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ON THE TOPOLOGICAL STRUCTURE OF THE FERMAT SURFACE OF DEGREE 5

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Abstract

We will give the monodromy representation of a certain fibration of the Fermat surface of degree 5 explicitly in terms of Dehn twists about concrete curves. This paper is a sequel to Ahara's work [1]

1. Introduction

Let V_5 be the complex projective hypersurface in CP_3 defined by the equation of degree 5

$$z_0^5 - z_1^5 = z_2^5 - z_3^5$$

where z_0,z_1,z_2,z_3 are the homogenuous coordinates. From topological viewpoint, this surface is a simply-connected 4-manifold. The Euler characteristic and the signature of V_5 are equal to 55 and -35, respectively (see [5]). By Freedman [4] V_5 is homeomorphic to $9CP_2\#44\overline{CP}_2$, but by Donaldson [3] it is not diffeomorphic to this connected sum.

Our motive is to understand the topological structure of V_5 through a holomorphic fibration over the Riemann sphere $CP_1 = C \cup \{\infty\}$.

The fibration $f : V_5 \to CP_1$ is defined as follows:

$$f : [z_0, z_1, z_2, z_3] \mapsto \begin{cases} z_2^4/z_0^4, & \text{if } z_0 = z_1 \text{ and } z_2 = z_3 \\ (z_0 - z_1)/(z_2 - z_3), & \text{otherwise.} \end{cases}$$

A general fiber of this fibration is a Riemann surface of genus 3. In unpublished notes (1990), the author determined the positions and the topological types of all singular fibers in $f : V_5 \to CP_1$; let F_σ denote the fiber over a point $\sigma \in CP_1$. Then F_σ is a singular fiber if and only if σ belongs to the following set SF consisting of 17 points:

$$SF = \{\sigma \mid \sigma^5 = -1/4, 1, or -4\} \cup \{0, \infty\}.$$

If $\sigma = 0$ or ∞ , F_σ is a union of 4 complex lines meeting in a point. If σ is a 5-th root of 1, F_σ is a union of two complex lines and a conic, meeting in 5 points transversely. If σ is a 5-th root of $-1/4$ or of-4, F_σ is an irreducible stable curve of virtual genus 3

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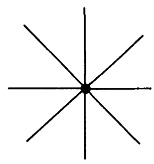
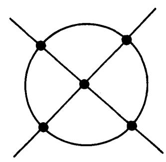
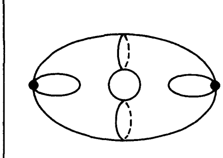
which has two transverse sef-intersection points. See the table below.

	$\sigma = 0$ or ∞	$\sigma^5 = 1$	$\sigma^5 = -\frac{1}{4}$ or -4
F_σ			

Remark. In the same unpublished notes, the author determined the positions and the types of all singular fibers appearing in a similar fibration of the Fermat surface of general degree n ($n \geq 2$). These results and the methods are summarized in [1, Section 2].

To accomplish the topological description of the fibration $f : V_5 \to CP_1$, it suffices to give the monodromy representation

$$\rho : \pi_1(CP_1 - SF, \sigma_0) \to \mathcal{M}_3$$

where $\mathcal{M}_3$ is the mapping class group of a closed oriented surface of genus 3, and σ_0 is a base point chosen in $CP_1 - SF$.

Ahara [1] has essentially determined ρ. He presents a general fiber F_σ as a 4-fold irregualar branched covering of CP_1 branched at 8 points. By numerical analysis using a computer, he has described motions of the branch points induced by movements of σ along paths in $CP_1 - SF$. He has also given an algorithm to compute the action of the monodromy on the fundamental group of F_{σ_0} up to inner automorphisms. Thus by Dehn-Nielsen's theorem [7] which states that $Aut(\pi_1 F_{\sigma_0})/Inn(\pi_1 F_{\sigma_0}) \cong \mathcal{M}_3$, it is possible in principle to describe the monodromy representation ρ explicitly in terms of Dehn twists. However, this last step remains undone in Ahara's paper . To carry it out actually is easy for some loops as Ahara asserts in Remark 4 of [1, Section 1], but for general loops it is not so immediate.

The purpose of this paper is to complete this last step ; we will take Ahara's computaion as a starting point and will give the monodromy representation ρ explicitly in terms of Dehn twists about concrete simple closed curves on F_{σ_0} (Theorem 2.1). Our method does not depend on Ahara's algorithm and is more pictorial.

2. Main results

Following Ahara [1], we choose $\sigma_0 = 11/10$ as a base point of $CP_1 - SF$. The fiber F_{σ_0} is presented as a 4-fold irregular branched covering of CP_1 branched at 8 points $\Sigma = \{A, \overline{A}, B, \overline{B}, C, \overline{C}, D, \overline{D}\}$, where $A \approx -0.9256 + 0.3786\sqrt{-1}$,$B \approx -0.3800 + 0.9250\sqrt{-1}$,$C \approx 0.2159 + 0.9764\sqrt{-1}$,$D \approx 0.3246 + 0.9458\sqrt{-1}$. See Ahara [1, Section 4].

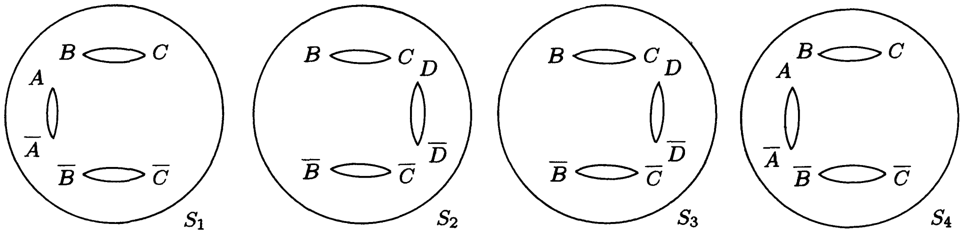


Figure 1

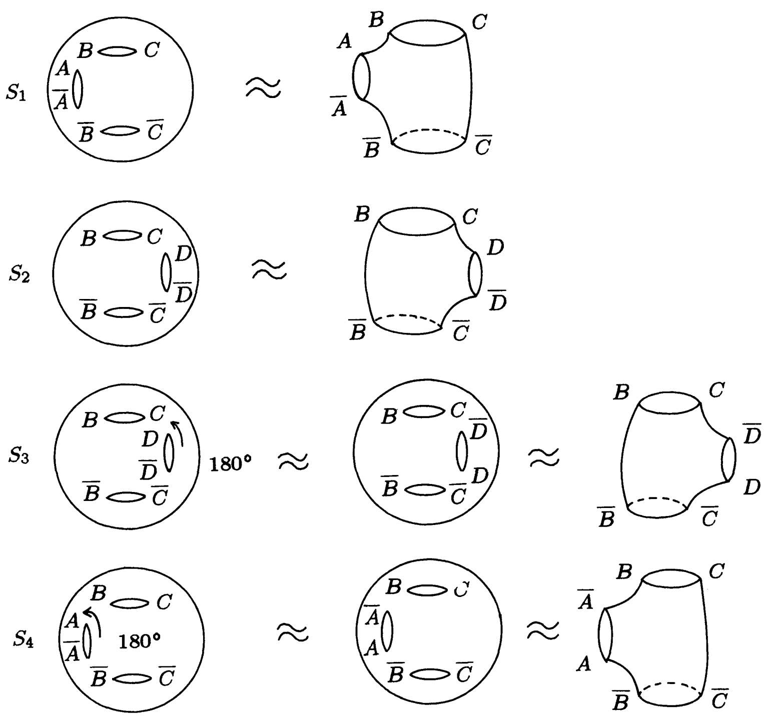


Figure 2

The branching monodromy $\xi : \pi_1(CP_1 - \Sigma, 0) \to \mathfrak{S}_4$ associated with the branched covering $F_{\sigma_0} \to CP_1$ is computed as follows:

$$\xi(a) = \xi(\overline{a}) = (14)$$
$$\xi(b) = \xi(c) = (12)(34)$$
$$\xi(\overline{b}) = \xi(\overline{c}) = (13)(24)$$
$$\xi(d) = \xi(\overline{d}) = (23)$$

where a, $\bar{a}$, b, $\bar{b}$,c, $\bar{c}$, d, and $\bar{d}$ are loops on $CP_1 - \Sigma$ which are based at 0 and go once around the corresponding point A,$\bar{A}$, B, $\bar{B}$, C, $\bar{C}$, D, and $\bar{D}$ respectively, and avoid the rest of the points. See Ahara [1, Section 4].

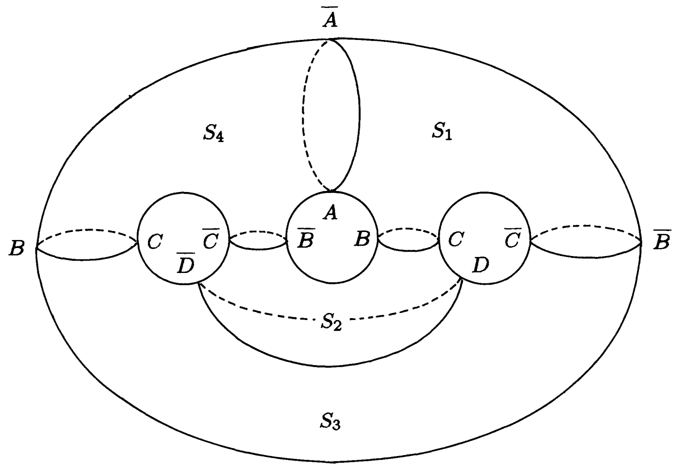


Figure 3

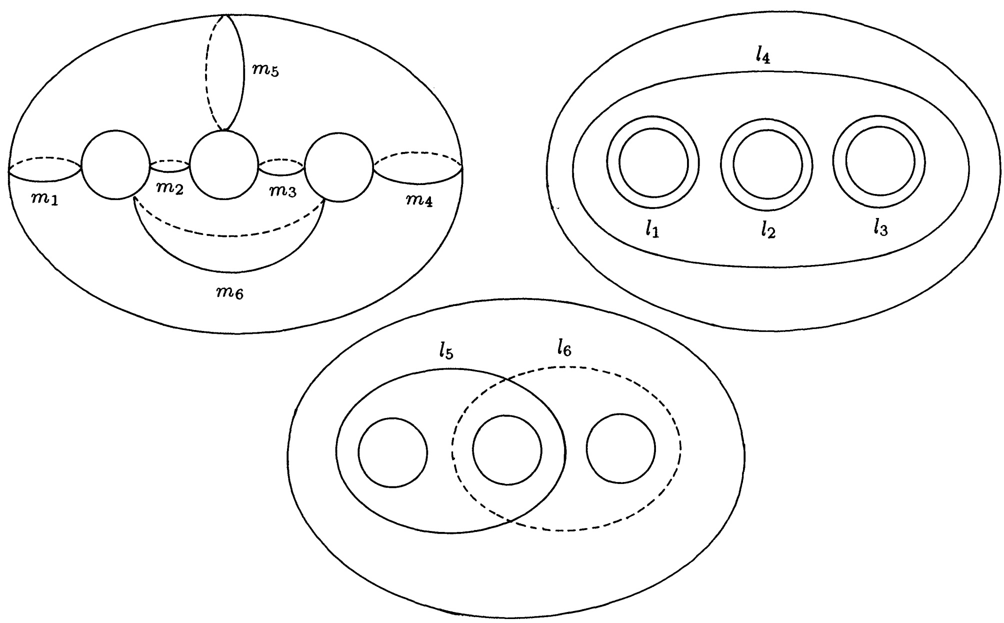


Figure 4

By the above facts,we can identify F_{σ_0} as the surface constructed as follows: Take 4 copies of CP_1 and call them S_1, S_2, S_3, S_4, and cut slits open on them as shown in

Figure 1.

Each of the resulting surfaces (again call S_i, $i = 1,2,3,4$) is homeomorphic to a pair of pants. See Figure 2, in which the homeomorphism between S_3 and a pair of pants is given as a 180° rotation of the slit $D\overline{D}$ followed by the natural identification. Similarly for S_4.

One obtains a surface of genus 3 by pasting these pairs of pants along their boundaries as shown in Figure 3. We will always identify F_{σ_0} as the surface of Figure 3.

Now take twelve simple closed curves $\{m_i, l_i\}_{i=1,2,\cdots,6}$ on F_{σ_0} as shown in Figure 4.

Dehn twists about these simple closed curves are denoted by the corresponding capital letters. Thus M_1 for example denotes the positive Dehn twist about m_1, and M_1^{-1} the negative twist. See Figure 5. Clearly, $M_iM_j = M_jM_i$ and $L_iL_j = L_jL_i$, for $i,j = 1,2,3,4,5,6$.

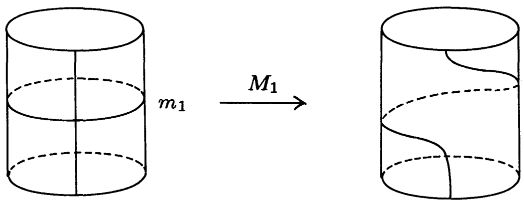


Figure 5

Let ω denote $exp(2\pi\sqrt{-1}/5)$. Then the cyclic group $C_5 = \{\omega^i \mid i = 0,1,2,3,4\}$ acts on V_5 by $\omega^i : [z_0, z_1, z_2, z_3] \mapsto [\omega^i z_0, \omega^i z_1, z_2, z_3]$ and on $CP_1 = C \cup \{\infty\}$ by the natural multiplication. Our fibration $f : V_5 \rightarrow CP_1$ is equivariant with respect to the actions of C_5. [1, Section1].

By the action of C_5 , we can identify F_{σ_0} with the fibers over $\omega^i\sigma_0$, $i = 1,2,3,4$. Let $\gamma : [0,1] \rightarrow CP_1 - SF$ be a path. Then there exists a continuous family of homeomorphisms $\{H_t : F_{\gamma(0)} \rightarrow F_{\gamma(t)}\}_{0\leq t\leq 1}$ such that $H_0 = id$ of $F_{\gamma(0)}$. If the path γ joins $\omega^i\sigma_0$ and $\omega^j\sigma_0$, we can define a monodromy homeomorphism $\rho(\gamma) : F_{\sigma_0} \rightarrow F_{\sigma_0}$ by setting

$$\rho(\gamma) := h_j^{-1} \circ H_1 \circ h_i$$

where $h_i : F_{\sigma_0} \rightarrow F_{\omega^i\sigma_0}$ and $h_j : F_{\sigma_0} \rightarrow F_{\omega^j\sigma_0}$ are homeomorphisms which give the identification by the action of C_5. The homeomorphism $\rho(\gamma)$ is determined up to isotopy by the homotopy class of γ (fixing the terminal points $\gamma(0)$ and $\gamma(1)$).

Ahara [1, Section5] takes four paths γ_0, γ_1, γ_2, γ_3 joining σ_0 and $\omega\sigma_0$, whose homotopy classes are depicted in Figure 6. Any loop in $CP_1 - SF$ based at σ_0 is homotopic to a composition of paths from the set of 20 paths

$$\{\omega^i\gamma_j\}_{i=0\cdots 4, j=0\cdots 3}$$

where $\omega^i\gamma_j$ denotes the path $\gamma_j : [0,1] \rightarrow CP_1 - SF$ rotated by the action of ω^i : $CP_1 \rightarrow CP_1$. Clearly, $\omega^i\gamma_j$ joins $\omega^i\sigma_0$ and $\omega^{i+1}\sigma_0$. The monodromy representation ρ : $\pi_1(CP_1 - SF, \sigma_0) \rightarrow \mathcal{M}_3$ is completely determined if we give the homeomorphisms $\{\rho(\omega^i\gamma_j) : F_{\sigma_0} \rightarrow F_{\sigma_0}\}_{i=0,\cdots,4,j=0,\cdots,3}$.

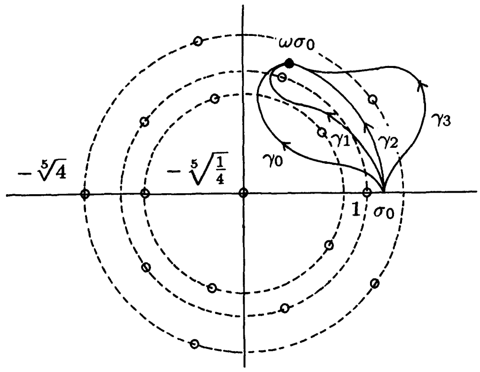


Figure 6

It is important to notice that the monodromy representation ρ is an anti-homomorphism and that the monodromy homeomorphism associated with a composite path $\gamma\gamma'$ (γ followed by γ') is computed by

$$\rho(\gamma\gamma') = \rho(\gamma') \circ \rho(\gamma).$$

Let $T : F_{\sigma_0} \to F_{\sigma_0}$ be an involution which is a 180° rotation of F_{σ_0} about the axis shown in Figure 7.

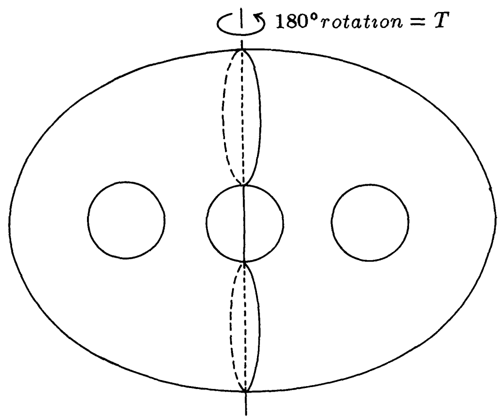


Figure 7

Now we are in a position to state our main result.

THEOREM 2.1. *The monodromy homeomorphisms associated with* $\omega^i\gamma_j$ $(i = 0, 1, 2, 3, 4, j = 0, 1, 2, 3)$ *are independent of* i, *and are given as follows :*

$$\rho(\omega^i\gamma_0) = \rho(\gamma_0) = M_1M_2M_3M_4L_1L_3M_6L_5L_6T,$$
$$\rho(\omega^i\gamma_1) = \rho(\gamma_1) = M_1M_2M_3M_4L_1L_3M_6T,$$
$$\rho(\omega^i\gamma_2) = \rho(\gamma_2) = M_5^{-1}L_1L_3M_6T,$$
$$\rho(\omega^i\gamma_3) = \rho(\gamma_3) = M_5^{-1}L_1L_3M_6L_2^{-1}L_4^{-1}T.$$

Remarks. 1) In the statement of Theorem 2.1, and in what follows, we regard self-homeo-morphisms of F_{σ_0} as elements of $\mathcal{M}_3$.

2) The involution T commutes with M_1M_4, M_2M_3, M_5, M_6, L_1L_3, L_2, L_4, and L_5L_6.

3) Using Lemma 3.3 of Section 3, one can show

$$\rho(\gamma_0)^2 = T, \quad \rho(\gamma_3)^2 = ST,$$

where $S : F_{\sigma_0} \to F_{\sigma_0}$ is the involution whose construction is indicated by Figure 8. Note that $ST = TS$. It follows that $\rho(\gamma_0)$ and $\rho(\gamma_3)$ have order 4 in $\mathcal{M}_3$.

4) Let $Q \in \mathcal{M}_3$ be defined by

$$Q = L_1^{-1}M_4L_3M_3L_3M_4.$$

Then using Lemma 3.3, one can show

$$Q\rho(\gamma_3)Q^{-1} = \rho(\gamma_0)^{-1}.$$

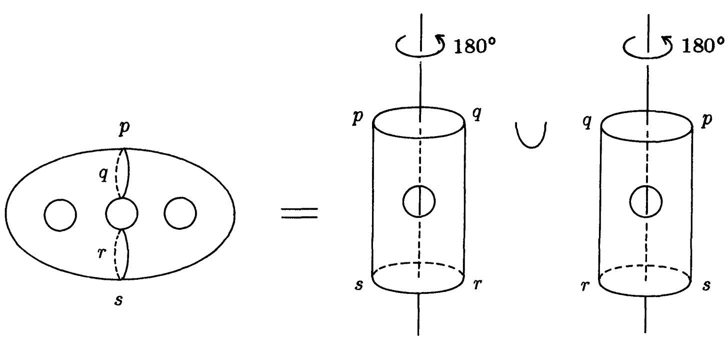


Figure 8

Here are some examples of monodromy calculations based on Theorem 2.1. The monodromy associated with loops $\gamma_1\gamma_0^{-1}$, $\gamma_2\gamma_1^{-1}$, $\gamma_3\gamma_2^{-1}$ (based at σ_0) are calculated as

$$\begin{aligned}
&\rho(\gamma_0)^{-1}\rho(\gamma_1) = L_5^{-1}L_6^{-1}\\
&\rho(\gamma_1)^{-1}\rho(\gamma_2) = M_6^{-1}L_3^{-1}L_1^{-1}(M_1^{-1}M_2^{-1}M_3^{-1}M_4^{-1}M_5^{-1})L_1L_3M_6\\
&\rho(\gamma_2)^{-1}\rho(\gamma_3) = L_2^{-1}L_4^{-1},
\end{aligned}$$

respectively.

The monodromy around F_1 is given by

$$\begin{aligned}
\rho((\omega^{-1}\gamma_1)^{-1}(\omega^{-1}\gamma_2)) &= \rho(\gamma_2)\rho(\gamma_1)^{-1}\\
&= M_1^{-1}M_2^{-1}M_3^{-1}M_4^{-1}M_5^{-1},
\end{aligned}$$

which coincides up to the sign convention for Dehn twists with the equation in the last example of [1, Section 5].

The monodromy around F_0 is given by

$$\begin{aligned}\rho(\gamma_0(\omega\gamma_0)(\omega^2\gamma_0)(\omega^3\gamma_0)(\omega^4\gamma_0)) &= \rho(\gamma_0)^5 = \rho(\gamma_0)\\ &= M_1M_2M_3M_4L_1L_3M_6L_5L_6T,\end{aligned}$$

and the monodromy around F_∞ is given by

$$\begin{aligned}\rho((\omega^4\gamma_3)^{-1}(\omega^3\gamma_3)^{-1}(\omega^2\gamma_3)^{-1}(\omega\gamma_3)^{-1}\gamma_3^{-1}) &= \rho(\gamma_3)^{-5} = \rho(\gamma_3)^{-1}\\ &= T^{-1}L_4L_2M_6^{-1}L_3^{-1}L_1^{-1}M_5,\end{aligned}$$

both of which have period 4 as noted in Remark 3).

Notice that these results also exemplify the correspondence between topological types of singular fibers and the monodromy homeomorphisms around them. See Theorem 1 of [6].

3. Outline of proof

Let B_8 denote the braid group whose elements are isotopy classes of motions of the 8 points $p_1 = \overline{D}$, $p_2 = \overline{C}$, $p_3 = \overline{B}$, $p_4 = \overline{A}$, $p_5 = A$, $p_6 = B$, $p_7 = C$, $p_8 = D$ in $CP_1 - \{0, \infty\}$. Following [1], we define elements β_i, $i = 1, \cdots, 8$ of B_8 as in Figure 9.

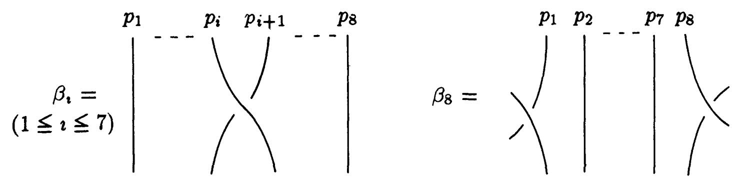


Figure 9

PROPOSITION 3.1([1, Prop.5.3]). *Suppose σ moves along the path γ_i ($i = 0, 1, 2,$ or 3). Then the corresponding motion $\beta(\gamma_i)$ of the points $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$ is represented as follows:*

$$\begin{aligned}\beta(\gamma_0) &= \beta_1^{-1}\beta_2\beta_1\beta_3^{-1}\beta_2^{-1}\beta_7^{-1}\beta_6\beta_7\beta_5^{-1}\beta_6^{-1}\beta_8\beta_4^{-1}\beta_1\beta_7,\\ \beta(\gamma_1) &= \beta_1^{-2}\beta_3^{-1}\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_5^{-1}\beta_7^{-2}\beta_2^{-1}\beta_4^{-1}\beta_6^{-1},\\ \beta(\gamma_2) &= \beta_1^{-2}\beta_3^{-1}\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_5^{-1}\beta_7^{-2},\\ \beta(\gamma_3) &= (\beta_1^{-1}\beta_3^{-1}\beta_5^{-1}\beta_7^{-1})^2\end{aligned}$$

Remarks. 1) The motion of the 8 points described by a product $\beta\beta'$ is isotopic to the motion β followed by β'.

2) The motion β_i with even i is lifted to Dehn twist(s) of F_{σ_0}, but with odd i it is not lifted to any self-homeomorphism of F_{σ_0}. This point makes Theorem 2.1 non-trivial.

3) As we remarked in Section 2, the homeomorphisms $\rho(\gamma_0)$ and $\rho(\gamma_3)$ have order 4 in $\mathcal{M}_3$, but $\beta(\gamma_0)$ and $\beta(\gamma_3)$ are not of finite order in B_8.

Let $\{x_1, x_2, x_3, x_4\}$ be the preimage of 0 under the branced covering $F_{\sigma_0} \to CP_1$, x_i being in the pair of pants S_i (see Section 2). The monodromy homeomorphism $\rho(\gamma_i) : F_{\sigma_0} \to F_{\sigma_0}$ can be taken so that they preserve $\{x_1, x_2, x_3, x_4\}$. Then they cause permutations $\pi(\gamma_i)$ of the indices $(1, 2, 3, 4)$.

PROPOSOTION 3.2 ([1, Prop.5.6]). *The permutations* $\pi(\gamma_i)$ *are given as follows:* $\pi(\gamma_0) = (14), \pi(\gamma_1) = \pi(\gamma_2) = \pi(\gamma_3) = (1243)$.

Recall that we took 12 simple closed curves $\{m_i, l_i\}_{i=1,\cdots,6}$ on F_{σ_0} in Section 2. The curve l_1 projects onto a curve on CP_1 shown in Figure 10, where the numbers 2,3,4 by the curve are the "sheet numbers "which indicate the pair of pants onto which the portion of the curve is lifted.

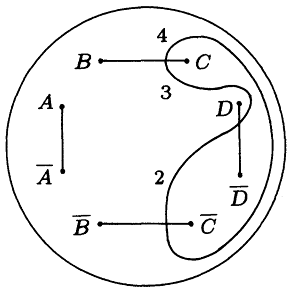


Figure 10

Applying the motion $\beta(\gamma_0)$ we move the projected curve, and as its final position we get a new curve. See Figure 11.

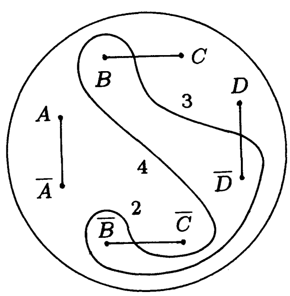


Figure 11

The sheet numbers attached to the new curves are determined by Proposition 3.2. The curve in Figure 11 is lifted to a curve on F_{σ_0} as shown in Figure 12.

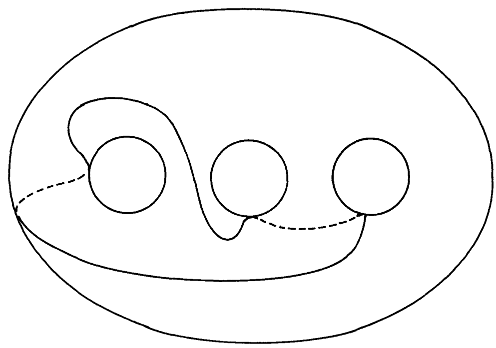


Figure 12

Therefore, the monodromy homeomorphism ρ (γ_0) should send l_1 to the curve ρ $(\gamma_0)(l_1)$ of Figure 12. By the same method, we can draw the images ρ $(\gamma_0)(l_2)$, ρ $(\gamma_0)(l_3)$, ρ $(\gamma_0)(m_1)$, ρ $(\gamma_0)(m_2)$, ρ $(\gamma_0)(m_3)$, ρ $(\gamma_0)(m_4)$, ρ $(\gamma_0)(m_5)$, ρ $(\gamma_0)(m_6)$ as in Figure 13.

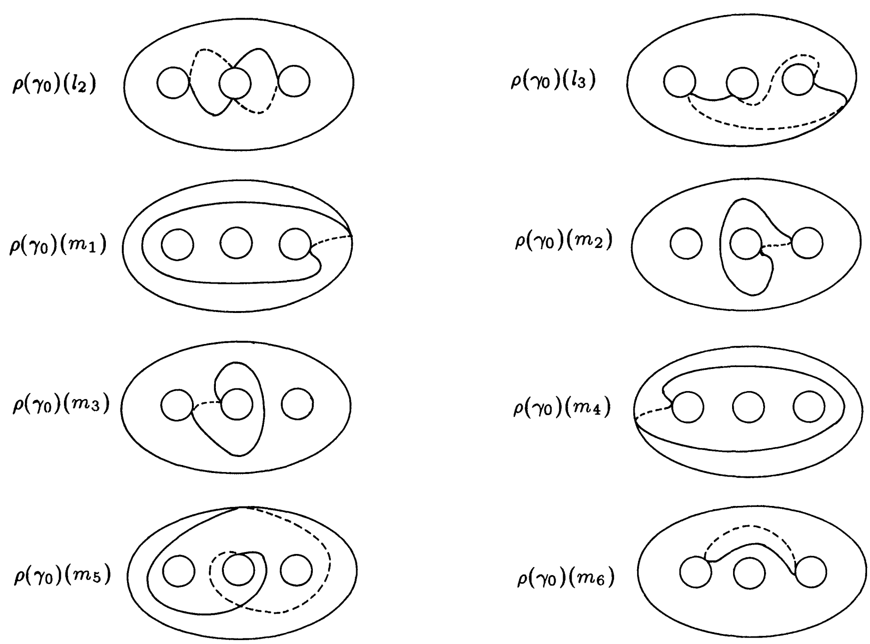


Figure 13

LEMMA 3.3. *Let $h, h' : F_{\sigma_0} \to F_{\sigma_0}$ be orientation preserving homeomorphisms. Suppose for each curve C in $\{l_1, l_2, l_3, m_1, m_2, m_3, m_4, m_5, m_6\}$ the image $h(C)$ is freely homotopic to $h'(C)$. Then h is isotopic to h'.*

Observe that each component of the complement of the union of the curves $l_1, l_2, l_3, m_1, m_2, m_3, m_4, m_5, m_6$ is an open disk and that no point is common to 3 or more curves. Then to prove Lemma 3.3 one can follow the arguments in the proofs of Lemmas 2.6 and 2.7 of Casson-Bleiler [2].

The first equality in Theorem 2.1 is proved by checking that the homeomorphism on the right hand side has the same images of $l_1, l_2, l_3, m_1, m_2, m_3, m_4, m_5, m_6$ as $\rho(\gamma_0)$, and then applying Lemma 3.3. Similarly for the rest of the equalities.

The author found the products of Dehn twists on the right hand sides of these equalities simply by trial and error.

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