2)(\alpha - a_3)(\alpha - a_4) + P(\alpha) e^H$$

does not reduce to a non-zero constant excepting $\alpha = a_1$. Hence there appear three possibilities:

(a) $P(\alpha) = k(\alpha - a_1)$, (b) $P(\alpha) = k(\alpha - a_1)^2$, (c) $P(\alpha) = k(\alpha - a_1)^3$

with a non-zero constant k. Here

$$P(\alpha) = \alpha^3(-x + y - z + u)$$

$$+ \alpha^2 \{(a_2 + a_3 + a_4) x - (a_3 + a_4) y + (a_2 + a_4) z - (a_2 + a_3) u\}$$

$$+ \alpha \{-(a_2 a_3 + a_2 a_4 + a_3 a_4) x + a_3 a_4 y - a_2 a_4 z + a_2 a_3 u\} + a_2 a_3 a_4 x.$$

CASE (a). Then

$$-x+y-z+u=0,$$

$$(a_2+a_3+a_4)x-(a_3+a_4)y+(a_2+a_4)z-(a_2+a_3)u=0$$

and

$$-a_1\{-(a_2a_3+a_2a_4+a_3a_4)x+a_3a_4y-a_2a_4z+a_2a_3u\}=a_2a_3a_4x.$$

Hence we have

$$\begin{cases} S_1=a_2+a_3+a_4, \\ S_2=a_2a_3+a_2a_4+a_3a_4, \\ S_3=\dfrac{a_2a_3a_4}{a_1}xe^H+a_2a_3a_4=\dfrac{\beta_1}{a_1}e^H+a_2a_3a_4, \\ S_4=a_2a_3a_4xe^H=\beta_1e^H. \end{cases}$$

We put $x_0=\beta_1/a_1$. Then

$$\begin{cases} S_1=a_2+a_3+a_4, \\ S_2=a_2a_3+a_2a_4+a_3a_4, \\ S_3=x_0e^H+a_2a_3a_4, \\ S_4=a_1x_0e^H. \end{cases}$$

This surface is denoted by R_{28}.

CASE (b). Then

$$x-y+z-u=0$$

$$(a_2+a_3+a_4)x-(a_3+a_4)y+(a_2+a_4)z-(a_2+a_3)u=\frac{a_2a_3a_4}{a_1^2}x,$$

$$-(a_2a_3+a_2a_4+a_3a_4)x+a_3a_4y-a_2a_4z+a_2a_3u=-2a_1\frac{a_2a_3a_4}{a_1^2}x.$$

We make use of $a_2a_3a_4x=\beta_1$. Then we have with $x_0=\beta_1/a_1^2$

$$\begin{cases} S_1=a_2+a_3+a_4, \\ S_2=x_0e^H+a_2a_3+a_2a_4+a_3a_4, \\ S_3=2a_1x_0e^H+a_2a_3a_4, \\ S_4=a_1^2x_0e^H. \end{cases}$$

This surface is denoted by R_{29}.

CASE (c). Then

$$-3a_1(-x+y-z+u)=(a_2+a_3+a_4)x-(a_3+a_4)y+(a_2+a_4)z-(a_2+a_3)u\,,$$

$$3a_1^2(-x+y-z+u)=-(a_2a_3+a_2a_4+a_3a_4)x+a_3a_4y-a_2a_4z+a_2a_3u\,,$$

$$-a_1^3(-x+y-z+u)=a_2a_3a_4x\equiv\beta_1\,.$$

Hence

$$x-y+z-u=\beta_1/a_1^3\,,$$

$$(a_2+a_3+a_4)x-(a_3+a_4)y+(a_2+a_4)z-(a_2+a_3)u=\frac{3\beta_1}{a_1^2}\,,$$

$$(a_2a_3+a_2a_4+a_3a_4)x-a_3a_4y+a_2a_4z-a_2a_3u=\frac{3\beta_1}{a_1}\,.$$

Then we have

$$\begin{cases} S_1=x_0e^H+a_2+a_3+a_4\,,\\ S_2=3a_1x_0e^H+a_2a_3+a_2a_4+a_3a_4\,,\\ S_3=3a_1^2x_0e^H+a_2a_3a_4\,,\\ S_4=a_1^3x_0e^H\,. \end{cases}$$

This surface is denoted by R_{30}.

$$F(z,\ \alpha Y+\beta)\equiv\alpha^4G(z,\ Y)\equiv\alpha^4(Y^4-T_1Y^3+T_2Y^2-T_3Y+T_4)\,.$$

$$F(z,\ y)\equiv y^4-S_1y^3+S_2y^2-S_3y+S_4=0\,.$$

$$T_1=\frac{1}{\alpha}(S_1-4\beta)\,,$$

$$T_2=\frac{1}{\alpha^2}(S_2-3S_1\beta+6\beta^2)\,,$$

$$T_3=\frac{1}{\alpha^3}(S_3-2S_2\beta+3S_1\beta^2-4\beta^3)\,,$$

$$T_4=\frac{1}{\alpha^4}(S_4-S_3\beta+S_2\beta^2-S_1\beta^3+\beta^4)\,.$$

Here we put

$$\begin{cases} \alpha A_1+\beta=0, \qquad \beta=a_1\,,\\ \alpha A_2=a_2-a_1\,,\\ \alpha A_3=a_3-a_1\,,\\ \alpha A_4=a_4-a_1\,. \end{cases}$$

Then we have

$$R_{15}\sim R_{28}, \qquad R_{14}\sim R_{29}, \qquad R_{13}\sim R_{30}.$$

§4. Surface with $P(y)=6$ (continued. bis)

We now consider the case (ii). Then $S_4=c_1$ and

$$\begin{cases} a_1^4-S_1a_1^3+S_2a_1^2-S_3a_1+c_1=c_2, \\ a_2^4-S_1a_2^3+S_2a_2^3-S_3a_2+c_1=\beta_1e^{H_1}, \\ a_3^4-S_1a_3^3+S_2a_3^2-S_3a_3+c_1=\beta_2e^{H_2}, \\ a_4^4-S_1a_4^3+S_2a_4^2-S_3a_4+c_1=\beta_3e^{H_3}. \end{cases}$$

Then

$$S_1=x_1e^{H_1}+x_2e^{H_2}+x_3e^{H_3}+y+a_2+a_3+a_4,$$

$$S_2=(a_3+a_4)x_1e^{H_1}+(a_2+a_4)x_2e^{H_2}+(a_2+a_3)x_3e^{H_3}$$
$$+(a_2+a_3+a_4)y+a_2a_3+a_3a_4+a_2a_4,$$

$$S_3=a_3a_4x_1e^{H_1}+a_2a_4x_2e^{H_2}+a_2a_3x_3e^{H_3}+(a_2a_3+a_3a_4+a_2a_4)y+a_2a_3a_4,$$

where

$$x_1=\frac{\beta_1}{a_2(a_2-a_3)(a_4-a_2)}, \qquad x_2=\frac{\beta_2}{a_3(a_2-a_3)(a_3-a_4)},$$

$$x_3=\frac{\beta_3}{a_4(a_3-a_4)(a_4-a_2)}, \qquad y=\frac{c_1}{a_2a_3a_4}.$$

Substituting these into $F(z, a_1)=c_2$, we have $H_1=H_2=H_3(\equiv H)$ and

$$(a_1-a_3)(a_1-a_4)x_1+(a_1-a_2)(a_1-a_4)x_2+(a_1-a_2)(a_1-a_3)x_3=0$$

and

$$c_2+y(a_1-a_2)(a_1-a_3)(a_1-a_4)=a_1(a_1-a_2)(a_1-a_3)(a_1-a_4).$$

Let us consider $F(z, \alpha)$. Then

$$F(z, \alpha)=(\alpha-a_2)(\alpha-a_3)(\alpha-a_4)(\alpha-y)$$
$$-\alpha(\alpha-a_1)\{(\alpha+a_1-a_3-a_4)x_1+(\alpha+a_1-a_2-a_4)x_2$$
$$+(\alpha+a_1-a_2-a_3)x_3\}e^H$$
$$=(\alpha-a_2)(\alpha-a_3)(\alpha-a_4)(\alpha-y)-\frac{\alpha(\alpha-a_1)}{a_2-a_3}\{A\alpha-B\}e^H,$$

where

$$A=\frac{\beta_1}{a_2(a_1-a_2)}-\frac{\beta_2}{a_3(a_1-a_3)},$$

$$B=\frac{a_3\beta_1}{a_2(a_1-a_2)}-\frac{a_2\beta_2}{a_3(a_1-a_3)}.$$

We now impose the conditions: $F(z, \alpha)$ does not reduce to a non-zero constant D except for $\alpha=0$, a_1 and further it does not reduce to De^H except for $\alpha=a_2$, a_3, a_4. Hence we have $(\alpha) y=a_2$ or $(\beta) y=a_3$ or (γ) $y=a_4$ and (1) $A=0$ or (2) $B=0$ or (3) $A\neq 0$, $B=Aa_1$.

If $A=0$, then $x_1+x_2+x_3=0$ and

$$(a_3+a_4)x_1+(a_2+a_4)x_2+(a_2+a_3)x_3=-\frac{\beta_1}{a_2(a_1-a_2)},$$

$$a_3a_4x_1+a_2a_4x_2+a_2a_3x_3=-\frac{a_1\beta_1}{a_2(a_1-a_2)}.$$

If $B=0$, then

$$x_1+x_2+x_3=\frac{\beta_1}{a_2^2(a_1-a_2)},$$

$$(a_3+a_4)x_1+(a_2+a_4)x_2+(a_2+a_3)x_3=\frac{a_1\beta_1}{a_2^2(a_1-a_2)},$$

$$a_3a_4x_1+a_2a_4x_2+a_2a_3x_3=0.$$

If $B=Aa_1$, $A\neq 0$, then

$$x_1+x_2+x_3=-\frac{\beta_1}{a_2(a_1-a_2)^2},$$

$$(a_3+a_4)x_1+(a_2+a_4)x_2+(a_2+a_3)x_3=-\frac{2a_1\beta_1}{a_2(a_1-a_2)^2},$$

$$a_3a_4x_1+a_2a_4x_2+a_2a_3x_3=-\frac{a_1^2\beta_1}{a_2(a_1-a_2)^2}.$$

Case (α)(1). Then with $x_0=-\beta_1/a_2(a_1-a_2)$

$$\begin{cases} S_1=2a_2+a_3+a_4, \\ S_2=x_0e^H+a_2^2+2a_2a_3+a_3a_4+2a_2a_4, \\ S_3=a_1x_0e^H+a_2^2a_3+2a_2a_3a_4+a_2^2a_4 \\ S_4=a_2^2a_3a_4. \end{cases}$$

This surface is denoted by R_4.

CASE (β)(1). Then with $x_0=-\beta_1/a_2(a_1-a_2)$

$$\begin{cases} S_1=a_2+2a_3+a_4, \\ S_2=x_0e^H+a_3^2+2a_2a_3+a_2a_4+2a_3a_4, \\ S_3=a_1x_0e^H+a_2a_3^2+a_3^2a_4+2a_2a_3a_4, \\ S_4=a_2a_3^2a_4. \end{cases}$$

This surface is denoted by R_5.

CASE (γ) (1). Then with $x_0=-\beta_1/a_2(a_1-a_2)$

$$\begin{cases} S_1=a_2+a_3+2a_4\,, \\ S_2=x_0e^H+a_2a_3+2a_2a_4+2a_3a_4+a_4^2\,, \\ S_3=a_1x_0e^H+a_3a_4^2+a_2a_4^2+2a_2a_3a_4\,, \\ S_4=a_2a_3a_4^2\,. \end{cases}$$

This surface is denoted by R_6.

CASE (α) (2). Then with $x_0=\beta_1/a_2^2(a_1-a_2)$

$$\begin{cases} S_1=x_0e^H+2a_2+a_3+a_4\,, \\ S_2=a_1x_0e^H+a_2^2+2a_2a_3+a_3a_4+2a_2a_4\,, \\ S_3=a_2^2a_3+2a_2a_3a_4+a_2^2a_4\,, \\ S_4=a_2^2a_3a_4\,. \end{cases}$$

This surface is denoted by R_7.

CASE (β) (2). Then with $x_0=\beta_1/a_2^2(a_1-a_2)$

$$\begin{cases} S_1=x_0e^H+a_2+2a_3+a_4\,, \\ S_2=a_1x_0e^H+2a_2a_3+a_3^2+2a_3a_4+a_2a_4\,, \\ S_3=a_2a_3^2+a_3^2a_4+2a_2a_3a_4\,, \\ S_4=a_2a_3^2a_4\,. \end{cases}$$

This surface is denoted by R_8.

CASE (γ) (2). Then with $x_0=\beta_1/a_2^2(a_1-a_2)$

$$\begin{cases} S_1=x_0e^H+a_2+a_3+2a_4\,, \\ S_2=a_1x_0e^H+a_4^2+2a_2a_4+2a_3a_4+a_2a_3\,, \\ S_3=2a_2a_3a_4+a_3a_4^2+a_2a_4^2\,, \\ S_4=a_2a_3a_4^2\,. \end{cases}$$

This surface is denoted by R_9.

CASE (α) (3). Then with $x_0=-\beta_1/a_2(a_1-a_2)^2$

$$\begin{cases} S_1=x_0e^H+2a_2+a_3+a_4\,, \\ S_2=2a_1x_0e^H+a_2^2+2a_2a_3+a_3a_4+2a_2a_4\,, \\ S_3=a_1^2x_0e^H+a_2^2a_3+2a_2a_3a_4+a_2^2a_4\,, \\ S_4=a_2^2a_3a_4\,. \end{cases}$$

This surface is denoted by R_{10}.

CASE (β)(3). Then with $x_0=-\beta_1/a_2(a_1-a_2)^2$

$$\begin{cases} S_1=x_0e^H+a_2+2a_3+a_4, \\ S_2=2a_1x_0e^H+2a_2a_3+a_3^2+2a_3a_4+a_2a_4, \\ S_3=a_1^2x_0e^H+a_2a_3^2+a_3^2a_4+2a_2a_3a_4, \\ S_4=a_2a_3^2a_4. \end{cases}$$

This surface is denoted by R_{11}.

CASE (γ)(3). Then with $x_0=-\beta_1/a_2(a_1-a_2)^2$

$$\begin{cases} S_1=x_0e^H+a_2+a_3+2a_4, \\ S_2=2a_1x_0e^H+a_4^2+2a_2a_4+2a_3a_4+a_2a_3, \\ S_3=a_1^2x_0e^H+2a_2a_3a_4+a_3a_4^2+a_2a_4^2, \\ S_4=a_2a_3a_4^2. \end{cases}$$

This surface is denoted by R_{12}.

CASE (v). Then $S_4=\beta_1e^{H_1}$ and

$$\begin{cases} a_1^4-S_1a_1^3+S_2a_1^2-S_3a_1+\beta_1e^{H_1}=c_1, \\ a_2^4-S_1a_2^3+S_2a_2^2-S_3a_2+\beta_1e^{H_1}=c_2, \\ a_3^4-S_1a_3^3+S_2a_3^2-S_3a_3+\beta_1e^{H_1}=\beta_2e^{H_2}, \\ a_4^4-S_1a_4^3+S_2a_4^2-S_3a_4+\beta_1e^{H_1}=\beta_3e^{H_3}. \end{cases}$$

Let us put

$$x_1=\frac{\beta_1}{a_1a_2a_3}, \qquad x_2=\frac{\beta_2}{a_2(a_1-a_3)(a_2-a_3)}$$

and

$$y_1=\frac{c_1}{a_1(a_1-a_2)(a_1-a_3)}, \qquad y_2=\frac{c_2}{a_2(a_1-a_2)(a_2-a_3)}.$$

From the first three equations we have

$$\begin{aligned} S_1&=x_1e^{H_1}-x_2e^{H_2}-y_1+y_2+a_1+a_2+a_3, \\ S_2&=(a_1+a_2+a_3)x_1e^{H_1}-(a_1+a_2)x_2e^{H_2}-(a_2+a_3)y_1+(a_1+a_3)y_2 \\ &\qquad +a_1a_2+a_1a_3+a_2a_3, \\ S_3&=(a_1a_2+a_1a_3+a_2a_3)x_1e^{H_1}-a_1a_2x_2e^{H_2}-a_2a_3y_1+a_1a_3y_2+a_1a_2a_3. \end{aligned}$$

Substituting these into the fourth equation, we have

$$H_1=H_2=H_3(\equiv H),$$

$$(a_4-a_1)(a_4-a_2)(a_4-a_3)x_1-(a_4-a_1)(a_4-a_2)a_4x_2+\beta_3=0,$$

$$(a_4-a_2)y_1-(a_4-a_1)y_2+(a_4-a_1)(a_4-a_2)=0.$$

Now we consider

$$F(z,\alpha)=e^H(\alpha-a_1)(\alpha-a_2)\{-x_1(\alpha-a_3)+x_2\alpha\}$$
$$+\alpha(\alpha-a_3)P(\alpha),$$

where

$$P(\alpha)=\alpha^2-(a_1+a_2-y_1+y_2)\alpha+a_1a_2-a_2y_1+a_1y_2.$$

There are several possibilities by the postulate that $F(z,\alpha)$ does not reduce to a non-zero constant D except for $\alpha=a_1$, $\alpha=a_2$ and further it does not reduce to De^H except for $\alpha=0$, $\alpha=a_3$, $\alpha=a_4$. Hence

$$(\alpha)\quad x_1=x_2 \quad\text{or}\quad (\beta)\quad \alpha(x_2-x_1)+x_1a_3=k(\alpha-a_1) \quad\text{or}$$

$$(\gamma)\quad \alpha(x_2-x_1)+x_1a_3=k(\alpha-a_2)$$

and

$$(1)\quad P(\alpha)=\alpha(\alpha-a_4) \quad\text{or}\quad (2)\quad P(\alpha)=(\alpha-a_4)^2 \quad\text{or}$$

$$(3)\quad P(\alpha)=(\alpha-a_3)(\alpha-a_4).$$

CASE (1). Then $a_1+a_2-y_1+y_2=a_4$ and $a_1a_2-a_2y_1+a_1y_2=0$. Hence

$$y_2=(a_2^2-a_2a_4)/(a_1-a_2) \quad\text{and}\quad y_1=(a_1^2-a_1a_4)/(a_1-a_2).$$

Thus

$$-y_1+y_2+a_1+a_2+a_3=a_3+a_4,$$

$$-(a_2+a_3)y_1+(a_1+a_3)y_2+a_1a_2+a_1a_3+a_2a_3=a_3a_4$$

and

$$-a_2a_3y_1+a_1a_3y_2+a_1a_2a_3=0.$$

CASE (2). Then $a_1+a_2-y_1+y_2=2a_4$, $a_1a_2-a_2y_1+a_1y_2=a_4^2$. This gives

$$y_1=(a_1-a_4)^2/(a_1-a_2) \quad\text{and}\quad y_2=(a_2-a_4)^2/(a_1-a_2).$$

Then

$$-y_1+y_2+a_1+a_2+a_3=a_3+2a_4,$$

$$-(a_2+a_3)y_1+(a_1+a_3)y_2+a_1a_2+a_1a_3+a_2a_3=2a_3a_4+a_4^2,$$

$$-a_2a_3y_1+a_1a_3y_2+a_1a_2a_3=a_3a_4^2.$$

CASE (3). Then $a_1+a_2-y_1+y_2=a_3+a_4$, $a_1a_2-a_2y_1+a_1y_2=a_3a_4$. This gives

$$y_1=(a_1-a_3)(a_1-a_4)/(a_1-a_2) \quad\text{and}\quad y_2=(a_2-a_3)(a_2-a_4)/(a_1-a_2).$$

Then

$$-y_1+y_2+a_1+a_2+a_3=2a_3+a_4,$$
$$-(a_2+a_3)y_1+(a_1+a_3)y_2+a_1a_2+a_1a_3+a_2a_3=a_3^2+2a_3a_4,$$
$$-a_2a_3y_1+a_1a_3y_2+a_1a_2a_3=a_3^2a_4.$$

CASE (α). Then $x_1=x_2$. Hence

$$(a_1+a_2+a_3)x_1-(a_1+a_2)x_2=a_3x_1=\beta_1/a_1a_2,$$
$$(a_1a_2+a_1a_3+a_2a_3)x_1-a_1a_2x_2=(a_1+a_2)a_3x_1=(a_1+a_2)\beta_1/a_1a_2.$$

We put $\beta_1/a_1a_2=x_0$.

CASE (β). Then $x_2=(a_1-a_3)x_1/a_1$. Hence

$$x_1-x_2=a_3x_1/a_1=\beta_1/a_1^2a_2,$$
$$(a_1+a_2+a_3)x_1-(a_1+a_2)x_2=a_3(2a_1+a_2)x_1/a_1=(2a_1+a_2)\beta_1/a_1^2a_2,$$
$$(a_1a_2+a_1a_3+a_2a_3)x_1-a_1a_2x_2=(a_1+2a_2)\beta_1/a_1a_2.$$

We put $\beta_1/a_1^2a_2=x_0$.

CASE (γ). Then $x_2=(a_2-a_3)x_1/a_2$. Hence

$$x_1-x_2=\beta_1/a_1a_2^2,$$
$$(a_1+a_2+a_3)x_1-(a_1+a_2)x_2=(a_1+2a_2)\beta_1/a_1a_2^2,$$
$$(a_1a_2+a_1a_3+a_2a_3)x_1-a_1a_2x_2=(2a_1+a_2)\beta_1/a_1a_2.$$

We put $\beta_1/a_1a_2^2=x_0$.

CASE (α)(1). Then

$$\begin{cases} S_1=a_3+a_4, \\ S_2=x_0e^H+a_3a_4, \\ S_3=(a_1+a_2)x_0e^H, \\ S_4=a_1a_2x_0e^H. \end{cases}$$

This surface is denoted by R_{19}.

CASE (β)(1). Then

$$\begin{cases} S_1=x_0e^H+a_3+a_4, \\ S^2=(2a_1+a_2)x_0e^H+a_3a_4, \\ S_3=a_1(a_1+2a_2)x_0e^H, \\ S_4=a_1^2a_2x_0e^H. \end{cases}$$

This surface is denoted by R_{20}.

CASE (γ) (1).

$$\begin{cases} S_1=x_0e^H+a_3+a_4\,, \\ S_2=(a_1+2a_2)x_0e^H+a_3a_4\,, \\ S_3=a_2(2a_1+a_2)x_0e^H\,, \\ S_4=a_1a_2^2x_0e^H\,. \end{cases}$$

This surface is denoted by R_{21}.

CASE (α) (2).

$$\begin{cases} S_1=a_3+2a_4\,, \\ S_2=x_0e^H+2a_3a_4+a_4^2\,, \\ S_3=(a_1+a_2)x_0e^H+a_3a_4^2\,, \\ S_4=a_1a_2x_0e^H\,. \end{cases}$$

This surface is denoted by R_{22}.

CASE (β) (2).

$$\begin{cases} S_1=x_0e^H+a_3+2a_4\,, \\ S_2=(2a_1+a_2)x_0e^H+2a_3a_4+a_4^2\,, \\ S_3=a_1(a_1+2a_2)x_0e^H+a_3a_4^2\,, \\ S_4=a_1^2a_2x_0e^H\,. \end{cases}$$

This surface is denoted by R_{23}.

CASE (γ) (2).

$$\begin{cases} S_1=x_0e^H+a_3+2a_4\,, \\ S_2=(a_1+2a_2)x_0e^H+2a_3a_4+a_4^2\,, \\ S_3=a_2(2a_1+a_2)x_0e^H+a_3a_4^2\,, \\ S_4=a_1a_2^2x_0e^H\,. \end{cases}$$

This surface is denoted by R_{24}.

CASE (α) (3).

$$\begin{cases} S_1=2a_3+a_4\,, \\ S_2=x_0e^H+a_3^2+2a_3a_4\,, \\ S_3=(a_1+a_2)x_0e^H+a_3^2a_4\,, \\ S_4=a_1a_2x_0e^H\,. \end{cases}$$

This surface is denoted by R_{25}.

CASE (β) (3).

$$\begin{cases} S_1 = x_0 e^H + 2a_3 + a_4, \\ S_2 = (2a_1 + a_2) x_0 e^H + a_3^2 + 2a_3 a_4, \\ S_3 = a_1(a_1 + 2a_2) x_0 e^H + a_3^2 a_4, \\ S_4 = a_1^2 a_2 x_0 e^H. \end{cases}$$

This surface is denoted by R_{26}.

CASE (γ) (3).

$$\begin{cases} S_1 = x_0 e^H + 2a_3 + a_4, \\ S_2 = (a_1 + 2a_2) x_0 e^H + a_3^2 + 2a_3 a_4, \\ S_3 = a_2(2a_1 + a_2) x_0 e^H + a_3^2 a_4, \\ S_4 = a_1 a_2^2 x_0 e^H. \end{cases}$$

This surface is denoted by R_{27}.

How many different surfaces are there among eighteen surfaces listed up in this section? As in § 2, § 3 we put

$$\begin{cases} \alpha A_1 + \beta = 0, \qquad \beta = a_2 \\ \alpha A_2 = a_1 - a_2, \\ \alpha A_3 = a_3 - a_2, \\ \alpha A_4 = a_4 - a_2. \end{cases}$$

Then we can prove that

$$\begin{array}{lll} R_4 \sim R_{19}, & R_5 \sim R_{25}, & R_6 \sim R_{22}, \\ R_7 \sim R_{20}, & R_8 \sim R_{26}, & R_9 \sim R_{23}, \\ R_{10} \sim R_{21}, & R_{11} \sim R_{27}, & R_{12} \sim R_{24}. \end{array}$$

Further we put

$$\begin{cases} \alpha A_1 = -\beta, \qquad \beta = a_1 \\ \alpha A_3 = a_2 - a_1, \\ \alpha A_4 = a_3 - a_1, \\ \alpha A_2 = a_4 - a_1. \end{cases}$$

Then we can prove that $R_4 \sim R_5$, $R_7 \sim R_8$ and $R_{10} \sim R_{11}$.

If we put

$$\begin{cases} \alpha A_4+\beta=0, & \beta=a_3 \\ \alpha A_3=a_4-a_3, \\ \alpha A_1=a_1-a_3, \\ \alpha A_2=a_2-a_3, \end{cases}$$

then we can prove that $R_{22}\sim R_{25}$, $R_{23}\sim R_{26}$ and $R_{24}\sim R_{27}$.

If we put

$$\begin{cases} \alpha A_2+\beta=0, & \beta=a_2 \\ \alpha A_3=a_3-a_2, \\ \alpha A_4=a_4-a_2, \\ \alpha A_1=a_1-a_2, \end{cases}$$

then $R_7\sim R_{21}$ is able to prove.

Hence we have only two different surfaces in the sense of $\sim$ in the cases (ii) and (v).

§ 5. Discriminant

We shall first decide the form of discriminant of R_{13}, which is defined by

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=x_0e^H+x_1, \\ S_2=x_2, \\ S_3=x_3, \\ S_4=x_4, \end{cases}$$

where $x_1=a_1+a_2+a_3+a_4$, $x_2=a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4$, $x_3=a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4$, $x_4=a_1a_2a_3a_4$.

For simplicity's sake we put $X=x_0e^H$. Then

$$L=-\frac{3}{8}S_1^2+S_2=-\left(\frac{3}{8}X^2+\alpha_1X+\alpha_2\right),$$

$$M=-\frac{1}{8}S_1^3+\frac{1}{2}S_1S_2-S_3=-\left(\frac{1}{8}X^3+\beta_1X^2+\beta_2X+\beta_3\right),$$

$$N=-\frac{3}{256}S_1^4+\frac{1}{16}S_1^2S_2-\frac{1}{4}S_1S_3+S_4$$

$$=-\left(\frac{3}{256}X^4+\gamma_1X^3+\gamma_2X^2+\gamma_3X+\gamma_4\right),$$

where

$$\alpha_1=\frac{3}{4}x_1,\quad \alpha_2=\frac{3}{8}x_1^2-x_2,$$

$$\beta_1=\frac{3}{8}x_1,\quad \beta_2=\frac{3}{8}x_1^2-\frac{1}{2}x_2,\quad \beta_3=\frac{1}{8}x_1^3-\frac{1}{2}x_1x_2+x_3,$$

$$\gamma_1=\frac{3}{64}x_1,\quad \gamma_2=\frac{9}{128}x_1^2-\frac{1}{16}x_2,\quad \gamma_3=\frac{3}{64}x_1^3-\frac{1}{8}x_1x_2+\frac{1}{4}x_3,$$

$$\gamma_4=\frac{3}{256}x_1^4-\frac{1}{16}x_1^2x_2+\frac{1}{4}x_1x_3-x_4.$$

Hence we evidently have

$$2\beta_1=\alpha_1,\qquad 16\gamma_1=\alpha_1,\qquad \alpha_2=4\beta_2-16\gamma_2.$$

Further we have

$$\beta_2-8\gamma_2=-3x_1^2/16=-\alpha_1^2/3$$

and

$$\beta_3-4\gamma_3=-x_1^3/16=-4\alpha_1^3/27.$$

The discriminant D of R_{13} is at most sixth degree for X, which was prowed in § 3 in [1]. The coefficient of X^6 is given in § 3 in [1]. Then we have

$$-\frac{27}{16}\cdot\frac{16}{27\cdot 27}\alpha_1^6+\frac{9}{2}\alpha_1\frac{\alpha_1^2}{3}\cdot\frac{4}{27}\alpha_1^3+\alpha_1^3\cdot\frac{-4}{27}\alpha_1^3+4\frac{-\alpha_1^6}{27}+\alpha_1^2\cdot\frac{\alpha_1^4}{9}=0.$$

Hence the coefficient of X^6 is equal to zero.

Next we consider the coefficient of X^5, which is given in § 3 in [1]. Firstly we consider the coefficient of γ_4, which is equal to

$$\frac{27}{2}(\beta_3-4\gamma_3)-18\alpha_1(\beta_2-8\gamma_2)-4\alpha_1^3.$$

This is equal to

$$-\frac{27}{2}\cdot\frac{4}{27}\alpha_1^3+18\alpha_1\frac{\alpha_1^3}{3}-4\alpha_1^3=0.$$

Then the remaining terms are equal to

$$-\frac{9}{2}\alpha_1(3\beta_3-8\gamma_3)(\beta_3-4\gamma_3)+\frac{9}{2}\beta_3(\beta_2+8\gamma_2)(\beta_2-8\gamma_\circ)$$

$$-6(11\beta_2-40\gamma_2)(\beta_2-8\gamma_2)\gamma_3$$

$$+30\alpha_1^2\beta_2\beta_3-24\cdot 8\alpha_1^2\gamma_2\beta_3-32\cdot 4\alpha_1^2\beta_2\gamma_3$$

$$+26\cdot 32\alpha_1^2\gamma_2\gamma_3+4\alpha_1^4(\beta_3-4\gamma_3)$$

$$+2\alpha_1(\beta_2-8\gamma_2)^2(13\beta_2-88\gamma_2)+4\alpha_1^3(\beta_2-8\gamma_2)^2.$$

We have

$$-\frac{9}{2}\alpha_1(3\beta_3-8\gamma_3)(\beta_3-4\gamma_3)$$

$$=\frac{9}{2}\alpha_1\cdot\frac{4}{27}\alpha_1^3\left(\beta_3-\frac{8}{27}\alpha_1^3\right)=\frac{2}{3}\alpha_1^4\beta_3-\frac{16}{81}\alpha_1^7,$$

$$\frac{9}{2}\beta_3(\beta_2+8\gamma_2)(\beta_2-8\gamma_2)-6(11\beta_2-40\gamma_2)(\beta_2-8\gamma_2)\gamma_3$$

$$=\frac{9}{2}\beta_3\left(2\beta_2+\frac{1}{3}\alpha_1^2\right)\frac{-1}{3}\alpha_1^2-6\left(6\beta_2-\frac{5}{3}\alpha_1^2\right)\frac{-1}{3}\alpha_1^2\frac{1}{4}\left(\beta_3+\frac{4}{27}\alpha_1^3\right)$$

$$=-\frac{4}{3}\alpha_1^4\beta_3+\frac{4}{9}\alpha_1^5\beta_2-\frac{10}{81}\alpha_1^7$$

and

$$30\alpha_1^2\beta_2\beta_3-3\cdot 64\alpha_1^2\gamma_2\beta_3-32\cdot 4\alpha_1^2\beta_2\gamma_3+26\cdot 32\alpha_1^2\gamma_2\gamma_3+4\alpha_1^4(\beta_3-4\gamma_3)$$

$$=\frac{2}{3}\alpha_1^4\beta_3-\frac{8}{9}\alpha_1^5\beta_2+\frac{56}{81}\alpha_1^7.$$

The last two terms are equal to

$$\frac{4}{9}\alpha_1^5\beta_2-\frac{22}{27}\alpha_1^7+\frac{4}{9}\alpha_1^7.$$

Summing up all these terms, we have that the coefficients of $\alpha_1^4\beta_3$, $\alpha_1^5\beta_2$ and α_1^7 vanish. Hence we have that the coefficient of X^5 vanishes. Hence

$$D_{R_{13}}=A_4x_0^4e^{4H}+A_3x_0^3e^{3H}+A_2x_0^2e^{2H}+A_1x_0e^H+A_0$$

with non-zero constants A_0, A_4. Why $A_0\neq 0$ and $A_4\neq 0$? This is due to Ullrich-Selberg's remification theorem. See [1] in § 3. From now on we shall not repeat this reason.

Next we shall consider the discriminant of R_{16}, which is defined by $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with

$$S_1=y_0e^H+a_4\equiv X+a_4,$$

$$S_2=(a_1+a_2+a_3)y_0e^H\equiv y_1X,$$

$$S_3=(a_1a_2+a_1a_3+a_2a_3)y_0e^H\equiv y_2X.$$

$$S_4=a_1a_2a_3y_0e^H\equiv y_3X.$$

Then

$$L=-\frac{3}{8}S_1^2+S_2=-\left(\frac{3}{8}X^2+\alpha_1X+\alpha_2\right)$$

$$M=-\frac{1}{8}S_1^3+\frac{1}{2}S_1S_2-S_3=-\left(\frac{1}{8}X^3+\beta_1X^2+\beta_2X+\beta_3\right)$$

$$N=-\frac{3}{256}S_1^4+\frac{1}{16}S_1^2S_2-\frac{1}{4}S_1S_3+S_4$$

$$=-\left\{\frac{3}{256}X^4+\gamma_1X^3+\gamma_2X^2+\gamma_3X+\gamma_4\right\},$$

where

$$\alpha_1=\frac{6}{8}a_4-y_1,\quad \alpha_2=\frac{3}{8}a_4^2,$$

$$\beta_1=\frac{3}{8}a_4-\frac{1}{2}y_1,\quad \beta_2=\frac{3}{8}a_4^2-\frac{1}{2}a_4y_1+y_2,\quad \beta_3=\frac{a_4^3}{8},$$

$$\gamma_1=\frac{3}{64}a_4-\frac{1}{16}y_1,\quad \gamma_2=\frac{9}{128}a_4^2-\frac{1}{8}a_4y_1+\frac{1}{4}y_2,$$

$$\gamma_3=\frac{3}{64}a_4^3-\frac{1}{16}a_4^2y_1+\frac{1}{4}a_4y_2-y_3,\quad \gamma_4=\frac{3}{256}a_4^4.$$

Evidently we have $2\beta_1=\alpha_1$, $16\gamma_1=\alpha_1$ and $\alpha_2=4\beta_2-16\gamma_2$. Hence the discriminant $D_{R_{16}}$ is a polynomial of X of sixth degree. The constant term of $D_{R_{16}}$ is equal to

$$-27\beta_3^4+144\alpha_2\beta_3^2\gamma_4-128\alpha_2^2\gamma_4^2-256\gamma_4^3+4\alpha_2^3\beta_3^2-16\alpha_2^4\gamma_4=0.$$

Let us consider the coefficient of X of $D_{R_{16}}$. Then it is just the following expression:

$$-27\cdot 4\beta_2\beta_3^3+144\cdot 16\gamma_1\beta_3^2\gamma_4+144\alpha_2\gamma_3\beta_3^2+144\cdot 2\alpha_2\beta_2\beta_3\gamma_4-128\cdot 32\gamma_1\gamma_4^2\alpha_2$$

$$-128\cdot 2\alpha_2^2\gamma_3\gamma_4-256\cdot 3\gamma_3\gamma_4^2+8\alpha_2^3\beta_2\beta_3+3\cdot 64\gamma_1\alpha_2^2\beta_3^2-16\alpha_2^4\gamma_3$$

$$-16\cdot 64\gamma_1\alpha_2^3\gamma_4.$$

It is very easy to prove that $144\alpha_2\gamma_3\beta_3^2-128\cdot 2\alpha_2^2\gamma_3\gamma_4-256\cdot 3\gamma_3\gamma_4^2-16\alpha_2^4\gamma_3=0$, $-27\cdot 4\beta_2\beta_3^3+144\cdot 2\alpha_2\beta_2\beta_3\gamma_4+8\alpha_2^3\beta_2\beta_3=0$ and $144\alpha_1\beta_3^2\gamma_4-128\cdot 2\alpha_1\gamma_4^2\alpha_2+3\cdot 4\alpha_1\alpha_2^2\beta_3^2-16\cdot 4\alpha_1\alpha_2^3\gamma_4=0$. Therefore

$$D_{R_{16}}=A_6y_0^6e^{6H}+A_5y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}$$

with non-zero constants A_2, A_6.

Now we shall consider the discriminant of R_{17}, which is defined by $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$, with

$$\begin{cases} S_1 = y_0 e^H + 2a_4 \equiv X + 2a_4, \\ S_2 = y_1 y_0 e^H + a_4^2 \equiv y_1 X + a_4^2, \\ S_3 = y_2 y_0 e^H \equiv y_2 X, \\ S_4 = y_3 y_0 e^H \equiv y_3 X, \end{cases}$$

where $y_1 = a_1 + a_2 + a_3$, $y_2 = a_1 a_2 + a_1 a_3 + a_2 a_3$ and $y_3 = a_1 a_2 a_3$. Then

$$L = -\left(\frac{3}{8} X^2 + \alpha_1 X + \alpha_2\right),$$

$$M = -\left(\frac{1}{8} X^3 + \beta_1 X^2 + \beta_2 X\right),$$

$$N = -\left(\frac{3}{256} X^4 + \gamma_1 X^3 + \gamma_2 X^2 + \gamma_3 X + \gamma_4\right),$$

where

$$\alpha_1 = \frac{3}{2} a_3 - y_1, \qquad \alpha_2 = \frac{1}{2} a_4^2,$$

$$\beta_1 = \frac{3}{4} a_4 - \frac{1}{2} y_1, \qquad \beta_2 = a_4^2 - a_4 y_1 + y_2,$$

$$\gamma_1 = \frac{3}{32} a_4 - \frac{1}{16} y_1, \qquad \gamma_2 = \frac{7}{32} a_4^2 - \frac{1}{4} a_4 y_1 + \frac{1}{4} y_2,$$

$$\gamma_3 = -\frac{1}{16} a_4^3 - \frac{1}{4} a_4^2 y_1 + \frac{1}{2} a_4 y_2 - y_3, \qquad \gamma_4 = -\frac{1}{16} a_4^4.$$

Evidently $2\beta_1 = \alpha_1$, $16\gamma_1 = \alpha_1$ and $\alpha_2 = 4\beta_2 - 16\gamma_2$. Hence the degree of $D_{R_{17}}$ is at most six. The constant term of $D_{R_{17}}$ is just equal to

$$\begin{aligned} &-128\alpha_2^2\gamma_4^2 - 256\gamma_4^3 - 16\alpha_2^4\gamma_4 \\ &= a_4^8\gamma_4(2-1-1) = 0. \end{aligned}$$

The coefficient of X of $D_{R_{17}}$ is equal to the following expression:

$$-256\alpha_1\gamma_4^2\alpha_2 - 256\alpha_2^2\gamma_3\gamma_4 - 256\cdot 3\gamma_3\gamma_4^2 - 16\alpha_2^4\gamma_3 - 64\alpha_1\alpha_2^3\gamma_4.$$

This is equal to zero, which is very easy to prove. Hence

$$D_{R_{17}} = A_6 y_0^6 e^{6H} + A_5 y_0^5 e^{5H} + A_4 y_0^4 e^{4H} + A_3 y_0^3 e^{3H} + A_2 y_0^2 e^{2H}$$

with non-zero coefficients A_2, A_6.

We consider the discriminant of R_{20}. R_{20} is defined by $y^4 - S_1 y^3 + S_2 y^2 - S_3 y + S_4 = 0$ with

$$\begin{cases} S_1 = x_0 e^H + a_3 + a_4 \equiv X + x_1 , \\ S_2 = (2a_1 + a_2) x_0 e^H + a_3 a_4 \equiv (2a_1 + a_2) X + x_2 , \\ S_3 = (a_1^2 + 2a_1 a_2) x_0 e^H \equiv (a_1^2 + 2a_1 a_2) X , \\ S_4 = a_1^2 a_2 x_0 e^H \equiv a_1^2 a_2 X . \end{cases}$$

Then

$$L = -\left(\frac{3}{8} X^2 + \alpha_1 X + \alpha_2\right),$$

$$M = -\left(\frac{1}{8} X^3 + \beta_1 X^2 + \beta_2 X + \beta_3\right),$$

$$N = -\left(\frac{3}{256} X^4 + \gamma_1 X^3 + \gamma_2 X^2 + \gamma_3 X + \gamma_4\right),$$

where

$$\alpha_1 = \frac{3}{4} x_1 - 2a_1 - a_2, \qquad \alpha_2 = \frac{3}{8} x_1^2 - x_2 ,$$

$$\beta_1 = \frac{3}{8} x_1 - \frac{1}{2}(2a_1 + a_2), \quad \beta_2 = \frac{3}{8} x_1^2 - \frac{1}{2}(2a_1 + a_2) x_1 - \frac{1}{2} x_2 + a_1^2 + 2a_1 a_2 ,$$

$$\beta_3 = \frac{1}{8} x_1^3 - \frac{1}{2} x_1 x_2 ,$$

$$\gamma_1 = \frac{3}{64} x_1 - \frac{1}{16}(2a_1 + a_2), \quad \gamma_2 = \frac{9}{128} x_1^2 - \frac{1}{8}(2a_1 + a_2) x_1 - \frac{1}{16} x_2 + \frac{1}{4}(a_1^2 + 2a_1 a_2),$$

$$\gamma_3 = \frac{3}{64} x_1^3 - \frac{1}{16}(2a_1 + a_2) x_1^2 - \frac{1}{8} x_1 x_2 + \frac{1}{4}(a_1^2 + 2a_1 a_2) x_1 - a_1^2 a_2 ,$$

$$\gamma_4 = \frac{3}{256} x_1^4 - \frac{1}{16} x_1^2 x_2 .$$

Evidently $2\beta_1 = \alpha_1$, $16\gamma_1 = \alpha_1$, $\alpha_2 = 4\beta_2 - 16\gamma_2$. Hence the discriminant $D_{R_{20}}$ is of at most sixth degree of X. The coefficient of X^6 is given in § 3 in [1]. This is just the following form:

$$-\frac{27}{16}(\beta_3 - 4\gamma_3)^2 + \frac{9}{2}\alpha_1(\beta_2 - 8\gamma_2)(\beta_3 - 4\gamma_3) + \alpha_1^3(\beta_3 - 4\gamma_3)$$
$$+ 4(\beta_2 - 8\gamma_2)^3 + \alpha_1^2(\beta_2 - 8\gamma_2)^2 .$$

We have

$$\beta_2 - 4\gamma_2 = -\frac{3}{16} x_1^2 + \frac{1}{2}(2a_1 + a_2) x_1 - (a_1^2 + 2a_1 a_2)$$

$$\equiv -\frac{3}{16}x_1^2+\frac{1}{2}y_1x_1-y_2$$

and

$$\beta_3-4\gamma_3=-\frac{1}{16}x_1^3+\frac{1}{4}y_1x_1^2-y_2x_1+y_3, \quad y_3=4a_1^2a_2$$

and

$$\alpha_1=\frac{3}{4}x_1-y_1.$$

Hence the coefficient of X^6 is equal to

$$-\frac{27}{16}\Big(\frac{1}{16}x_1^3-\frac{1}{4}y_1x_1^2+y_2x_1-y_3\Big)^2$$
$$+\frac{9}{2}\Big(\frac{3}{4}x_1-y_1\Big)\Big(\frac{3}{16}x_1^2-\frac{1}{2}y_1x_1+y_2\Big)\Big(\frac{1}{16}x_1^3-\frac{1}{4}y_1x_1^2+y_2x_1-y_3\Big)$$
$$+\Big(\frac{3}{4}x_1-y_1\Big)^3\Big(-\frac{1}{16}x_1^3+\frac{1}{4}y_1x_1^2-y_2x_1+y_3\Big)$$
$$-4\Big(\frac{3}{16}x_1^2-\frac{1}{2}y_1x_1+y_2\Big)^3+\Big(\frac{3}{4}x_1-y_1\Big)^2\Big(\frac{3}{16}x_1^2-\frac{1}{2}y_1x_1+y_2\Big)^2.$$

It is easy to prove that the above expression vanishes identically. The constant term of $D_{R_{20}}$ is

$$-27\beta_3^4+144\alpha_2\gamma_4\beta_3^2-128\alpha_2^2\gamma_4^2-256\gamma_4^3+4\alpha_2^3\beta_3^2-16\alpha_2^4\gamma_4.$$

By a simple computation we can prove that this is equal to zero. Hence we have

$$D_{R_{20}}=A_5x_0^5e^{5H}+A_4x_0^4e^{4H}+A_3x_0^3e^{3H}+A_2x_0^2e^{2H}+A_1x_0e^H$$

with non-zero coefficients A_1, A_5.

We consider the case R_{22}, which is defined by

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=y_1, \\ S_2=y_0e^H+y_2\equiv X+y_2, \\ S_3=(a_1+a_2)y_0e^H+y_3\equiv(a_1+a_2)X+y_3, \\ S_4=a_1a_2y_0e^H\equiv a_1a_2X. \end{cases}$$

Here $y_1=a_3+2a_4$, $y_2=2a_3a_4+a_4^2$ and $y_3=a_3a_4^2$.

Then

$$L=X+\alpha_1,$$

$$M=\beta_0 X+\beta_1,$$

$$N=\gamma_0 X+\gamma_1,$$

where

$$\alpha_1=-\frac{3}{8}y_1^2+y_2,\qquad \beta_0=\frac{1}{2}y_1-a_1-a_2,$$

$$\beta_1=-\frac{1}{8}y_1^3+\frac{1}{2}y_1y_2-y_3,\quad \gamma_0=\frac{1}{16}y_1^2-\frac{1}{4}(a_1+a_2)y_1+a_1a_2,$$

$$\gamma_1=-\frac{3}{256}y_1^4+\frac{1}{16}y_1^2y_2-\frac{1}{4}y_1y_3.$$

In this case $D_{R_{22}}$ is

$$-27M^4+144LM^2N-128L^2N^2+256N^3-4L^3M^2+16L^4N.$$

Hence $D_{R_{22}}$ is a polynomial of X of at most fifth degree. The coefficient of X^5 is

$$-4\beta_0^2+16\gamma_0=-4(a_1-a_2)^2\neq 0.$$

We shall compute the constant term of $D_{R_{22}}$. This is

$$-27\beta_1^4+144\alpha_1\beta_1^2\gamma_1-128\alpha_1^2\gamma_1^2+256\gamma_1^3-4\alpha_1^3\beta_1^2+16\alpha_1^4\gamma_1.$$

Hence we should compute the following expression:

$$\begin{aligned}
&-27\Big(-\frac{1}{8}y_1^3+\frac{1}{2}y_1y_2-y_3\Big)^4\\
&+144\Big(-\frac{3}{8}y_1^2+y_2\Big)\Big(-\frac{3}{256}y_1^4+\frac{1}{16}y_1^2y_2-\frac{1}{4}y_1y_3\Big)\Big(-\frac{1}{8}y_1^3+\frac{1}{2}y_1y_2-y_3\Big)^2\\
&-128\Big(\frac{3}{8}y_1^2-y_2\Big)^2\Big(\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2+\frac{1}{4}y_1y_3\Big)^2\\
&-256\Big(\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2+\frac{1}{4}y_1y_3\Big)^3\\
&+4\Big(\frac{3}{8}y_1^2-y_2\Big)^3\Big(\frac{1}{8}y_1^3-\frac{1}{2}y_1y_2+y_3\Big)^2\\
&-16\Big(\frac{3}{8}y_1^2-y_2\Big)^4\Big(\frac{3}{256}y_1^2-\frac{1}{16}y_1^2y_2+\frac{1}{4}y_1y_3\Big)^2.
\end{aligned}$$

We can prove that coefficients of y_1^{12}, $y_1^{10}y_2$, $y_1^9y_3$, $y_1^8y_2^2$, $y_1^7y_2y_3$, $y_1^6y_3^2$, $y_1^6y_2^3$, $y_1^5y_2^2y_3$, $y_1^4y_2^4$ and $y_1^4y_2y_3^2$ are all equal to zero. Hence the above expression reduce to

$$B_1y_1^3y_2^3y_3+B_2y_1^2y_2^2y_3^2+B_3y_1y_2y_3^3+B_4y_1^3y_3^3+B_5y_2^3y_3^2+B_6y_3^4$$

with $B_1=0$, $B_2=1$, $B_3=18$, $B_4=-4$, $B_5=-4$, $B_6=-27$. Hence we have

$$y_3^2(y_1^2y_2^2+18y_1y_2y_3-4y_1^3y_3-4y_2^3-27y_3^2).$$

Let us put $y_1=a_3+2a_4$, $y_2=(2a_3+a_4)a_4$, $y_3=a_3a_4^2$. Then

$$\begin{aligned} y_1^2y_2^2-18y_1y_2y_3-4y_1^3y_3-4y_2^3-27y_3^2 \\ &=(a_3+2a_4)^2a_4^2(2a_3+a_4)^2+18(a_3+2a_4)(2a_3+a_4)a_3a_4^3 \\ &\quad-4(a_3+2a_4)^3a_3a_4^2-4(2a_3+a_4)^3a_4^3-27a_3^2a_4^4 \\ &=0. \end{aligned}$$

Therefore

$$D_{R_{22}}=-4(a_1-a_2)^2y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H$$

with non-zero coefficient A_1.

We consider $D_{R_{28}}$. R_{28} is defined by

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=y_1, \\ S_2=y_2, \\ S_3=y_0e^H+y_3\equiv X+y_3, \\ S_4=a_1y_0e^H\equiv a_1X. \end{cases}$$

Here $y_1=a_2+a_3+a_4$, $y_2=a_2a_3+a_2a_4+a_3a_4$, $y_3=a_2a_3a_4$. Then

$$\begin{aligned} L&=-\frac{3}{8}y_1^2+y_2\equiv\alpha_1, \\ M&=-X+\beta_1, \\ N&=\gamma_0X+\gamma_1, \end{aligned}$$

where

$$\beta_1=-\frac{1}{8}y_1^3+\frac{1}{2}y_1y_2-y_3, \qquad \gamma_0=-\frac{1}{4}y_1+a_1,$$

$$\gamma_1=-\frac{3}{256}y_1^4+\frac{1}{16}y_1^2y_2-\frac{1}{4}y_1y_3.$$

Therefore $D_{R_{28}}$ is equal to

$$\begin{aligned} &-27(X-\beta_1)^4+144\alpha_1(X-\beta_1)^2(\gamma_0X+\gamma_1)-128\alpha_1^2(\gamma_0X+\gamma_1)^2 \\ &+256(\gamma_0X+\gamma_1)^3-4\alpha_1^3(X-\beta_1)^2+16\alpha_1^4(\gamma_0X+\gamma_1). \end{aligned}$$

Hence

$$D_{R_{28}}=-27X^4+A_3X^3+A_2X^2+A_1X+A_0$$

with $X=y_0e^H$ and a non-zero coefficient A_0.

Finally we consider $D_{R_{29}}$. R_{29} is defined by

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=y_1, \\ S_2=y_0e^H+y_2\equiv X+y_2, \\ S_3=2a_1y_0e^H+y_3\equiv 2a_1X+y_3, \\ S_4=a_1^2y_0e^H\equiv a_1^2X. \end{cases}$$

Here $y_1=a_2+a_3+a_4$, $y_2=a_2a_3+a_2a_4+a_3a_4$, $y_3=a_2a_3a_4$. Then

$$L=X+\alpha_1,$$

$$M=\beta_0X+\beta_1,$$

$$N=\gamma_0X+\gamma_1$$

with

$$\alpha_1=-\frac{3}{8}y_1^2+y_2,\quad \beta_0=\frac{1}{2}y_1-2a_1,\quad \beta_1=-\frac{1}{8}y_1^3+\frac{1}{2}y_1y_2-y_3,$$

$$\gamma_0=\frac{1}{16}y_1^2-\frac{a_1}{2}y_1+a_1^2,\quad \gamma_1=-\frac{3}{256}y_1^4+\frac{1}{16}y_1^2y_2-\frac{1}{4}y_1y_3.$$

Then the coefficient of X^5 of $D_{R_{29}}$ is equal to

$$-4\beta_0^2+\gamma_0=-4\Big(\frac{1}{4}y_1^2-2a_1y_1+4a_1^2\Big)+16\Big(\frac{1}{16}y_1^2-\frac{a_1}{2}y_1+a_1^2\Big)$$

$$=0.$$

Therefore $D_{R_{29}}$ is a polynomial of X of fourth degree, Hence

$$D_{R_{29}}=A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0$$

with non-zero coefficients A_0, A_4.

§6. Remarks

Let us put

$$F(z,\ y)\equiv y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

and

$$\alpha^4 G(z, Y) \equiv F(z, \alpha Y+\beta)$$
$$\equiv \alpha^4[Y^4-T_1Y^3+T_2Y^2-T_3Y+T_4].$$

Evidently

$$\{(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_2-y_3)(y_2-y_4)(y_3-y_4)\}^2$$
$$=\alpha^{12}\{(Y_1-Y_2)(Y_1-Y_3)(Y_1-Y_4)(Y_2-Y_3)(Y_2-Y_4)(Y_3-Y_4)\}^2.$$

Hence $R_1 \sim R_2$ implies $D_{R_1}=\alpha^{12}D_{R_2}$. Therefore the non-vanishing property or the vanishing property of coefficients is completely preserved. Hence the forms of discriminants of all surfaces listed in § 2, 3 and 4 are completely determind.

We shall not give any proof of the following fact: Let R be the Riemann surface R_{13}. Let F be a regular function on R. Then F is representable as

$$F=f_1+f_2y+f_3y^2+f_4y^3$$

where f_1, f_2, f_3 and f_4 are meromorphic functions in $|z|<+\infty$, all of which are regular at any points satisfying $H'(z)\neq 0$.

We can prove this quite similarly as in § 6 in [1]. And the similar facts for R_{16}, R_{17}, R_{20}, R_{22}, R_{28} and R_{29} hold.

Further we can make use of transformation formula of discriminants established in § 7 in [1].

§ 7. Theorems

We now introduce an assumption that $H(z)$ is a polynomial.

Let R be the surface R_{13}: $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with $S_1=x_0e^H+x_1$, $S_2=x_2$, $S_3=x_3$ and $S_4=x_4$, where $x_1=a_1+a_2+a_3+a_4$, $x_2=a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4$, $x_3=a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4$, $x_4=a_1a_2a_3a_4$. Then $P(y)=6$. Suppose that $P(R_{13})>6$.

If $P(R_{13})=7$, then there is non-constant regular function F on R_{13} such that $P(F)=7$ and

$$F=f_1+f_2y+f_3y^2+f_4y^3,$$

where f_1, f_2, f_3, f_4 are meromorphic in $|z|<\infty$ and regular excepting at most at points satisfying $H'=0$. We assume that F defines the surface R_4^*: $F^4-U_1F^3+U_2F^2-U_3F+U_4=0$ with $U_1=y_1$, $U_2=y_0e^L+y_2$, $U_3=b_1y_0e^L+y_3$, $U_4=y_4$, where $y_1=b_2+b_3+b_4+b_5$, $y_2=b_2b_3+b_2b_4+b_2b_5+b_3b_4+b_3b_5+b_4b_5$, $y_3=b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5$ and $y_4=b_2b_3b_4b_5$. Let us denote the discriminants of R_{13} and R_4^* by Δ and D, respectively. Then we have

$$D=\Delta\cdot G^2,$$

where G may have poles, whose number is finite. Let us denote

$$D=-4b_1^2(y_0e^L-\delta_1)(y_0e^L-\delta_2)(y_0e^L-\delta_3)(y_0e^L-\delta_4)(y_0e^L-\delta_5)$$

and

$$\Delta=A_4(x_0e^H-\gamma_1)(x_0e^H-\gamma_2)(x_0e^H-\gamma_3)(x_0e^H-\gamma_4).$$

CASE 1). The counting function of simple zeros of Δ satisfies

$$N_2(r, 0, \Delta)\sim 4T(r, e^H),$$

that is, $\gamma_i\neq\gamma_j$ for $i\neq j$. Then

$$N_2(r, 0, \Delta)=N_2(r, 0, D)\sim m\cdot T(r, e^L)$$

with $m=1, 2, 3, 5$. Then L should be a polynomial such that $\deg L=\deg H$. In this case we can return back y from F. Then we have

$$\Delta=D\cdot I^2.$$

The number of poles of I is finite again. Hence $G^2\cdot I^2=1$. The zeros of G coincides with poles of I. Hence $G=\beta e^M$ with rational β and M entire with $M(0)=0$. M may reduce to constant 0. In this case $\delta_i\neq\delta_j$ for $i\neq j$.

CASE 2). $N_2(r, 0, \Delta)\sim 2T(r, e^H)$, that is, $\gamma_1\neq\gamma_2$, $\gamma_1\neq\gamma_3$, $\gamma_2\neq\gamma_3$ but $\gamma_3=\gamma_4$. Then $N_2(r, 0, \Delta)=N_2(r, 0, D)\sim m\cdot T(r, e^L)$ with $m=1, 2, 3, 5$. Then L should be a polynomial such that $\deg L=\deg H$. We can return back y from F. Then $\Delta=D\cdot I^2$. Then number of poles of I is finite. Hence $G^2\cdot I^2=1$. The zeros of G conincides with poles of I. Then we can count the multiple zeros. The counting function of multiple zeros

$$N_0(r, 0, \Delta)=(1+o(1))N_0(r, 0, D),$$

where $N_0(r, 0, \Delta)=N(r, 0, \Delta)-N_2(r, 0, \Delta)$. Hence

$$N_1(r, 0, \Delta)=2\cdot T(r, e^H)$$

and

$$N(r, 0, \Delta)=(1+o(1))(5-m)T(r, e^L).$$

Then $m=5/2$, which is absurd, since m is an integer.

CASE 3). $N_0(r, 0, \Delta)\sim T(r, e^H)$, that is, $\gamma_1\neq\gamma_2=\gamma_3=\gamma_4$. Then $N_0(r, 0, D)\sim 2\cdot T(r, e^L)$ and the counting functions of triple zeros $N_3(r, 0, \Delta)$, $N_3(r, 0, D)$ satisfies $N_3(r, 0, \Delta)=N_3(r, 0, D)$ and $N_3(r, 0, \Delta)=3\cdot T(r, e^H)$, $N_3(r, 0, D)=3\cdot T(r, e^L)$. This is a contradiction.

CASE 4). Δ does not have any simple zero. Then either

$$N_2(r, 0, D)\sim T(r, e^L) \quad\text{or}\quad N_3(r, 0, D)\sim 3\cdot T(r, e^L)$$

but

$$N_3(r, 0, \Delta)=o(1) \quad\text{or}\quad N_5(r, 0, D)\sim 5\cdot T(r, e^L)$$

where N_5 is the counting function of multiplicity 5. All of these lead to a contradiction.

Therefore we have

$$D=\Delta\beta^2 e^{2M}$$

with a rational function β. Further D, Δ must have only simple factors. Hence we have

$$5\cdot T(r, e^L)\sim N(r, 0, D)=N(r, 0, \Delta)\sim 4T(r, e^H).$$

Hence

$$T(r, e^L)\sim\frac{4}{5}T(r, e^H).$$

We have

$$-4b_1^2 y_0^5 e^{5L}+B_4 y_0^4 e^{4L}+B_3 y_0^3 e^{3L}+B_2 y_0^2 e^{2L}+B_1 y_0 e^L+B_0$$
$$=(A_4 x_0^4 e^{4H}+A_3 x_0^3 e^{3H}+A_2 x_0^2 e^{2H}+A_1 x_0 e^H+A_0)\beta^2 e^{2M}$$

with non-zero constants B_0, A_4, A_0. By Borel's unicity theorem we have only two possibilities: either

$$M\equiv 0,\ -4b_1^2 y_0^5=A_4 x_0^4\beta^2,\ B_0=A_0\beta^2,\ 5L=4H\ \text{ and }\ B_4=B_3=B_2=B_1=A_3$$
$$=A_2=A_1=0$$

or

$$M=-2H,\ 5L=-4H,\ -4b_1^2 y_0^5=A_0\beta^2,\ B_0=A_4 x_0^4\beta^2\ \text{ and }\ B_4=B_3=B_2=B_1$$
$$=A_3=A_2=A_1=0.$$

If F defines the surface R_6^*, then the same proof does work. So we shall omit its detail.

We assume that F defines the surface R_7^*: $F^4-U_1F^3+U_2F^2-U_3F+U_4=0$ with $U_1=y_0e^L+y_1$, $U_2=\alpha_1 y_0 e^L+y_2$, $U_3=\alpha_2 y_0 e^L$, $U_4=\alpha_3 y_0 e^L$, where $y_1=b_4+b_5$, $y_2=b_4b_5$, $\alpha_1=b_1+b_2+b_3$, $\alpha_2=b_1b_2+b_1b_3+b_2b_3$, $\alpha_3=b_1b_2b_3$. Then we have

$$D_{R_7*}=\Delta_{R_{13}}\cdot G^2,$$

where G may have poles, whose number is finite. Let us denote

$$D_{R_7*}=B_6(y_0e^L-\delta_1)(y_0e^L-\delta_2)(y_0e^L-\delta_3)(y_0e^L-\delta_4)(y_0e^L-\delta_5)y_0e^L$$
$$\equiv B_6 y_0^6 e^{6L}+B_5 y_0^5 e^{5L}+B_4 y_0^4 e^{4L}+B_3 y_0^3 e^{3L}+B_2 y_0^2 e^{2L}+B_1 y_0 e^L$$

with non-zero coefficients B_1, B_6. Quite similarly we have

$$D_{R_7*}=\Delta_{R_{13}}\cdot\beta^2 e^{2M}$$

wtih a rational function β. Further D_{R_7*} and $\Delta_{R_{13}}$ must have only simple factors. Therefore

$$5T(r, e^L)\sim 4T(r, e^H).$$

By

$$B_6 y_0{}^6 e^{6L} + B_5 y_0{}^5 e^{5L} + B_4 y_0{}^4 e^{4L} + B_3 y_0{}^3 e^{3L} + B_2 y_0{}^2 e^{2L} + B_1 y_0 e^L$$
$$= (A_4 x_0{}^4 e^{4H} + A_3 x_0{}^3 e^{3H} + A_2 x_0{}^2 e^{2H} + A_1 x_0 e^H + A_0)\beta^2 e^{2M}$$

and by Borel's unicity theorem we have two possibilities:

$2M - L = 0$, $5L = 4H$, $B_6 y_0{}^6 = A_4 x_0{}^4 \beta^2$, $B_1 y_0 = A_0 \beta^2$ and $B_5 = B_4 = B_3 = A_3 = A_2 = A_1 = 0$

or

$4H + 2M - L = 0$, $2M - L = 5L (5L = -4H)$, $A_0 \beta^2 = B_6 y_0{}^6$, $A_4 x_0{}^4 \beta^2 = B_1 y_0$ and $B_5 = B_4 = B_3 = B_2 = A_3 = A_2 = A_1 = 0$.

If $P(R_{13}) = 8$, then there is a non-constant regular function F on R_{13} such that $P(F) = 8$ and $F = f_1 + f_2 y + f_3 y^2 + f_4 y^3$ defines the surface X_1. Then

$$D_{X_1} = \Delta_{R_{13}} \cdot G^2.$$

And we can prove that

$$D_{X_1} = \Delta_{R_{13}} \cdot \beta^2 e^{2M}$$

with a rational function β. Then Borel's unicity theorem implies that $A_3 = A_2 = A_1 = 0$. Hence we have the following result. In the following theorems we always assume that e^H is an entire function of finite order.

THEOREM 1. *Let us denote the discriminant of* R_{13}

$$\Delta_{R_{13}} = A_4 x_0{}^4 e^{4H} + A_3 x_0{}^3 e^{3H} + A_2 x_0{}^2 e^{2H} + A_1 x_0 e^H + A_0$$

with non-zero coefficients A_0, A_4. *If at least one of coefficients* A_3, A_2, A_1 *does not vanish. Then* $P(R_{13}) = 6$.

THEOREM 2. *Let us denote the discriminant of* R_{16}

$$\Delta_{R_{16}} = A_6 x_0{}^6 e^{6H} + A_5 x_0{}^5 e^{5H} + A_4 x_0{}^4 e^{4H} + A_3 x_0{}^3 e^{3H} + A_2 x_0{}^2 e^{2H}$$

with non-zero coefficients A_6, A_2. *If at least one of coefficients* A_5, A_4, A_3 *does not vanish, then* $P(R_{16}) = 6$.

THEOREM 3. *Let us denote the discriminant of* R_{17}

$$\Delta_{R_{17}} = A_6 x_0{}^6 e^{6H} + A_5 x_0{}^5 e^{5H} + A_4 x_0{}^4 e^{4H} + A_3 x_0{}^3 e^{3H} + A_2 x_0{}^2 e^{2H}$$

with non-zero coefficients A_6, A_2. *If at least one of coefficients* A_5, A_4, A_3 *does not vanish, then* $P(R_{17}) = 6$.

THEOREM 4. *Let us put the discriminant of* R_{20}

$$\Delta_{R_{20}}=A_5x_0{}^5e^{5H}+A_4x_0{}^4e^{4H}+A_3x_0{}^3e^{3H}+A_2x_0{}^2e^{2H}+A_1x_0e^H$$

with non-zero coefficients A_5, A_1. *If at least one of coefficients* A_4, A_3, A_2 *does not vanish, then* $P(R_{20})=6$.

The same holds for R_{22}.

THEOREM 5. *Let us put the discriminant of* R_{28}

$$\Delta_{R_{28}}=A_4x_0{}^4e^{4H}+A_3x_0{}^3e^{3H}+A_2x_0{}^2e^{2H}+A_1x_0e^H+A_0$$

with $A_4=-27$ *and* $A_0\neq 0$. *If at least one of coefficients* A_3, A_2, A_1 *does not vanish, then* $P(R_{28})=6$.

The same holds for R_{29}.

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