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MODIFIED NASH TRIVIALITY OF A FAMILY OF ZERO-SETS OF WEIGHTED HOMOGENEOUS POLYNOMIAL MAPPINGS

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§0. Introduction.

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be an n-tuple of positive integers. Assume that the greatest common divisor of α_j's is 1. Let $\mathbf{N}$ denote the set of positive integers, and let $\mathbf{R}$ denote the set of real numbers. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a polynomial function defined by

$$f(x) = \sum_{\beta} A_\beta x_1^{\beta_1} \cdots x_n^{\beta_n} \quad (A_\beta \neq 0,\ \beta_1, \cdots, \beta_n \in \mathbf{N} \cup \{0\}).$$

We say that f is *weighted homogeneous of type* $(\alpha_1, \cdots, \alpha_n; L)$ $(\alpha_1, \cdots, \alpha_n, L \in \mathbf{N})$, if

$$\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = L \quad \text{for any } \beta = (\beta_1, \cdots, \beta_n).$$

Let J be an open interval, and $t_0 \in J$. Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a polynomial mapping where each $f_{t,i}$ is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n; L_i)$ $(1 \leqq i \leqq p)$ for $t \in J$. We define a mapping $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ by $F(x,t) = f_t(x)$. Assume that F is a polynomial mapping (or of class C^2). It is well-known that the following fact holds under these assumptions:

FACT. *If $f_t^{-1}(0) \cap \sum f_t = \{0\}$ for any $t \in J$ (where $\sum f_t$ denotes the singular points set of f_t), then $(\mathbf{R}^n \times J, F^{-1}(0))$ is topologically trivial i.e. there exists a t-level preserving homeomorphism $\sigma : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^n \times J, \{0\} \times J)$ such that*

$$\sigma((R^n \times J, F^{-1}(0))) = (\mathbf{R}^n \times J, f_{t_0}^{-1}(0) \times J).$$

Remark 1. Results generalizing this fact have been obtained in [2], [5]. But it seems that the fact itself was recognized by many mathematicians a good while ago.

Since we consider the weighted homogeneous case with an isolated singularity, it seems natural that stronger triviality than topological one holds. In fact, such triviality called "modified Nash triviality" holds under the above assumptions (see Theorem in §2). On the other hand, we have introduced the notion of "strong C^0 triviality" for a family of analytic functions in [6]. Roughly speaking, strong C^0 equivalence is a C^0 equivalence which preserves the tangency of analytic arcs at $0 \in \mathbf{R}^n$. In §4, we discuss the relation between modified Nash triviality and strong C^0 triviality for a family of zero-sets of weighted homogeneous polynomials.

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§1. Some properties of Nash manifolds.

In this section, we recall some important results on Nash manifolds. A *semi-algebraic set* of $\mathbf{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbf{R}^n \mid f_1(x) = \cdots = f_k(x) = 0,\ g_1(x) > 0, \cdots, g_m(x) > 0\},$$

where $f_1, \cdots, f_k, g_1, \cdots, g_m$ are polynomial functions on $\mathbf{R}^n$. Let $r = 0, 1, 2, \cdots, \infty, \omega$. A semi-algebraic set of $\mathbf{R}^n$ is called a C^r (affine) *Nash manifold* if it is a regular C^r submanifold of $\mathbf{R}^n$. Let $M \subset \mathbf{R}^m$ and $N \subset \mathbf{R}^n$ be C^r Nash manifolds. A C^s mapping $f : M \to N$ $(s \leqq r)$ is called a C^s *Nash mapping* if the graph of f is semi-algebraic in $\mathbf{R}^m \times \mathbf{R}^n$.

THEOREM 1 (B. Malgrange [10]). (1) *A C^∞ Nash manifold is a C^ω Nash manifold.*

(2) *A C^∞ Nash mapping between C^ω Nash manifolds is a C^ω Nash mapping.*

After this, a Nash manifold and a Nash mapping mean a C^ω Nash manifold and a C^ω Nash mapping, respectively.

THEOREM 2 (M. Shiota [11]). *Let $M_1 \supset N_1$, $M_2 \supset N_2$ be compact Nash manifolds and compact Nash submanifolds. If the pairs (M_1, N_1) and (M_2, N_2) are C^∞ diffeomorphic, then they are Nash diffeomorphic.*

Remark 2. In Theorem 2, we can replace the assumption of "C^∞ diffeomorphic" by "C^1 diffeomorphic" ([12]).

THEOREM 3 (M. Shiota [12]). *There exist two (affine) Nash manifolds which are C^ω diffeomorphic but not Nash diffeomorphic.*

In general, Nash diffeomorphism is stronger than the notion of C^ω diffeomorphism.

§2. Result.

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be an n-tuple of positive integers. Put $\rho = \alpha_1 \cdots \alpha_n$ and $\rho_i = \rho/\alpha_i$ $(1 \leqq i \leqq n)$. Set

$$S_1(\alpha) = \{(X_1, \cdots, X_n) \in \mathbf{R}^n \mid X_1^{2\rho_1} + \cdots + X_n^{2\rho_n} = 1\}.$$

We define $\pi_\alpha : S_1(\alpha) \times \mathbf{R} \to \mathbf{R}^n$ by

$$\pi_\alpha(X_1, \cdots, X_n; u) = (u^{\alpha_1} X_1, \cdots, u^{\alpha_n} X_n).$$

Put $E = S_1(\alpha) \times \mathbf{R}$ and $E_0 = \pi_\alpha^{-1}(0) = S_1(\alpha) \times \{0\}$. Then E is a Nash manifold and E_0 is a Nash submanifold. The restricted mapping $\pi_\alpha|_{E-E_0} : E - E_0 \to \mathbf{R}^n - \{0\}$ is a $2:1$ mapping. Therefore $\pi_\alpha : (E, E_0) \to (\mathbf{R}^n, 0)$ is a finite Nash modification. Let J be an open interval and $t_0 \in J$, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in J)$ be a weighted homogeneous polynomial mapping. We define $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ by $F(x, t) = f_t(x)$.

DEFINITION. We say that $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a π_α-*modified Nash trivial-*

ization, if there exists a t-level preserving Nash diffeomorphism $\Phi : (E \times J, E_0 \times J) \to (E \times J, E_0 \times J)$ which induces a t-level preserving homeomorphism $\phi : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^n \times J, \{0\} \times J)$ such that

$$\phi((\mathbf{R}^n \times J, F^{-1}(0))) = (\mathbf{R}^n \times J, f_{t_0}^{-1}(0) \times J).$$

THEOREM. *Let J be an open interval, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a polynomial mapping where each $f_{t,i}$ is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n; L_i)$ $(1 \leqq i \leqq p)$ for $t \in J$. Assume that $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ is a polynomial mapping. If $f_t^{-1}(0) \cap \sum f_t = \{0\}$ for any $t \in J$, then $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a π_α-modified Nash trivialization.*

Remark 3. In the case $n \leqq p$, $f_t^{-1}(0) \cap \sum f_t = \{0\}$ implies $f_t^{-1}(0) = \{0\}$.

Example 1. Let $f_t : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ $(t \in \mathbf{R})$ be a weighted homogeneous polynomial of type $\alpha = (1, 2, 3; 13)$ defined by

$$f_t(x, y, z) = x^{13} + xy^6 + xz^4 + ty^5 z.$$

Then $\dfrac{\partial f_t}{\partial x} = 13x^{12} + y^6 + z^4$. Therefore each f_t has an isolated singularity. It follows from the Theorem that $(\mathbf{R}^3 \times \mathbf{R}, F^{-1}(0))$ admits a π_α-modified Nash trivialization.

PROBLEM 1. *Several kinds of topological triviality theorems for a family of analytic varieties are known. Do modified Nash triviality theorems hold under the same assumption for a family of algebraic varieties?*

§3. Outline of the proof of the theorem.

The proof of the Theorem consists of three parts.

STEP 1. Remark that $S_1(\alpha) \cap f_t^{-1}(0)$ $(t \in J)$ and $S_1(\alpha) \times J \cap F^{-1}(0)$ are Nash submanifolds of $S_1(\alpha)$ and $S_1(\alpha) \times J$, respectively.

PROPOSITION 1. *Under the same assumption as the Theorem, there exists a t-level preserving C^∞ diffeomorphism*

$$H : S_1(\alpha) \times J \cap F^{-1}(0) \to (S_1(\alpha) \cap f_{t_0}^{-1}(0)) \times J.$$

EULER'S THEOREM. *If $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n; L)$, then*

$$\alpha_1 x_1 \frac{\partial f}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial f}{\partial x_n} = Lf.$$

Many singularists would know that such a t-level preserving C^∞ diffeomorphism in Proposition 1 exists. But, thanks to Euler's Theorem, we can concretely construct a C^∞ vector field on $S_1(\alpha) \times J \cap F^{-1}(0)$, called the Kuo vector field ([8], [9]), whose flow gives the diffeomorphism.

STEP 2. (semi-algebraic triviality theorem)

PROPOSITION 2. *Let $M \supset N$ be a Nash manifold and a Nash (regular) submanifold such that N is closed in M. Let $p : M \to \mathbf{R}$ be a proper Nash submersion such that the restriction of p to N is also a proper Nash submersion. Then there exists a Nash diffeomorphism*

$$\Phi : (M, N) \to (M_0, N_0) \times \mathbf{R}$$

such that $p \circ \Phi^{-1}$ is the canonical projection onto $\mathbf{R}$, namely, $p \circ \Phi^{-1}(m_0, t) = t$ for $(m_0, t) \in M_0 \times \mathbf{R}$, where $M_0 = p^{-1}(0)$ and $N_0 = (p|_N)^{-1}(0)$.

Remark 4. We can replace $\mathbf{R}$ by an open interval J in Proposition 2.

We can prove this proposition, by using the following results :
(i) Proposition 2 holds in the case where $N = \emptyset$ ([3] Theorem 1).
(ii) For $0 < r < \infty$, Proposition 2 holds in the case where M is a C^r Nash manifold with boundary N ([3] Theorem 3).
(iii) C^r Nash tubular neighbourhood theorem holds for $r > 1$ ([12] Lemma I.3.2).
(iv) C^r Nash partition of unity holds for $0 < r < \infty$ ([12] Corollaries II.2.8, 2.9 and Remark II.2.13).
(v) C^r Nash approximation theorem holds for $r \leqq \omega$ ([12] Corollary II.5.7).

STEP 3. By Propositions 1 and 2, there exists a t-level preserving Nash diffeomorphism $G : S_1(\alpha) \times J \to S_1(\alpha) \times J$ such that

$$G(S_1(\alpha) \times J \cap F^{-1}(0)) = (S_1(\alpha) \cap f_{t_0}^{-1}(0)) \times J.$$

We write $G(x,t) = (\sigma_t(x), t)$ for $(x,t) \in S_1(\alpha) \times J$. We define a mapping $\Phi : (E \times J, E_0 \times J) \to (E \times J, E_0 \times J)$ by

$$\Phi((x; u), t) = ((\sigma_t(x); u), t).$$

Then this Φ gives the modified Nash trivialization in the Theorem.

§4. Strong C^0 equivalence.

First, we define the notion of strong C^0 equivalence.

NOTATION. (1) By an analytic arc at $0 \in \mathbf{R}^n$, we mean the germ of an analytic map $\lambda \cdot [0, \varepsilon) \to \mathbf{R}^n$ with $\lambda(0) = 0$, $\lambda(s) \neq 0$, $s > 0$. The set of all such arcs is denoted by $A(\mathbf{R}^n, 0)$.

(2) For $\lambda, \mu \in A(\mathbf{R}^n, 0)$, $O(\lambda, \mu) > 1$ (resp. $O(\lambda, \mu) = 1$) means that arcs λ, μ are *tangent* (resp. *crossing without touching*) at $0 \in \mathbf{R}^n$.

Let $E_{|\omega|}(n, 1)$ be the set of analytic function germs : $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, and let $S(\mathbf{R}^n, 0)$ be the set of set germs at $0 \in \mathbf{R}^n$.

DEFINITION. Given $f, g \in E_{|\omega|}(n, 1)$, we say that $(\mathbf{R}^n, f^{-1}(0))$, $(\mathbf{R}^n, g^{-1}(0)) \in S(\mathbf{R}^n, 0)$ are *strongly C^0 equivalent*, if there exists a local homeomorphism $\sigma : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that
(I) $\sigma(f^{-1}(0)) = g^{-1}(0)$,

(II) if $\lambda \in A(\mathbf{R}^n, 0)$ with $\lambda \subset f^{-1}(0)$ (resp. $g^{-1}(0)$), then $\sigma(\lambda)$ (resp. $\sigma^{-1}(\lambda)$) $\in A(\mathbf{R}^n, 0)$, and

(III) for any $\lambda, \mu \in A(\mathbf{R}^n, 0)$ with λ, $\mu \subset f^{-1}(0)$, $O(\lambda, \mu) = 1$ if and only if $O(\sigma(\lambda), \sigma(\mu)) = 1$.

Let J be an open interval, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ ($t \in J$) be a weighted homogeneous polynomial of type $\alpha = (\alpha_1, \cdots, \alpha_n)$ with an isolated singularity. In this section, we discuss the relation between π_α-modified Nash triviality and strong C^0 triviality of the family $\{(\mathbf{R}^n, f_t^{-1}(0))\}_{t \in J}$.

(A) Consider the homogeneous case i.e. $\alpha_1 = \cdots = \alpha_n = 1$. Recall the notations $E = S_1(\alpha) \times \mathbf{R}$ and $E_0 = S_1(\alpha) \times \{0\}$. We say that $(X; u) = (X_1, \cdots, X_n; u)$, $(Y; s) = (Y_1, \cdots, Y_n; s) \in E$ are *equivalent*, if

(i) $X_i = Y_i$ $(1 \leqq i \leqq n)$ and $u = s$, or

(ii) $X_i = -Y_i$ $(1 \leqq i \leqq n)$ and $u = -s$.

Then this relation is an equivalence relation. We denote by $\widetilde{E}$ and $\widetilde{E_0}$ the quotient sets of E and E_0 by the relation $\sim$, respectively. Let $\pi : (E, E_0) \to (\widetilde{E}, \widetilde{E_0})$ be the quotient map, and let $\widetilde{\pi_\alpha} : (\widetilde{E}, \widetilde{E_0}) \to (\mathbf{R}^n, 0)$ be the blow up at $0 \in \mathbf{R}^n$. Then the following diagram commutes :

$$\begin{tikzcd}
(E, E_0) \arrow[d, "\pi"] \arrow[dr, "\pi_\alpha"] & \\
(\widetilde{E}, \widetilde{E_0}) \arrow[r, "\widetilde{\pi_\alpha}"] & (\mathbf{R}^n, 0)
\end{tikzcd}$$

By the Theorem, $\{(\mathbf{R}^n, f_t^{-1}(0))\}_{t \in J}$ admits a π_α-modified Nash trivialization i.e. there exists a t-level preserving Nash diffeomorphism $\Phi : (E \times J, E_0 \times J) \to (E \times J, E_0 \times J)$ which induces a t-level preserving homeomorphism $\phi : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^n \times J, \{0\} \times J)$ such that

$$\phi((\mathbf{R}^n \times J, F^{-1}(0))) = (\mathbf{R}^n \times J, f_{t_0}^{-1}(0) \times J) \text{ for } t_0 \in J.$$

In this case, the Nash diffeomorphism Φ induces a t-level preserving Nash diffeomorphism $\widetilde{\Phi} : (\widetilde{E} \times J, \widetilde{E_0} \times J) \to (\widetilde{E} \times J, \widetilde{E_0} \times J)$ such that the following diagram commutes :

$$\begin{tikzcd}
(E \times J, E_0 \times J) \arrow[r, "(\pi, \mathrm{id})"] \arrow[d, "\Phi"] & (\widetilde{E} \times J, \widetilde{E_0} \times J) \arrow[r, "(\widetilde{\pi_\alpha}, \mathrm{id})"] \arrow[d, "\widetilde{\Phi}"] & (\mathbf{R}^n \times J, \{0\} \times J) \arrow[d, "\phi"] \\
(E \times J, E_0 \times J) \arrow[r, "(\pi, \mathrm{id})"] & (\widetilde{E} \times J, \widetilde{E_0} \times J) \arrow[r, "(\widetilde{\pi_\alpha}, \mathrm{id})"] & (\mathbf{R}^n \times J, \{0\} \times J)
\end{tikzcd}$$

Therefore π_α-modified Nash triviality implies strong C^0 triviality.

(B) Consider the case where $\alpha_1 = \cdots = \alpha_s < \alpha_{s+1} = \cdots = \alpha_n$ $(1 \leqq s < n)$.

PROBLEM 2. *Is the family $\{(\mathbf{R}^n, f_t^{-1}(0))\}_{t \in J}$ strongly C^0 trivial?*

The Theorem in §2 has come from this problem.

(C) Consider the case where $n = 3$ and $\alpha_1 < \alpha_2 < \alpha_3$.

PROPOSITION 3. *Let $f, g : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ be weighted homogeneous polynomials of type $(\alpha_1, \alpha_2, \alpha_3; L)$ $(\alpha_1 < \alpha_2 < \alpha_3)$ with an isolated singularity. Assume that two set germs $(\mathbf{R}^3, f^{-1}(0))$, $(\mathbf{R}^3, g^{-1}(0)) \in S(\mathbf{R}^3, 0)$ are strongly C^0 equivalent. If $f^{-1}(0) \supset \{x_1 = 0\}$, then $g^{-1}(0) \supset \{x_1 = 0\}$.*

Example 1. Let $f_t : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$, $(t \in \mathbf{R})$ be a weighted homogeneous polynomial defined in §2. Then, by Proposition 3, $\{(\mathbf{R}^3, f_t^{-1}(0))\}_{t \in \mathbf{R}}$ is not strongly C^0 trivial at $0 \in \mathbf{R}$.

Example 2 (Briançon-Speder family [1]). Let $f_t : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ be a weighted homogeneous polynomial defined by

$$f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$$

for $|t| < 1 + \varepsilon$, where ε is a sufficiently small positive number. Then each f_t has an *algebraically* isolated singularity. But two set germs $(\mathbf{R}^3, f_0^{-1}(0))$, $(\mathbf{R}^3, f_{-1}^{-1}(0)) \in S(\mathbf{R}^3, 0)$ are not strongly C^0 equivalent (Theorem A in [6]).

It follows from the Theorem and the above examples that modified Nash triviality does not imply strong C^0 triviality in this case.

Remark 5. In [7], the author formulated a necessary condition for a family of weighted homogeneous polynomials of three variables to be strongly C^0 trivial. Recently T. Fukui has given a new approach to strong C^0 triviality of a family of polynomial functions of three variables, by using toric resolution ([4]).

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