

ON LENGTH-MINIMIZING STEINER NETWORKS

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Introduction

Let M be a set of points in R^n . The Steiner problem can be stated as follows: to find a network of least length in the class of networks with fixed ends M . There are two approaches to this problem. In the first case, the search for an absolutely minimal network is carried out in the class of networks, whose vertices all belong to M . This approach is developed in the papers by Du, Hwang and others (see, for example, [5], [6]). In the second case, the set of vertices of the networks may be larger than the set M . The vertices do not belong to M are called Steiner points and the points in M are called boundary points. In this paper we study globally minimal network in the second case. By using calibration systems we prove that each locally minimal network is also globally minimal in the class of networks with the same topological type. The method of calibrations was developed in works of Federer, Dao Trong Thi, Lawson, Harvey, Morgan and others (see, for example, [1]-[3]). Calibration systems were used first in [4].

1. Steiner networks

DEFINITION 1.1. A Steiner network in R^n is any connected complex of one-dimensional simplexes, whose vertices have degree at most three. A Steiner network without vertices of degree two is said to be nondegenerate Steiner network.

Henceforth we shall study only acyclic nondegenerate Steiner networks with boundary points coinciding with the vertices of degree one. Such networks are called simply Steiner networks.

LEMMA 1.2. *Let a Steiner network N has k boundary points. Then N has $(k - 2)$ Steiner points and $(2k - 3)$ sides.*

Proof. Assume that the network N has m Steiner points and c sides. By calculating we have $k = 3m - 2(m - 1)$. Hence, we obtain $m = k - 2$ and $c = 2k - 3$. The lemma is proved.

DEFINITION 1.3. Two boundary points of a Steiner network are called adjacent boundary points if they are ends of two adjacent sides (i.e. sides, which have a common end).

LEMMA 1.4. *A Steiner network always has at least two adjacent boundary points.*

Proof. Suppose that the given network N has not any two adjacent boundary points. We consider a boundary point A . The network N has only one side a with the end A . The second end of a is denoted by A_1 . Then N has two sides (different from a) with the end A_1 . We denote by B_1, B_2 the other ends of the sides a_1, a_2 respectively. Among B_1, B_2 there is at least one point, for example, B_1 . Further, the N has two sides b_1, b_2 (different from a_1) joining B_1 with C_1 and C_2 respectively (Fig. 1). Go on, because the network N is acyclic we obtain infinite number of Steiner points. That is impossible. The lemma is proved.

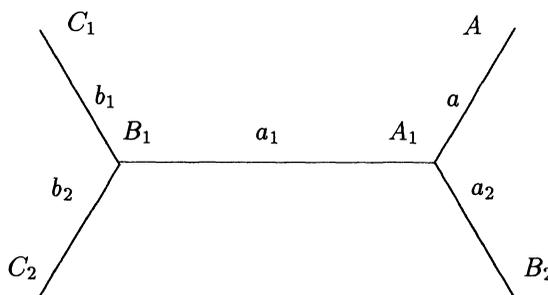


Fig. 1

THEOREM 1.5. *Every Steiner network with k boundary points has a finite series of subnetworks N_1, N_2, \dots, N_{k-2} such that*

$$N_1 \subset N_2 \subset \dots \subset N_{k-2} = N$$

Here N_1 has only one Steiner point, $N_{i+1} = N_i \cup \{a_i, b_i\}$ $1 \leq i \leq k-3$ and a_i, b_i are sides of N that do not belong to N_i .

Proof. Let a_{k-3}, b_{k-3} be sides of N joining two adjacent boundary points. We put $N_{k-3} = N \setminus \{a_{k-3}, b_{k-3}\}$. Then N_{k-3} is a Steiner network. By Lemma 1.4 N_{k-3} has two sides joining adjacent boundary points, for example, a_{k-4}, b_{k-4} . We put $N_{k-4} = N_{k-3} \setminus \{a_{k-4}, b_{k-4}\}$. Go on, after $k-3$ steps we get a series of subnetworks we need.

DEFINITION 1.6. A Steiner network is said to be oriented if its sides can be oriented so that every two adjacent sides are oriented opposite to each other.

COROLLARY 1.7. *A Steiner network has exactly two orientations.*

Proof. Suppose that the given network N has k boundary points. By Corollary 1.5 N has a series of subnetworks $N_1 \subset N_2 \subset \dots \subset N_k = N$. We consider N_1 . This subnetwork has only one point. Clearly, N_1 has exactly two following orientations (Fig. 2).

Further, we assume that the subnetwork N_i has exactly two orientations and consider the subnetwork $N_{i+1} = N_i \cup \{a_i, b_i\}$. Suppose that the sides a_i, b_i meet at A and denote by c_i the side in N_i with the end A . Clearly, each orientation in c_i , corresponding a certain orientation in N_i , completely defines a orientation on a_i and b_i (Fig. 3). Thus,

there are exactly two orientations in N_{i+1} . Proceeding the process we can prove the corollary.

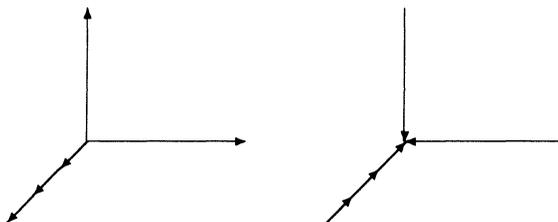


Fig. 2

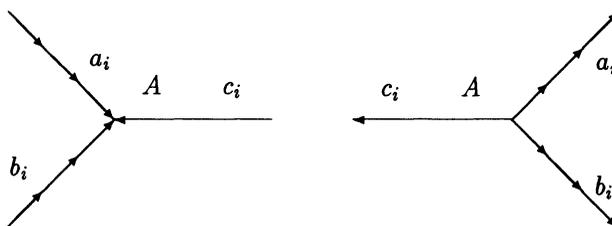


Fig. 3

2. Globally minimal Steiner networks

We remember that a path of a network is any continuous series of sides (with orientation) joining two vertices. If these vertices are boundary points then it is called a maximal path. A system of maximal paths is said to be independent if every one of them is not a combination of other paths from the system.

DEFINITION 2.1. A system of mximal paths $\{P_j\}_j$ in N is called a basis of maximal paths if it satisfies the following conditions:

- (1) The union of all paths from $\{P_j\}_j$ overlaps N
- (2) The system $\{P_j\}_j$ is independent
- (3) Every maximal path in N is a combination of path from $\{P_j\}_j$

THEOREM 2.2. *Every oriented Steiner network N with k boundary points has a basis of mximal paths consisting of $(k - 1)$ paths.*

Proof. We shall prove the theorem by induction for number k . At first we consider the case $k = 3$, i.e. the given network has three boundary points with three sides a, b, c

oriented as in Fig. 4.

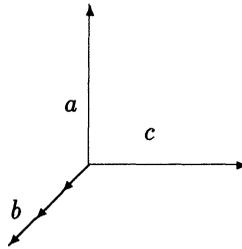


Fig. 4

We put $P_1 = a - b, P_2 = c - b$. Clearly, the system $\{P_1, P_2\}$ is a basis of maximal paths in N . Now, we assume that a Steiner network with i ($i > 3$) boundary points has a basis of maximal paths consisting of $(i - 1)$ paths. There is in N_i a side c_i ended at A_i . Assume that the side c_i belongs to $P_{i_1}, \dots, P_{i_k}, 1 \leq i_1, \dots, i_k \leq i - 1$. We put:

$$\begin{aligned} \bar{P}_\alpha &= P_\alpha, \alpha \neq i_1, i_2, \dots, i_k \\ \bar{P}_{i_j} &= P_{i_j} - \epsilon_{i_j}(c_i)b_i \\ \bar{P}_i &= a_i - b_i \end{aligned}$$

Where ϵ_{i_j} is the sign of the side c_i in the path P_{i_j} . To prove that the system $\{\bar{P}_1, \dots, \bar{P}_i\}$ is a basis of maximal paths in N_{i+1} we need only to check the condition (3) in definition 2.1. Let \bar{P} be a path joining two boundary points A and B in N_{i+1} . The first case: A, B belong to N_i . Suppose that P is a path joining A, B in N_i . Then P is the combination, for example, of P_{j_1}, \dots, P_{j_m} from $\{P_1, \dots, P_{i-1}\}$.

Hence, \bar{P} is the combination of $\bar{P}_{j_1}, \dots, \bar{P}_{j_m}$ in N_{i+1} .

The second case: A belongs to N_i, B is an end of a_i . We assume that \tilde{P} is the path joining A, A_i in N_i and \tilde{P} is the combination of $\bar{P}_{\alpha_1}, \dots, \bar{P}_{\alpha_n}$ from $\{P_1, \dots, P_{i-1}\}$. Then \bar{P} is the combination of $\bar{P}_{\alpha_1}, \dots, \bar{P}_{\alpha_n}, \bar{P}_i$ in N_{i+1} . The third case: A belongs to N_i, B is the end of b_i . We suppose that P is the path joining A, A_i in N_i and P is the combination of $P_{\beta_1}, \dots, P_{\beta_t}$ from (P_1, \dots, P_{i-1}) . Then \bar{P} is the combination of $\bar{P}_{\beta_1}, \dots, \bar{P}_{\beta_t}$ in N_{i+1} . The fourth case: A is the end of a_i and B is the end of b_i . Then $\bar{P} = \bar{P}_i$. The theorem is proved.

DEFINITION 2.3. Let N, N' be Steiner networks in R^n with the same boundary points A_1, \dots, A_k . We say that N and N' are of the same topological type if there is a homeomorphism $f: R^n \rightarrow R^n$ such that $f(A_i) = A_i, i = 1, \dots, k$ and $f(N) = N'$.

THEOREM 2.4. Let N be an oriented Steiner network with k boundary points in R^n and with a basis of maximal paths $\{P_1, \dots, P_k\}$. Suppose that there is a system of closed differential 1-forms $\omega_1, \dots, \omega_k$ on R^n such that

$$(1) \quad \sum_{j \in J_a} \omega_j(\epsilon_j(a)\vec{N}_x) = 1]$$

$$(2) \quad \left\| \sum_{j \in J_a} \epsilon_j(a) \omega_j \right\| \leq 1$$

Where $J_a = \{j : a \in P_j\}$ for any side $a \in N$ and \vec{N}_x is the unit tangent vector to N at $x \in a$ with the same orientations a . Then N is a length-minimizing network in the class of networks with fixed topological type.

Proof. Let N' be any Steiner network belonging to the topological type of N . Assume that $f : R^n \rightarrow R^n$ is homeomorphism such that $f(N) = N'$ and $f(A_i) = A_i$ for any i , where A_i are the vertices of N . The orientation on N' is induced by the orientation on N under f . We have $\{f(P_1), \dots, f(P_k)\}$ is a basis of the maximal paths in N' . Denote the length of N and N' by $|N|, |N'|$ respectively, and put $a' = f(a), x' = f(x)$. We have

$$\begin{aligned} |N| &= \sum_{a \in N} |a| = \sum_a \int_a \sum_{j \in J_a} \omega_j (\epsilon_j(a) \vec{N}_x) \\ &= \sum_j \int_{P_j} \omega_j = \sum_j \int_{P_{j'}} \omega_j \\ &= \sum_{a'} \int_{a'} \sum_{j \in J_{a'}} \omega_j (\epsilon_j(a') \vec{N}_{x'}) \\ &= \sum_{a'} \int_{a'} \sum_{j \in J_a} \omega_j (\epsilon_j(a) \vec{N}_{x'}) \\ &\leq \sum_{a'} \left| \int_{a'} 1 \right| \\ &= \sum_{a'} |a'| \\ &= |N'| \end{aligned}$$

Where $P'_j = f(P_j)$ and $|a|, |a'|$ are the length of a and a' , respectively. The theorem is proved.

Such a system $\{\omega_j\}$ is called a calibration system on N .

Now we remember that locally minimal Steiner networks have following properties:

- (1) The network consists of strightline segments
- (2) At every vertice the segments meet at angles of 120° .

By using theorem 2.4 we obtain the following result.

THEOREM 2.5. *Every locally minimal oriented Steiner network is also length-minimizing in the class of the Steiner networks with the same topological type.*

Proof. We shall prove this theorem by induction for the number of boundary points of N . At first we consider the case when N has only three boundary points A_1, A_2, A_3 .

The network is oriented as in Fig. 5.

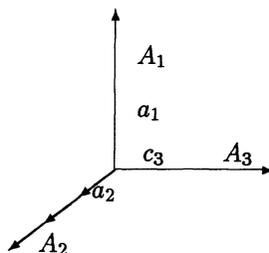


Fig. 5

Denote sides of N by a_1, a_2, a_3 . We put

$$P_1 = a_1 - a_3$$

$$P_2 = a_2 - a_3$$

Clearly, the system $\{P_1, P_2\}$ is a basis of maximal paths of N .

Let $\vec{a}_1, \vec{a}_2, \vec{a}_3$ be the unit tangent vectors to N on a_1, a_2, a_3 , respectively and a_1^*, a_2^*, a_3^* are the unit co-vectors dual to them. Let ω_1, ω_2 be constant differential 1-forms induced by a_1^*, a_2^* respectively. Then $\{\omega_1, \omega_2\}$ is a calibration system of N . Indeed, for any $x \in N$ we have:

$$\text{If } x \in a_1 \text{ then } \omega_1(\vec{N}_x) = a_1^*(\vec{a}_1) = 1$$

$$\text{If } x \in a_2 \text{ then } \omega_2(\vec{N}_x) = a_2^*(\vec{a}_2) = 1$$

$$\text{If } x \in a_3 \text{ then } (\omega_1\omega_2)(\vec{N}_x) = a_3^*(\vec{a}_3) = 1$$

By theorem 2.4 N is a length-minimizing network.

Further, assume that the statement of the theorem is true for any network with k boundary points and consider an arbitrary locally minimal oriented network N with $k + 1$ boundary points.

By Corollary 1.5 we have $N = N_k \cup \{a_k, b_k\}$. Since N_k is a locally minimal oriented network with k boundary points there are a basis of maximal paths $\{P_1, \dots, P_{k-1}\}$ and a calibration system $\{\omega_1, \dots, \omega_{k-1}\}$ of N_k such that

$$\left\| \sum_{j \in J_a} \epsilon_j(a) \omega_j \right\| \leq 1$$

$$\sum_{j \in J_a} \omega_j (\epsilon_j(a) \vec{N}_x) = 1$$

for any $x \in a \in N$, where \vec{a} denotes the unit tangent vector to a and a^* denotes the unit co-vector dual to \vec{a} . This means that $\sum_{j \in J_a} \epsilon_j(a) \omega_j = a^*$.

Now we choose the basis of maximal paths $\{\bar{P}_1, \dots, \bar{P}_k\}$ of N as in the proof of Theorem 2.2 and let ω_k be the unit constant differential 1-form, dual to a_k . We shall prove that $\{\omega_1, \dots, \omega_{k-1}, \omega_k\}$ is a calibration system of N . Really, denoting by $\bar{\epsilon}_j(a)$ the sign of the side a in the path \bar{P}_j and putting $\bar{J}_a = \{j : a \in \bar{P}_j\}$ we have :

a. If $x \in a \in N_k$ then

$$\begin{aligned} \left\| \sum_{j \in \bar{J}_a} \bar{\epsilon}_j(a) \omega_j \right\| &= \left\| \sum_{j \in J_a} \epsilon_j(a) \omega_j \right\| = \|a^*\| = 1 \\ \sum_{j \in \bar{J}_a} \omega_j \left(\bar{\epsilon}_j(a) \vec{N}_x \right) &= \sum_{j \in J_a} \omega_j \left(\epsilon_j(a) \vec{N}_x \right) = 1 \end{aligned}$$

b. If $x \in b_k$ then

$$\left\| \sum_{j \in \bar{J}_{b_k}} \epsilon_j(b_k) \omega_j \right\| = \left\| - \sum_{j \in J_{c_k}} \epsilon_j(c_k) \omega_j - \omega_k \right\| = \|-c_k^* - a_k^*\| = \|b_k^*\| = 1$$

$$\begin{aligned} \left[\sum_{j \in \bar{J}_{b_k}} \bar{\epsilon}_j(b_k) \omega_j \right] \left(\vec{N}_x \right) &= \left[- \sum_{j \in J_{c_k}} \epsilon_j(c_k) \omega_j - \omega_k \right] \left(\vec{N}_x \right) \\ &= [-c_k^* - a_k^*] \left(\vec{b}_k \right) \\ &= b_k^* \left(\vec{b}_k \right) \\ &= 1 \end{aligned}$$

c. If $x \in a_k$ then

$$\begin{aligned} \left\| \sum_{j \in \bar{J}_{a_k}} \bar{\epsilon}_j(a_k) \omega_j \right\| &= \|\omega_k\| = 1 \\ \sum_{j \in \bar{J}_{a_k}} \bar{\epsilon}_j(a_k) \omega_j \left(\vec{N}_x \right) &= \omega_k \left(\vec{a}_k \right) = 1 \end{aligned}$$

Thus, we have constructed a calibration system $\{\omega_1, \dots, \omega_{k-1}, \omega_k\}$ of N . The theorem is proved.

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