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DISCRETE MEASURES AND THE RIEMANN HYPOTHESIS

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1. Introduction

The purpose of this paper is to show that the Riemann hypothesis is equivalent to a problem of the rate of convergence of certain discrete measures defined on the positive real numbers to the measure $\frac{6}{\pi^2}udu$, where du is Lebesgue measure.

As a motivation consider the following: For each positive real number y, let μ_y be the infinite measure on the real line defined by

$$\mu_y = \sum_{n\in\mathbb{Z}} y\,\delta_{ny},$$

where $\mathbb{Z}$ denotes the integers and δ_x denotes the Dirac mass at the point $x \in \mathbb{R}$. It follows by the Poisson summation formula that if $f \in C_c^\infty(\mathbb{R})$ ($C_c^\infty(\mathbb{R})$ = functions $f : \mathbb{R} \to \mathbb{R}$, of class C^∞ and with compact support), then for every $\beta > 0$:

$$\mu_y(f) = \int_{\mathbb{R}} f(t)\,dt + o(y^\beta), \qquad \text{as } y \to 0.$$

This is so because by the Poisson summation formula [B],

$$y\sum_{n\in\mathbb{Z}} f(ny) = \sum_{n\in\mathbb{Z}} \widehat{f}(ny^{-1})$$

where $\widehat{f}$ is the Fourier transform of f and, since f is smooth with compact support we have that $\widehat{f}$ is of rapid decay at infinity. Hence

$$y\sum_{n\in\mathbb{Z}} f(ny) = \widehat{f}(0) + o(y^\beta) \quad \text{as } y \to 0 \text{ for all } \beta > 0.$$

So, as $y \to 0$, the atoms of μ_y cluster uniformly and $\mu_y(f)$ gives a very good approximation of integrals of smooth functions with compact support.

Now let $\mathbb{R}^\bullet$ denote the multiplicative group of positive real numbers. For each $y \in \mathbb{R}^\bullet$, let us consider the infinite measure, m_y, defined on smooth functions with compact support in $\mathbb{R}^\bullet$, by the formula:

$$m_y(f) = \sum_{n\in\mathbb{N}} y\varphi(n) f(y^{\frac{1}{2}}n) \tag{1}$$

where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers and $\varphi(n) = n\prod_{p|n}(1 - \frac{1}{p})$ is Euler's totient function, which counts the number of integers which are relatively prime to a given integer, and are lesser or equal to that integer. In fact, for every $r \geqq 0$, r an

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integer or $r = \infty$, we can consider m_y as an element in the dual space of $C_c^r(\mathbb{R}^\bullet) =$ complex-valued functions $f : \mathbb{R}^\bullet \to \mathbb{C}$, of class C^r, with compact support.

We will prove the following theorems:

THEOREM A. *For every* $f \in C_c^0(\mathbb{R}^\bullet)$:

$$\begin{aligned} m_y(f) &= \frac{1}{\zeta(2)} \int_0^\infty u f(u) du + \mathcal{O}(y^{1/2} \log y) \\ &= \frac{6}{\pi^2} \int_0^\infty u f(u) du + \mathcal{O}(y^{\frac{1}{2}} \log y) \quad \text{as } y \to 0. \end{aligned} \tag{2}$$

DEFINITION. Let $m_0(f) = \int_0^\infty \frac{6}{\pi^2} u f(u) du$.

THEOREM B. 1) *The Riemann hypothesis is true if and only if for every function* $f \in C_c^r(\mathbb{R}^\bullet)$, *with* $2 \leqq r \leqq \infty$

$$m_y(f) = m_0(f) + o(y^{\frac{3}{4} - \epsilon}),$$

as $y \to 0$, *for all* $\epsilon > 0$.

2) *Furthermore, if* $\alpha \in (\frac{1}{2}, \frac{3}{4})$ *is such that, for all functions* $f \in C_c^2(\mathbb{R}^\bullet)$ *one has,*

$$m_y(f) = m_0(f) + o(y^{\alpha - \epsilon}), \tag{3}$$

as $y \to 0$, *for all* $\epsilon > 0$, *then the Riemann zeta-function has no zeroes in the half-plane* $\Re(s) > 2(1 - \alpha)$. *Conversely, if the Riemann zeta-function has no zeroes in the half-plane* $\Re(s) > 2(1 - \alpha)$ *then (3) holds for all functions* $f \in C_c^2(\mathbb{R}^\bullet)$.

3) *If f is the characteristic function of an interval then:*

$$\overline{\lim_{y \to 0}} y^{-\alpha} \left| m_y(f) - m_0(f) \right| = \infty, \quad \text{if } \alpha > \frac{1}{2}.$$

Hence $\frac{1}{2}$ *is the best possible exponent of the error for some nonsmooth functions.*

4) *Let the function F, with domain in the positive reals be defined by*

$$F(x) = \begin{cases} 1 - x & \text{for } x \leqq 1 \\ 0 & \text{for } x > 1 \end{cases}$$

Then:

$$m_y(F) = \frac{1}{\pi^2} + o(y^{\frac{1}{2}}) \quad \text{as } y \to 0,$$

and,

$$\overline{\lim_{y \to 0}} y^{-\alpha} \left| m_y(F) - m_0(F) \right| = \infty, \quad \text{for all } \alpha > \frac{1}{2} \tag{4}$$

if and only if the Riemann hypothesis is false in the strongest possible sense: there exist zeroes of Riemann's ζ*-function arbitrarily close to the critical line* $\Re(s) = 1$.

If $f = \chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, $0 < a < b$, then we can also define $m_y(f)$ in the obvious manner:

$$m_y(f) = \sum_{a y^{-\frac{1}{2}} \leq n \leq b y^{-\frac{1}{2}}} y \varphi(n).$$

The measures m_y and their connection to the Riemann hypothesis were discovered by the author as a consequence of studying geometrically the beautiful paper [Z] by Don Zagier. The author wrote [V] inspired by this paper which contains a remarkable connection obtained by Zagier between the Riemann Hypothesis and horocyclic measures on the modular orbifold (see also P. Sarnak [S] and E. Ghys [G]). The present paper can be thought of as a continuation of [V].

In order to be as self-contained as possible we will recall some classical and fundamental results.

2. Preliminaries

First, let us start by proving formula (2) for characteristic functions. Let $f = \chi_{[a,b]}$ be the characteristic function of the interval $[a,b]$, where $0 < a < b$. Then:

$$\begin{aligned} m_y(f) &= \sum_{ay^{-\frac{1}{2}} \leq n \leq by^{-\frac{1}{2}}} y\varphi(n) = \frac{1}{2\zeta(2)}[b^2 - a^2] + \mathcal{O}(y^{\frac{1}{2}} \log y) \\ &= \frac{1}{\zeta(2)} \int_0^\infty uf(u)du + \mathcal{O}(y^{\frac{1}{2}} \log y). \end{aligned} \tag{5}$$

The second equality follows from the well-known formula:

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + \mathcal{O}(x \log x), \quad x > 1. \tag{6}$$

This formula, due to Mertens (1874), can be found, for instance, in Hardy and Wright [HW] or Apostol [A], p. 70.

Thus,

$$m_y(f) = m_0(f) + \mathcal{O}(y^{\frac{1}{2}} \log y) \quad \text{as } y \to 0.$$

If $m_y(f) - m_0(f) = K_f(y)$, then $K_f(y) = h_f(y)y^{\frac{1}{2}} \log y$ and $h_f(y)$ remains bounded as $y \to 0$ and the bound depends only on the interval $[a,b]$. If $f \in C_c^1(\mathbb{R}^\bullet)$ then we can apply Abel's summation formula and (6) to obtain:

$$\begin{aligned} m_y(f) &= \sum_{n \in \mathbb{N}} y\varphi(n)f(y^{\frac{1}{2}}n) = -\frac{1}{2\zeta(2)} \int_0^\infty u^2 f'(u)\, du + \mathcal{O}(y^{\frac{1}{2}} \log y) \\ &= \frac{1}{\zeta(2)} \int_0^\infty uf(u)du + \mathcal{O}(y^{\frac{1}{2}} \log y), \quad \text{as } y \to 0. \end{aligned}$$

Since any continuous function with compact support in $\mathbb{R}^\bullet$ can be uniformly approximated by C^1 functions with compact support in $\mathbb{R}^\bullet$, and the error terms depend only on the support of the functions, we immediately obtain Theorem A. However, Theorem A will also be a consequence of what follows. It is interesting to note that the volume of $PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$, with respect to Haar measure, is $\pi^2/3$. As it turns out, Mertens Theorem corresponds to the statement that the ergodic measures of the horocyclic flow which are supported on the periodic orbits and uniformly distributed with respect to arc-length, converge vaguely to Haar measure as the period tends to infinity (see [V] and [Z]).

2.1 Mellin transform

To see how naturally the Riemann ζ function arises in connection with the measures m_y, let us first recall a classical formula:

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n\geq 1} \frac{\varphi(n)}{n^s}, \quad \text{for} \quad \Re(s) > 2 \tag{7}$$

(see, for instance, [A], p. 229). Let $r \geqq 0$ be an integer or infinity. For each $f \in C_c^r(\mathbb{R}^\bullet)$ consider the *Mellin transform* of $m_y(f)$:

$$\mathcal{M}_f(s) = \int_0^\infty m_y(f) y^{s-2}\, dy. \tag{8}$$

PROPOSITION 2.1.1. *The integral defining $\mathcal{M}_f(s)$ converges absolutely in the half-plane $\Re(s) > 1$ and uniformly in $\Re(s) > 1+\epsilon$ for all $\epsilon > 0$. Therefore, it defines a holomorphic function on the half plane $\Re(s) > 1$.*

Proof. Let us suppose that support $(f) \subset [a,b]$; $0 < a < b$. Let $\|f\|_\infty = \sup_{y\in\mathbb{R}^\bullet} |f(y)|$. Then if $\Re(s) > 1$, we have, since $|m_y(f)| \leqq A\|f\|_\infty$, for some $A > 0$:

$$|\mathcal{M}_f(s)| \leqq \|f\|_\infty A \left(\frac{b^{\sigma-1}}{\sigma-1}\right), \quad \text{where } \sigma = \Re(s). \tag{9}$$

Therefore, we have absolute convergence in $\Re(s) > 1$, and evidently the convergence is uniform in $\Re(s) > 1+\epsilon$ for $\epsilon > 0$. □

Remarks 2.1.2.

(a) Strictly speaking, equation (8) defines, in classical notation, the Mellin transform of $y^{-1}m_y(f)$; however, we will still call it the Mellin transform of $m_y(f)$. Let $[C_c^r(\mathbb{R}^\bullet)]^*$ denote the topological dual of $C_c^r(\mathbb{R}^\bullet)$. Then the function

$$\mathcal{M} :, \ \{\Re(s) > 1\} \to [C_c^r(\mathbb{R}^\bullet)]^*,$$

given by

$$s \mapsto \int_0^\infty m_y(\cdot) y^{s-2}\, dy, \quad \Re(s) > 1$$

defines a weakly holomorphic function. For every s such that $\Re(s) > 1$, $\mathcal{M}$ defines an infinite measure on $\mathbb{R}^\bullet$. When $r = \infty$, $\mathcal{M}$ defines a holomorphic function whose values are distributions of finite order. Compare [S]. We will be able to continue $\mathcal{M}$ analytically to obtain a weakly meromorphic function with values in the distribution space of $\mathbb{R}^\bullet$.

(b) We notice that for every $0 \leqq r \leqq \infty$, and $f \in C_c^r(\mathbb{R}^\bullet)$ we have $m_y(f) = 0$ if y is sufficiently large:

$$m_y(f) = 0 \quad \text{for all} \quad y > b, \quad \text{where support}(f) \subset [a,b].$$

(c) Via the logarithm, or the exponential, we can transport measures defined on $\mathbb{R}$ to measures defined on $\mathbb{R}^\bullet$, and vice versa. Let $m_y^+ = \exp^*(m_y)$ be the measure on the real line obtained by pulling back m_y by $\exp : \mathbb{R} \to \mathbb{R}^\bullet$. Then m_y^+ is supported on a discrete set of points which is irregularly distributed on the real line and the Dirac masses that define m_y^+ are weighted by Euler's function. This accounts for the difference between the measures μ_y defined at the beginning of the introduction and m_y, as far as error terms are concerned. This also establishes a connection between the measures $\{m_y\}_{y>0}$ and Farey sequences as in the well-known results of Franel ([F]) and Landau ([La]). See ([V]).

Now, let us combine equations (1) and (8) to obtain:

$$\mathcal{M}_f(s) = \int_0^\infty \Big(\sum_{n\in\mathbb{N}} y\varphi(n) f(y^{\frac{1}{2}}n)\Big) y^{s-2}\, dy; \quad \Re(s) > 1. \tag{10}$$

Fix $n \in \mathbb{N}$ and define $\psi_n : \mathbb{R}^\bullet \to \mathbb{C}$ by the formula

$$\psi_n(y) = y\varphi(n) f(y^{\frac{1}{2}}n). \tag{11}$$

Then:

$$\int_0^\infty \psi_n(y) y^{s-2}\, dy = \varphi(n) \int_0^\infty f(y^{\frac{1}{2}}n) y^{s-1}\, dy.$$

Changing variable: $u = y^{\frac{1}{2}}n$, we get:

$$\int_0^\infty \psi_n(y) y^{s-2}\, dy = 2\frac{\varphi(n)}{n^{2s}} \int_0^\infty f(u) u^{2s-1}\, du; \quad \Re(s) > 1. \tag{12}$$

Now, if $\sigma = \Re(s) > 2$, we have $\left|\frac{\varphi(n)}{n^{2s}}\right| \leq \frac{1}{n^3}$. Hence, by the Lebesgue dominated convergence theorem and formula (7) we obtain:

PROPOSITION 2.1.3.

$$\mathcal{M}_f(s) = 2\frac{\zeta(2s-1)}{\zeta(2s)} \int_0^\infty f(u) u^{2s-1}\, du; \quad \Re(s) > 2. \tag{13}$$

Furthermore if F is the function defined in Theorem B part 3), then all the above applies so:

$$\mathcal{M}_f(s) = \frac{\zeta(2s-1)}{s(2s+1)\zeta(2s)}.$$

Let

$$\varphi_f(s) = \int_0^\infty f(u) u^{2s-1}\, du. \tag{14}$$

Since f has compact support it follows that $\varphi_f(s)$ is an entire function with derivative

$$\frac{d}{ds}(\varphi_f(s)) = 2\int_0^\infty f(u)(\log u) u^{2s-1}\, du. \tag{15}$$

Therefore, we see from (13) that $\mathcal{M}_f(s) = \frac{2\zeta(2s-1)}{\zeta(2s)}\varphi_f(s)$, can be continued as a meromorphic function to all of $\mathbb{C}$ and we obtain, using the properties of ζ, the following:

PROPOSITION 2.1.4. a) *$\mathcal{M}_f(s)$ is a meromorphic function with a simple pole at $s = 1$ with residue:*

$$\mathrm{Res}_{s=1}(\mathcal{M}_f(s)) = \frac{1}{\zeta(2)} \int_0^\infty u f(u)\, du.$$

b) *All other possible poles of $\mathcal{M}_f(s)$ are the negative integers and the zeroes of $\zeta(2s)$ in the strip $0 \leq \Re(s) < 1/2$.*

From the functional equation of ζ,

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\zeta(1-s) \tag{16}$$

we have:

PROPOSITION 2.1.5 (Functional Equation). *The function $\mathcal{M}_f(s)$ satisfies the functional equation:*

$$\frac{\pi^{s-\frac{1}{2}}\mathcal{M}_f(\frac{1}{2}-s)}{\varphi_f(\frac{1}{2}-s)\Gamma(\frac{1}{2}-s)\zeta(-2s)} = \frac{\pi^{-s}\mathcal{M}_f(s)}{\varphi_f(s)\Gamma(s)\zeta(2s-1)}. \tag{17}$$

Suppose that $f \in C_c^k(\mathbb{R}^\bullet)$, for $k \geqq 1$. Then, integrating by parts we obtain:

$$\varphi_f(s) = \frac{(-1)^k}{2s(2s+1)\dots(2s+k-1)}\int_0^\infty f^{(k)}(u)u^{2s+k-1}\,du. \tag{18}$$

Therefore, if $f \in C_c^k(\mathbb{R}^\bullet)$, there exist positive constants A and B such that for all $\sigma \in \mathbb{R}$,

$$|\varphi_f(\sigma+it)| \leqq A\frac{B^{2|\sigma|}}{(1+|t|)^k}. \tag{19}$$

The constants A and B depend only on f and k. In fact, if support$(f) \subset [a,b]$ and $0<a<b$, then

$$|\varphi_f(\sigma+it)| \leqq \frac{((b-a)\|f^{(k)}\|_\infty b^{k-1})b^{|2\sigma|}}{|2\sigma+2it|\cdot|2\sigma+1+2it|\dots|2\sigma+k-1+2it|}. \tag{20}$$

Therefore, if $f \in C_c^\infty(\mathbb{R}^\bullet)$ it follows that f belongs to the Paley-Wiener space, $PW(\mathbb{C})$ (see Lang [L], p. 74), *i.e.*, there exists a constant $c>0$ such that for every natural number N and $\sigma \in \mathbb{R}$ we have:

$$|\varphi_f(\sigma+it)| \ll \frac{c^{|2\sigma|}}{(1+|t|)^N}, \quad \text{as } t \to \pm\infty. \tag{21}$$

That is, given N and σ there exists a positive constant $K = K(f,N)$, depending only on f and N such that

$$|\varphi_f(\sigma+it)| \leqq K\frac{c^{|2\sigma|}}{(1+|t|)^N}$$

for all $t \in \mathbb{R}$ such that $|t| \geqq t_0$, where t_0 depends only on f and N.

From (21) it follows that φ_f is of rapid decay in any fixed vertical strip, *i.e.*, $\varphi_f(\sigma+it)$ tends very rapidly to zero uniformly in any strip $\sigma_1 \leqq \sigma \leqq \sigma_2$, as $|t| \to \infty$. In particular, if $f \in C_c^2(\mathbb{R}^\bullet)$, we have that the function $g_f^\sigma : \mathbb{R}^\bullet \to \mathbb{C}$ defined by

$$g_f^\sigma(t) = \varphi_f(\sigma+it) \tag{22}$$

has the property that $g_f^\sigma \in \mathcal{L}_1(\mathbb{R},\mathbb{C})$, for all $\sigma \in \mathbb{R}$.

Now, let us recall the following facts about the order of growth of $\zeta(s)$ along vertical lines. Let $\mu(\sigma)$ be the lower bound of real numbers $\ell \geqq 0$ such that

$$\zeta(\sigma+it) = \mathcal{O}(|t|^\ell) \quad \text{as} \quad |t| \to \infty. \tag{23}$$

Then μ has the following properties (Titchmarsh [T], p. 95):

$$\begin{cases} \text{i)} & \mu \text{ is continuous non-increasing and never negative.} \\ \text{ii)} & \mu \text{ is convex downwards in the sense that the curve } y = \mu(\sigma) \\ & \text{has no points above the chord joining any two of its points.} \\ \text{iii)} & \mu(\sigma) = 0 \text{ if } \sigma \geqq 1 \text{ and } \mu(\sigma) = \frac{1}{2} - \sigma \text{ if } \sigma \leqq 0. \end{cases} \tag{24}$$

The Lindelöf hypothesis is equivalent to the statement that

$$\begin{cases} \mu(\sigma) = \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\ \mu(\sigma) = 0 & \text{if } \sigma \geqq \frac{1}{2} \end{cases} \tag{25}$$

which is equivalent to:

$$\zeta\left(\frac{1}{2} + it\right) = \mathcal{O}(t^{\epsilon}) \quad \text{for all } \epsilon > 0. \tag{26}$$

Now *suppose the Riemann hypothesis is true*; then $\log \zeta(s)$ is a holomorphic function in the half-plane $\Re(s) > \frac{1}{2}$ (except at $s = 1$) and we have the following estimates due to Littlewood:

$$\text{For } \epsilon > 0 \text{ and } \sigma \geqq \tfrac{1}{2}:$$
$$-\epsilon \log t < \log\left|\zeta(s)\right| < \epsilon \log t; \quad s = \sigma + it, \quad t \geqq t_0(\epsilon),$$

that is:

$$\begin{cases} \zeta(s) = \mathcal{O}(t^{\epsilon}) \\ \frac{1}{\zeta(s)} = \mathcal{O}(t^{\epsilon}) \end{cases} \quad \text{for every } \epsilon > 0,\ s = \sigma + it,\ \sigma > \frac{1}{2} \text{ as } |t| \to \infty. \tag{27}$$

The estimates (27), *valid under the Riemann hypothesis*, can be found in Titchmarsh [T], Chapter XIV, p. 337, formulæ (14.2.5) and (14.2.6). Furthermore, suppose that $\alpha > \frac{1}{2}$ is such that ζ has no zeroes in the half-plane $\Re(s) > \alpha$; then Littlewood has the following estimates:

$$\begin{cases} \zeta(s) = \mathcal{O}(t^{\epsilon}) \\ \frac{1}{\zeta(s)} = \mathcal{O}(t^{\epsilon}) \end{cases} \text{for every } \epsilon > 0,\ s = \sigma + it,\ \sigma \geqq \alpha. \tag{28}$$

Also if for each $\sigma > \frac{1}{2}$ (and $s = \sigma + it$ as before) we define $\nu(\sigma)$ as the lower bound of numbers a such that

$$\log \zeta(s) = \mathcal{O}(\log^{a} t)$$

then for $\beta < \sigma < 1$, $\beta = \sup\{\Re(\rho) \mid \zeta(\rho) = 0\}$, we have

$$1 - \sigma \leqq \nu(\sigma) \leqq 2(1 - \sigma). \tag{28$'$}$$

Also $\log \zeta(s)$ (for $\Re(s) > \beta$) has the same ν function as $\frac{\zeta'(s)}{\zeta(s)}$, *i.e.*, if we define $\nu'(\sigma)$ as the lower bound of numbers such a that

$$\frac{\zeta'(s)}{\zeta(s)} = \mathcal{O}(\log^{a} t)$$

then

$$1 - \sigma \leqq \nu'(\sigma) \leqq 2(1 - \sigma). \tag{28$''$}$$

Also, (see Titchmarsh, Chapter XIV, Theorem 14.5) we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \mathcal{O}((\log t)^{2-2\sigma})$$

uniformly for $\beta < \sigma_0 \leqq \sigma \leqq \sigma_1 < 1$, $\sigma \neq \frac{1}{2}$.

3. Proof of Theorems

In all that follows we will assume that $f \in C_c^\infty(\mathbb{R}^\bullet)$ but everything will still hold if we only assume that $f \in C_c^r(\mathbb{R}^\bullet)$, $r \geqq 2$.

3.1. *Proof of Theorem* A. By the Mellin inversion formula we have:

$$m_y(f) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{M}_f(s) y^{1-s}\, ds, \tag{29}$$

for an appropriate $a \in \mathbb{R}$. In our case we can take $a = \frac{1}{2}$ because the function $\Theta_{\frac{1}{2}}(t) = \mathcal{M}_f(\frac{1}{2}+it)$ satisfies $\Theta_{\frac{1}{2}} \in \mathcal{L}_1(\mathbb{R},\mathbb{C})$; this is so because the function $\varphi_f(\frac{1}{2}+it)$ is in the Paley-Wiener space and by (24) the function $Z_f(t) = \frac{\zeta(2it)}{\zeta(1+2it)}$ is $\mathcal{O}(|t|^{\frac{1}{2}+\epsilon})$ for all $\epsilon > 0$. Hence $\varphi_f|_{\Re(s)=\frac{1}{2}} \cdot Z_f \in \mathcal{L}_1(\mathbb{R},\mathbb{C})$.

The integral of $\mathcal{M}_f(s)y^{1-s}$ over the boundary of of the vertical strip $\frac{1}{2} \leqq \sigma \leqq 2$ exists and it is equal at the same time to $\mathcal{R}es_{s=1}(\mathcal{M}_f(s))$ and equal to

$$m_y(f) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right) y^{\frac{1}{2}} y^{-it}\, dt.$$

We have:

$$\left|\int_{-\infty}^{\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right) y^{\frac{1}{2}} y^{-it}\, dt\right| = y^{\frac{1}{2}} \left|\int_{-\infty}^{\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right) y^{-it}\, dt\right| = o(y^{\frac{1}{2}}),$$

because, by the Riemann-Lebesgue Theorem:

$$\lim_{y\to 0}\left|\int_{-\infty}^{\infty} \mathcal{M}_f\left(\frac{1}{2}+it\right) y^{-it}\, ds\right| = 0.$$

Thus

$$m_y(f) = \frac{1}{\zeta(2)}\int_0^{\infty} uf(u)\, du + o(y^{\frac{1}{2}}).$$

This proves theorem A. □

3.2. *Proof of Theorem* B. Suppose the Riemann hypothesis is true. Then we set in formula (29) $a = \frac{1}{4}+\epsilon$, for any fixed $\epsilon > 0$. Then the function $\Theta_{\frac{1}{4}+\epsilon}(t) = \mathcal{M}_f(\frac{1}{4}+it)$ has the property that $\Theta_{\frac{1}{4}+\epsilon} \in \mathcal{L}_1(\mathbb{R},\mathbb{C})$.

Therefore, the integral of $\mathcal{M}_f(s)y^{1-s}$ exists over the boundary of the band $\frac{1}{4}+\epsilon \leqq \sigma \leqq 2$. Therefore:

$$m_y(f) = \mathcal{R}es_{s=1}(\mathcal{M}_f(s)) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}_f\left(\frac{1}{4}+\epsilon+it\right) y^{-it} y^{\frac{3}{4}-\epsilon}\, dt. \tag{30}$$

Again, by the Riemann-Lebesgue Theorem:

$$m_y(f) = \frac{1}{\zeta(2)}\int_0^{\infty} uf(u)du + o(y^{\frac{3}{4}-\epsilon}). \tag{31}$$

If, on the other hand, $m_y(f) = \frac{1}{\zeta(2)} \int_0^\infty uf(u)du + o(y^{\frac{3}{4}-\epsilon})$ for all $\epsilon > 0$ and all functions $f \in C_c^\infty(\mathbb{R}^\bullet)$, then $\mathcal{M}_f(s)$ is holomorphic (except for a pole at $s = 1$) in the half-plane $\Re(s) > \frac{1}{4}+\epsilon$, for all $\epsilon > 0$. Since, under the hypotheses, $\mathcal{M}_f(s) = \frac{2\zeta(2s-1)}{\zeta(2s)}\varphi_f(s)$ is holomorphic in that half-plane and we can choose f so that $\varphi_f(s)$ does not vanish at any given zero of ζ, it follows that $\zeta(2s-1)/\zeta(2s)$ is holomorphic in the half-plane $\Re(s) > \frac{1}{4}$ and hence the Riemann hypothesis would be true. The reason that $\mathcal{M}_f(s)$ is holomorphic in the half-plane, under the hypothesis that $m_y(f) = m_0(f) + K(y)$, where $K(y) = o(y^{\frac{3}{4}-\epsilon})$, is the following:

$$\mathcal{M}_f(s) = \frac{m_0(f)}{s-1} + \int_0^\infty K(y)y^{s-2}\,dy. \tag{32}$$

The integral in the right-hand side of (32) converges absolutely and uniformly in the half-plane $\Re(s) > \frac{1}{4} + \epsilon$, so it defines a holomorphic function in that half-plane.

Suppose that $\beta = \sup\{\Re(\rho) \mid \zeta(\rho) = 0\}$. Then the function $\Theta_\epsilon(t) = \mathcal{M}_f(\frac{\beta}{2} + \epsilon + it)$ belongs to $\mathcal{L}_1(\mathbb{R}, \mathbb{C})$ for all $\epsilon > 0$.

This fact follows from (24), and the fact that $\varphi_f(s)$ is of rapid decay on vertical lines. As we know, $\frac{\beta}{2} \in [\frac{1}{4}, \frac{1}{2}]$ since Riemann's zeta-function has an infinite number of zeroes on the line $\Re(s) = \frac{1}{2}$ (Littlewood, Titchmarsh, Landau, Selberg) and no zeroes on the closed half-plane $\Re(s) \geq 1$ (by the prime number theorem). Then, by the Mellin inversion formula we have:

$$m_y(f) = m_0(f) + o(y^{1-\frac{\beta}{2}-\epsilon}) \quad \text{for all } \epsilon > 0. \tag{33}$$

If, on the other hand, (33) holds with $\alpha = 1-\frac{\beta}{2}-\epsilon$, $\frac{\beta}{2} \in [\frac{1}{4}, \frac{1}{2})$ then, proceeding as in the proof of formula (32) we obtain that ζ has no zeroes in the half-plane $\Re(s) > 2(1-\alpha)$.

Therefore, we have proven everything stated in Theorem B except for the fact that the exponent $\frac{1}{2}$ is optimal for characteristic functions and the assertion regarding the function F . To finish the proof we need the following: For $x > 0$, let $\Phi(x) = \sum_{n\leq x}\varphi(n)$ and set $\Phi(x) = 0$ for $0 < x < 1$. Then by Mertens theorem $\Phi(x) = \frac{3}{\pi^2}x^2 + (x\log x)b(x)$ for a bounded function $b(x)$: $-c < b(x) < c$ for all $x > 0$ and some constant $c > 0$.

LEMMA 3.2.1. *For all* $\alpha > 1$

$$\overline{\lim_{x\to\infty}}\, x^\alpha \left|\frac{\Phi(x)}{x^2} - \frac{3}{\pi^2}\right| = \infty.$$

Proof. Suppose the contrary. Then there exist $\alpha > 1$, $c > 0$ and a function $b_\alpha(x)$ defined on the positive reals and such that $-c < b_\alpha(x) < c$ for all $0 < x < \infty$ such that

$$\frac{\Phi(x)}{x^2} - \frac{3}{\pi^2} = x^{-\alpha}b_\alpha(x). \tag{34}$$

Let $H(x) = \Phi(x)/x^2$. Then

$$H(x+1) = \Phi(x)\frac{x^2}{(x+1)^2} + \frac{\varphi([[x+1]])}{(x+1)^2}, \tag{35}$$

where $[[\cdot]]$ denotes integral part.

By (34) and (35) and letting x run over the integers such that $x+1$ is a prime, we obtain:

$$L(x) = x^\alpha\left[\frac{3}{\pi^2}\left[\frac{2x-1}{(x+1)^2}\right] - \frac{x}{(x+1)^2}\right] = b_\alpha(x)\left[\frac{x^2}{(x+1)^2}\right] - b_\alpha(x+1)\left[\frac{x}{x+1}\right]^\alpha = R(x)$$

But this is an absurdity since $L(x)$ is unbounded when $\alpha > 1$ whereas the right-hand side remains bounded. This proves the statement in Theorem B for the characteristic function of the interval $(0,1]$. The proof for an arbitrary closed interval is similar. Now let F be the function given in Theorem B and suppose:

$$\overline{\lim_{y\to 0}} y^{-\alpha}\left|m_y(F) - m_0(F)\right| = \infty, \qquad \text{for all } \alpha > \frac{1}{2}$$

First we note that

$$m_0(F) = \frac{1}{\zeta(2)}\int_0^\infty uF(u)du = \frac{1}{\pi^2}$$

The Mellin transform $M_F(s)$ is:

$$\mathcal{M}_F(s) = \frac{\zeta(2s-1)}{s(2s+1)\,\zeta(2s)}$$

Hence its only poles in the half-plane $\Re(s) > 0$ are located at the zeroes of $\zeta(2s)$, since $s(2s+1)$ does not vanish in that half-plane. $\mathcal{M}_F(s)$ is *not* of rapid decay namely not of Paley-Wiener type, however it decays fast enough so as to be able to shift the vertical line of integration—in Mellin's inversion formula—to the vertical line $\Re(s) = \frac{\beta}{2} + \epsilon$ where β, as before, is the supremum of the real parts of the zeroes of Riemann's ζ-function. *Now suppose* $\beta < 1$. We want to arrive to a contradiction.

First we note that for any $\epsilon > 0$ the inequalities (24) through (28″) imply that the function

$$h(t) = \frac{\zeta(\beta - 1 + 2\epsilon + 2it)}{(\beta/2 + \epsilon + it)(1 + \beta + 2\epsilon + 2it)\zeta(\beta + 2\epsilon + 2it)}$$

has the property that

$$\lim_{|t|\to\infty} h(t) = 0 \tag{40}$$

and

$$\lim_{|t|\to\infty} h'(t) = 0 \tag{41}$$

for all $\epsilon > 0$. In fact we have if $s = \sigma + it$ and $-1 \leqq \sigma \leqq 2$:

$$|\zeta'(s)| \leqq K_1|\log t||t|^{\mu(\sigma)}$$

and

$$|\zeta(s)| \leqq K_2|t|^{\mu(\sigma)}$$

for some constants $K_1, K_2 > 0$ and $|t|$ sufficiently large. Hence under the hypotheses:

$$|h(t)| \leqq K_1|t|^{-\delta}$$

$$|h'(t)| \leqq K_2|t|^{-\delta}$$

for some $\delta > 1$. On the other hand, the improper integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)y^{1-\frac{\beta}{2}-\epsilon-it}dt$$

exists for all $y > 0$. Namely

$$\lim_{T_1,T_2\to\infty}\frac{1}{2\pi}\int_{-T_1}^{T_2} h(t)y^{1-\frac{\beta}{2}-\epsilon-it}dt$$

exists for all $y > 0$, This follows from Cauchy's Residue Theorem by integrating the function $H_y(s) = \frac{1}{2\pi i}\mathcal{M}_F(s)y^{1-s}$ for each fixed $y > 0$ along the rectangle $Q(T_1, T_2)$ with vertices:

$$\begin{aligned} A(T_1) &= \frac{\beta}{2} + \epsilon - iT_1 \\ B(T_2) &= \frac{\beta}{2} + \epsilon + iT_2 \\ C(T_2) &= 2 + iT_2 \\ D(T_1) &= 2 - iT_1. \end{aligned}$$

The integrals along the segments $[B(T_2), C(T_2)]$ and $[D(T_1), A(T_1)]$ tend uniformly to zero as T_1 and T_2 tend to infinity, hence

$$\lim_{T_1,T_2\to\infty} \frac{1}{2\pi}\int_{-T_1}^{T_2} h(t)y^{1-\frac{\beta}{2}-\epsilon-it}dt = -\frac{1}{\pi^2} + m_y(f).$$

Hence:

$$m_y(F) = \frac{1}{\pi^2} + \frac{1}{2\pi}y^{1-\frac{\beta}{2}-\epsilon}\int_{-\infty}^{\infty} h(t)y^{-it}dt. \tag{42}$$

Now consider the integral

$$G(y) = \int_{-\infty}^{\infty} h(t)e^{-(\log y)it}dt.$$

By (42), $G(y)$ is continuous. Also, $G(y) = \widehat{h}(\log y)$ where $\hat{h}$ is the Fourier transform of h. Since $h \in \mathcal{L}_1(\mathbb{R}, \mathbb{C})$ h has a well defined Fourier transform and all of the above is valid. In fact since h and h' vanish at infinity, we have for $y \neq 1$:

$$G(y) = i[\log y]^{-1} \lim_{a\to\infty}\int_{-a}^{a} e^{-(\log y)it}dh$$

Integration by parts is valid, since both h and h' vanish at infinity. Hence Riemann-Lebesgue is valid and we obtain:

$$\lim_{y\to 0} G(y) = 0.$$

Therefore, under the hypothesis $\beta < 1$ we obtain:

$$m_y(F) = \frac{1}{\pi^2} + \frac{1}{2\pi}y^{1-\frac{\beta}{2}-\epsilon}G(y) \quad \text{for all } \epsilon > 0$$

and therefore:

$$\lim_{y\to 0} y^{-1+\frac{\beta}{2}+\epsilon}\left|m_y(F) - \frac{1}{\pi^2}\right| = 0.$$

But this contradicts the hypothesis since $1 - \frac{\beta}{2} - \epsilon > \frac{1}{2}$ if ϵ is small enough. □

Remark. We have shown that:

$$m_y(F) = \frac{1}{\pi^2} + o(y^{1/2})$$

Now let $y = N^{-2}$ where N is a positive integer. Then,

$$m_y(F) = N^{-3}\sum_{n=1}^{N-1}\Phi(n)$$

From the last two equations we obtain:

$$\lim_{N\to\infty} N^{-3}\sum_{n=1}^{N-1}\Phi(n)=\frac{1}{\pi^2}$$

and,

$$\lim_{N\to\infty}\left[N^{-2}\sum_{n=1}^{N-1}\Phi(n)-\frac{N}{\pi^2}\right]=0$$

By Mertens Theorem, one has:

$$\sum_{n=1}^{N-1}\Phi(n)=\frac{3}{\pi^2}\sum_{n=1}^{N-1}n^2+\sum_{n=1}^{N-1}nb(n)\log(n)$$

For some bounded function $b(n)$. Hence, recalling that $\sum_{n=1}^{N-1}n^2 = \frac{2N^3-3N^2+N}{6}$, we obtain:

COROLLARY.

$$\lim_{N\to\infty} N^{-2}\sum_{n=1}^{N-1}nb(n)\log(n)=\frac{3}{2\pi^2}$$

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