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NOTE ON ESTIMATION OF THE NUMBER OF THE CRITICAL VALUES AT INFINITY

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1. Let $f(x,y)$ be a polynomial of degree d and we consider the polynomial function $f:\mathbf{C}^2 \to \mathbf{C}$. Let $\Sigma(f)$ be the critical values. The restriction

$$f:\mathbf{C}^2 - f^{-1}(\Sigma) \to \mathbf{C} - \Sigma$$

is not necessarily a locally trivial fibration. We say that $\tau \in \mathbf{C}$ is *a regular value at infinity* of the function $f:\mathbf{C}^2 \to \mathbf{C}$ if there exist positive numbers R and ε so that the restriction of f, $f: f^{-1}(D_\varepsilon(\tau)) - B_R^4 \to D_\varepsilon(\tau)$, is a trivial fibration over the disc $D_\varepsilon(\tau)$ where $D_\varepsilon(\tau) = \{\eta \in \mathbf{C}; |\eta - \tau| \le \varepsilon\}$ and $B_R^4 = \{(x,y); |x|^2 + |y|^2 \le R\}$. Otherwise τ is a called *a critical value at infinity.* We denote the set of the critical values at infinity by Σ_∞. It is known that Σ_∞ is finite ([23], [2]). The purpose of this note is to give an estimation on the number of critical values at infinity. The detail will be published elsewhere ([12]).

We first consider the canonical projective compactification $\mathbf{C}^2 \subset \mathbf{P}^2$. We denote the homogeneous coordinates of $\mathbf{P}^2$ by X, Y, Z so that $x = X/Z$ and $y = Y/Z$ Let L_∞ be the line at infinity: $L_\infty = \{Z = 0\}$. Write

$$f(x,y) = f_0 + f_1(x,y) + \cdots + f_d(x,y)$$

where $f_i(x,y)$ is a homogeneous polynomial of degree i for $i = 0, \ldots, d$. We can write

$$f_d(x,y) = c x^{\nu_0} y^{\nu_{k+1}} \prod_{j=1}^{k} (y - \lambda_j x)^{\nu_j} \tag{1.1}$$

where $c \in \mathbf{C}^*$ and $\lambda_1, \ldots, \lambda_k$ are non-zero distinct numbers and we assume that $\nu_i > 0$ for $1 \le i \le k$ and $\nu_0, \nu_{k+1} \ge 0$. Note that we have the equality

$$\nu_0 + \cdots + \nu_{k+1} = d \tag{1.2}$$

Let C_τ be the projective curve which is the closure of the fiber $f^{-1}(\tau)$. Then C_τ is defined by $C_\tau = \{(X;Y;Z) \in \mathbf{P}^2; F(X,Y,Z) - \tau Z^d = 0\}$ where $F(X,Y,Z)$ is the homogeneous polynomial defined by

$$F(X,Y,Z) = f(X/Z, Y/Z)Z^d = f_0 Z^d + f_1(X,Y)Z^{d-1} + \cdots + f_d(X,Y) \tag{1.3}$$

The intersection of C_τ and the line at infinity, $C_\tau \cap L_\infty$, is independent of $\tau \in \mathbf{C}^2$ and it is the base point locus of the family $\{C_\tau; \tau \in \mathbf{C}\}$. Obviously we have $C_\tau \cap L_\infty = \{Z = f_d(X,Y) = 0\}$. For brevity, let $A_i = (\alpha_i; \beta_i; 0) \in \mathbf{P}^2$ for $i = 0, \ldots, k+1$ where $A_0 = (0;1;0), A_{k+1} = (1;0;0)$ and $\beta_i/\alpha_i = \lambda_i$ for $1 \le i \le k$. Then under the assumption (1.1), $C_0 \cap L_\infty = \{A_i; \nu_i > 0\}$. Note that $A_i \in C_0 \cap L_\infty$ for $i = 1, \ldots, k$. We consider the family of germs of a curve at A_j: $\{(C_\tau, A_j); \tau \in \mathbf{C}\}$. Then it is known that τ is a

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regular value at infinity if and only if $\{(C_t, A_j); t \in \mathbf{C}\}$ is a topologically stable family near $t = \tau$ for any A_j with $\nu_j > 0$ ([2]). This is the case if $f(x,y) - \tau$ is reduced and the local Milnor number μ of the family $\{(C_t, A_j); t \in \mathbf{C}\}$ is constant in a neighborhood U of $\tau \in \mathbf{C}$. To study the stability of the local topological type at A_j, we will use the affine polar quotient along the polar curve at infinity.

2. Affine polar quotients and a toric compactification.

A. Affine polar quotients.

Let $\ell(x,y) = \alpha y - \beta x$ be a linear form. The polar curve $\Gamma_\ell(f)$ for f with respect to ℓ is defined by the Jacobian $\Gamma_\ell(f) = \{(x,y) \in \mathbf{C}^2; J(f,\ell)(x,y) = 0\}$ where

$$J(f,\ell)(x,y) = \alpha\frac{\partial f}{\partial x}(x,y) + \beta\frac{\partial f}{\partial y}(x,y) = 0$$

$\Gamma_\ell(f)$ is an affine curve of degree $d-1$ and equal to the critical locus of the mapping $(f,\ell) : \mathbf{C}^2 \to \mathbf{C}^2$. Let L_η be the projective line $\{\alpha Y - \beta X - \eta Z = 0\}$ which is the closure of the affine line $\ell^{-1}(\eta)$. The base point of this pencil $\{L_\eta; \eta \in \mathbf{C}\}$ is $B = (\alpha;\beta;0)$ in the homogeneous coordinates. We say that ℓ is *generic at infinity* for the polynomial f if $B \notin C_0 \cap L_\infty$. This is the case if and only if $f_d(\alpha,\beta) \neq 0$. We assume the genericity of ℓ hereafter. Let $\overline{\Gamma_\ell(f)}$ be the projective closure of $\Gamma_\ell(f)$ and let $\overline{\Gamma_\ell(f)} \cap L_\infty = \{Q_1, \dots, Q_\delta\}$. Let γ be a local analytic irreducible component of $\overline{\Gamma_\ell(f)}$ at Q_i. Consider an analytic parametrization $\Phi_\gamma : (D_\varepsilon(0), 0) \to (\gamma, Q_i)$ in a local coordinate system in a neighborhood of Q_i. In the original affine coordinates, this can be written as $\Phi_\gamma(t) = (x_\gamma(t), y_\gamma(t))$ where $x_\gamma(t)$ and $y_\gamma(t)$ are Laurent series in t. Consider the rational number $v_\gamma(f,\tau)$ defined by

$$v_\gamma(f,\tau) = \frac{\mathrm{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau)}{\mathrm{val}_t(\ell(x_\gamma(t), y_\gamma(t)))}$$

Here val_t is the standard valuation defined by the variable t. It is easy to see that this number depends only on τ, γ and f and it does not depend on the choice of the parametrization. So we call this number *the affine polar quotient of the the function* $f(x,y) - \tau$ ([9],[15]). This definition is an analogy of the local polar quotient defined in [10]. In the case of $f(x_\gamma(t), y_\gamma(t)) - \tau \equiv 0$, the valuation $\mathrm{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau)$ is $+\infty$ by definition. Let p be a positive integer. We use the convention $+\infty/\pm p = \pm\infty$ and $-\infty$ (resp. $+\infty$) is negative (resp. positive). Note that for the definition of the affine polar quotient, the compactification does not make any difference.

We generalize the notion of a regular value at infinity. Let $A_i \in C_0 \cap L_\infty$ (so $\nu_i > 0$) and let $\tau \in \mathbf{C}$. We say that τ is *a regular value at* A_i for $f : \mathbf{C}^2 \to \mathbf{C}$ if there exists an open neighborhood U of A_i in $\mathbf{P}^2$ and a positive number ε such that $f : U \cap f^{-1}(D_\varepsilon(\tau)) \to D_\varepsilon(\tau)$ is a trivial fibration. Here $f^{-1}(D_\varepsilon(\tau)) \subset \mathbf{C}^2$ and therefore $U \cap f^{-1}(D_\varepsilon(\tau)) \subset U - L_\infty$. Now the importance of the affine polar quotients is the following lemma:

LEMMA (2.1). *Assume that ℓ is generic. Then $\tau \in \mathbf{C}$ is a regular value at A_i for $f : \mathbf{C}^2 \to \mathbf{C}$ if the affine polar quotient $v_\gamma(f,\tau) \geq 0$ for any local irreducible component γ of $\overline{\Gamma_\ell(f)}$ at A_i.*

For the proof, we reffer to [12].

COROLLARY (2.1.1)([15]). *Assume that ℓ is generic. Then $\tau \in \mathbf{C}$ is a regular value at infinity for the function $f : \mathbf{C}^2 \to \mathbf{C}$ if (and only if) the affine polar quotient satisfies $v_\gamma(f, \tau) \geq 0$, for any local irreducible component γ of $\overline{\Gamma_\ell(f)}$ at A_i, $i = 0, \ldots, k+1$.*

Note that $\mathrm{val}_t(\ell(x_\gamma(t), y_\gamma(t))) < 0$ as $\|(x_\gamma(t), y_\gamma(t))\| \to \infty$. Thus for any irreducible component γ at infinity of $\Gamma_\ell(f)$, we have

$$v_\gamma(f, \tau) \geq 0 \iff \mathrm{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau) \leq 0 \tag{2.1.2}$$

LEMMA (2.2). *Assume that ℓ is generic. Choose A_i with $\nu_i \geq 2$ and let $(x_\gamma(t), y_\gamma(t))$ be a parametrization of a local irreducible component γ of $\overline{\Gamma_\ell(f)}$ at A_i.*

(i) *If $v_\gamma(f; 0) > 0$, then $v_\gamma(f; \tau) > 0$ for any any $\tau \in \mathbf{C}$.*

(ii) *If $v_\gamma(f; 0) \leq 0$, there exists a unique $\xi \in \mathbf{C}$ so that $v_\gamma(f; \xi) < 0$. For any other $\tau \neq \xi$, $v_\gamma(f; \tau) = 0$.*

Proof. Assume first that $v_\gamma(f; 0) > 0$. Then $\mathrm{val}_t(f(x_\gamma(t),\ y_\gamma(t))) < 0$ by (2.1.2) and therefore $\mathrm{val}_t(f(x_\gamma(t),\ y_\gamma(t)) - \tau) < 0$ for any τ.

Assume that $v_\gamma(f; 0) \leq 0$. This implies that $\mathrm{val}_t(f(x_\gamma(t),\ y_\gamma(t))) \geq 0$. Then $\lim_{t\to 0} f(x_\gamma(t),\ y_\gamma(t))$ is well defined. So we denote this limit by ξ. Then it is obvious that $\mathrm{val}_t(f(x_\gamma(t),\ y_\gamma(t)) - \tau) = 0$ for any $\tau \neq \xi$. This completes the proof.

DEFINITION (2.3). We call that a local irreducible component γ of $\overline{\Gamma_\ell(f)}$ at A_i is *stable* (respectively *unstable*) if $v_\gamma(f; 0) > 0$ (resp. $v_\gamma(f; 0) \leq 0$). We denote the set of unstable local irreducible components of $\overline{\Gamma_\ell(f)}$ at infinity by $\mathcal{US}(\Gamma_\ell)$. Assume that γ is a unstable local irreducible component and let ξ be the complex number characterized in (ii). Considering ξ as a function of γ, we write $\xi(\gamma)$. Thus we have a mapping $\xi : \mathcal{US}(\Gamma_\ell) \to \mathbf{C}$. $\xi(\gamma)$ is called *the limit critical value of f along γ.*

COROLLARY (2.3.1). *The number of the critical values at infinity $|\Sigma_\infty|$ is equal to the cardinality of the image $\xi(\mathcal{US}(\Gamma_\ell))$. In particular, it is less than or equal to the cardinality of $\mathcal{US}(\Gamma_\ell)$.*

We define *the projective degeneracy at infinity* $\nu_\infty^{pr}(f)$ by

$$\nu_\infty^{pr}(f) = \sum_{i=0}^{k+1} \max(\nu_i - 1, 0)$$

As the number of irreducible components of $\overline{\Gamma_\ell}$ at A_i is less than or equal to $\nu_i - 1$, we have the following estimation.

THEOREM (2.4). *The number of critical points at infinity $|\Sigma_\infty|$ is less than or equal to $\nu_\infty^{pr}(f)$. In particular, $|\Sigma_\infty| \leq d - 1$.*

This estimation can be obtained using the projective compactification but it is not so good when ν_0 or ν_{k+1} is big. It turns out that a suitable toric compactification is more convenient for our purpose.

B. Toric Compactification of $\mathbf{C}^2$.

Let $f(x,y) = \sum_{(m,n)} a_{m,n}x^m y^n$ be a given polynomial of degree d. As we are interested in the estimation of the number of critical values at infinity, we may assume that $f(0,0) \neq 0$ by adding a constant if necessary. We consider the Newton polygon $\Delta(f)$ of f which is the convex hull of the integral point (m,n) such that $a_{m,n} \neq 0$. By the assumption $f(0,0) \neq 0$, we have $O \in \Delta(f)$. Let N be the space of covectors. Any covector P defines a linear function on $\Delta(f)$. For any integral covector $P = {}^t(p,q)$, let $\Delta(P;f) \subset \Delta(f)$ be the locus where the linear function $P|\Delta(f)$ takes the minimal value. We denote this minimal value by $d(P;f)$ as usual. Let $f_P(x,y)$ be the partial sum

$$f_P(x,y) := \sum_{(m,n)\in\Delta(P;f)} a_{m,n}x^m y^n$$

and we call f_P the face function of the covector P. The dual Newton diagram $\Gamma^*(f)$ is defined by the following equivalence relation in N: $P \sim Q$ if and only if $\Delta(P;f) = \Delta(Q;f)$. Here $\Delta(P;f)$ is the locus where the linear function $P|\Delta(f)$ takes its minimal value. Let Σ^* be a regular simplicial cone subdivision of $\Gamma^*(f)$ and let X be the toric variety associated with Σ^*. Let $E_1 = {}^t(1,0), E_2 = {}^t(0,1)$. It is easy to see that $\mathrm{Cone}(E_1, E_2)$ is admissible with Σ^*. This is immediate from the assumption that $O \in \Delta(f)$. Thus we may assume that $\mathrm{Cone}(E_1,E_2)$ is a simplicial cone in Σ^*. Let $R_1, \dots, R_\mu$ be the vertices of Σ^* in the counter-clockwise orientation where $R_1 = E_1, R_2 = E_2$. Thus $\sigma_i := \mathrm{Cone}(R_i, R_{i+1}),\ i = 1,\dots,\mu$ be the two-dimensional simplicial cones in Σ^* where $R_{\mu+1} = R_1$. Here we assume $R_1 = E_1, R_2 = E_2, R_{\mu+1} = R_1$. Let $\sigma_1 = \mathrm{Cone}(E_1, E_2)$. Recall that X is a smooth compact toric variety of dimension 2 whose affine charts are $\mathbf{C}^2_{\sigma_i}; i = 1,\dots,\mu$ and it has the canonical decomposition

$$X = \mathbf{C}^{*2} \coprod_{i=1}^{\mu} \widehat{E}(R_i)$$

where $\widehat{E}(R_i)$ is a rational curve corresponding to the vertex $R_i \in \mathrm{Vertex}(\Sigma^*)$. The divisor $\widehat{E}(R_i)$ intersects with $\widehat{E}(R_{i-1})$ and $\widehat{E}(R_{i+1})$. So the dual graph of the divisors $\widehat{E}(R_i), i = 1,\dots,\mu$ makes a cycle. Taking a subdivision if necessary, we may assume that $H := {}^t(-1,-1)$ in $\mathrm{Vertex}(\Sigma^*)$. Thus we assume that $H = R_\theta$ for some $3 \le \theta \le \mu$. The projective compactification corresponds to the smallest simplicial cone Σ_0^* which has three vertices $\{E_1, E_2, H\}$. Let (u_i, v_i) be the corresponding coordinates of the chart $\mathbf{C}^2_{\sigma_i}$. Let us consider the unimodular matrix σ_i' corresponding to the vertices of the cone σ_i:

$$\sigma_i' = \begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix}$$

$(a_i b_{i+1} - a_{i+1} b_i = 1)$. Then the original affine space is identified with the coordinate space $\mathbf{C}^2_{\sigma_1}$ with $x = u_1, y = v_1$. Recall that $\mathbf{C}^2_{\sigma_i}$ is glued with the original affine space $\mathbf{C}^2$ by

$$\text{(2.5)} \qquad \begin{cases} x = u_i^{a_i} v_i^{a_{i+1}} \\ y = u_i^{b_i} v_i^{b_{i+1}} \end{cases}, \qquad \begin{cases} u_i = x^{b_{i+1}} y^{-a_{i+1}} \\ v_i = x^{-b_i} y^{a_i} \end{cases}$$

We consider the curve $C = \{(x,y) \in \mathbf{C}^2; f(x,y) = 0\}$ in the original affine space $\mathbf{C}^2$ and let $\widetilde{C}$ be the closure of C in X. The curve $\widetilde{C}$ is defined in $\mathbf{C}^2_{\sigma_i}$ by the equation

$f_{\sigma_i}(u_i, v_i) = 0$ where $f_{\sigma_i}(u_i, v_i)$ is defined by

$$f_{\sigma_i}(u_i, v_i) := f(u_i^{a_i} v_i^{a_{i+1}}, u_i^{b_i} v_i^{b_i+1})/u_i^{d(R_i;f)} v_i^{d(R_{i+1};f)}$$

In $\mathbf{C}^2_{\sigma_i}$, $\widehat{E}(R_i)$ is defined by $u_i = 0$. It is easy to see that $f_{\sigma_i}(0,0) \neq 0$ and

$$f_{\sigma_i}(0, v_i) = f_{R_i}(u_i^{a_i} v_i^{a_{i+1}}, u_i^{b_i} v_i^{b_i+1})/u_i^{d(R_i;f)} v_i^{d(R_{i+1};f)}$$

is non-constant if and only if $\dim \Delta(R_i; f) \geq 1$. Let $D_1, \ldots, D_m$ be the faces of $\Delta(f)$ in the counter-clockwise orientation so that D_1, D_m contains the origin O. Let $P_i = {}^t(p_i, q_i)$ be the corresponding primitive integral covector of D_i. Note that each P_i must be a vertex of Σ^* and therefore we can write $P_i = R_{\nu_i}$ for some $1 \leq \nu_i \leq \mu$. Then we can write

$$f_{P_i}(x, y) = \delta_i x^{r_i} y^{s_i} \prod_{j=1}^{\ell_i} (y^{p_i} - \xi_{i,j} x^{q_i})^{\nu_{i,j}} \tag{2.6}$$

where $\delta_i \in \mathbf{C}^*$ and $\xi_{i,j}$, $1 \leq j \leq \ell_i$ are mutually distinct non-zero complex numbers. By the above consideration, $\widehat{E}(R_i) \cap C \neq \emptyset$ if and only if $i = \nu_j$ for some $1 \leq j \leq m$. We consider the toric coordinate chart $\sigma_{\nu_i} = \mathrm{Cone}(R_{\nu_i}, R_{\nu_i+1})$. Then

$$h_{\sigma_i}(0, v_i) = \delta_i \prod_{j=1}^{\ell_i} (v_i - \xi_{i,j})^{\nu_{i,j}} \tag{2.7}$$

Thus $\widehat{E}(R_{\nu_i}) \cap \widetilde{C}$ consists of ℓ_i points $\{(0, \xi_{i,j}); j = 1, \ldots, \ell_i\} \subset \mathbf{C}^2_{\sigma_{\nu_i}}$. Put $A_{i,j} := (0, \xi_{i,j}) \in \widehat{E}(R_{\nu_i}) \cap \widetilde{C}$ for $1 \leq i \leq m, 1 \leq j \leq \ell_i$. See [20], [16], [7] for further information about the toric compactification.

Now we consider the limit of the value of the function f along an irreducible component γ of $o\overline{\Gamma_\ell(f)}$. Let $\Phi_\gamma(t)$ be a parametrization of γ in the coordinates (x, y) (namely in $\mathbf{C}^2_{\sigma_1}$) in the neighborhood of the infinity where $x_\gamma(t)$ and $y_\gamma(t)$ are Laurent series in the variable t. We assume that $x_\gamma(t), y_\gamma(t) \not\equiv 0$ and write them as

$$\begin{cases} x_\gamma(t) = \alpha_\gamma t^{p_\gamma} + & \text{(higher terms)} \\ y_\gamma(t) = \beta_\gamma t^{q_\gamma} + & \text{(higher terms)}, \quad t \in D_\varepsilon(0) \end{cases} \tag{2.8}$$

Let $Q_\gamma := {}^t(p_\gamma, q_\gamma) \in N$ and $A_\gamma := (\alpha_\gamma, \beta_\gamma)$. We assume that

$$\min(p_\gamma, q_\gamma) < 0, \quad \alpha_\gamma, \beta_\gamma \neq 0 \tag{2.8.1}$$

so that $x_\gamma(t) \not\equiv 0$, $y_\gamma(t) \not\equiv 0$ and $|x_\gamma(t)|^2 + |y_\gamma(t)|^2 \to \infty$. In this situation, we have

PROPOSITION (2.9). (i) *We have* $\mathrm{val}_t f(x_\gamma(t), y_\gamma(t)) \geq d(Q_\gamma; f)$ *and the inequality holds if and only if* $Q_\gamma \sim P_i$ *and* $\beta_\gamma^{p_i} - \xi_{i,j} x_\gamma^{q_i} = 0$ *for some* i, $1 \leq i \leq m$ *and* $j, 1 \leq j \leq \ell_i$.
(ii) *The limit* $\lim_{t\to 0} \Phi_\gamma(t)$ *in* X *always exists and we have*

$$\lim_{t\to 0} \Phi_\gamma(t) = \begin{cases} (0,0) \in \mathbf{C}^2_{\sigma_j} & \text{if } Q_\gamma \in \mathrm{IntCone}(R_j, R_{j+1}) \\ (0, \alpha_\gamma^{-b_j} \beta_\gamma^{b_j}) \in \mathbf{C}^2_{\sigma_j} & \text{if } Q_\gamma = cR_j, \text{ for some } c > 0 \end{cases}$$

Here $\mathrm{IntCone}(R_j, R_{j+1})$ *is the open cone generated by* R_j *and* R_{j+1}. *In particular, if* $Q_\gamma \sim P_i$ *and* $\beta_\gamma^{p_i} - \xi_{i,j} x_\gamma^{q_i} = 0$ *for some* i, $1 \leq i \leq m$, $\lim_{t\to 0} \Phi_\gamma(t) = (0, \xi_{i,j}) \in \mathbf{C}^2_{\sigma_{\nu_i}}$.

We refer to [12] for the further detail. Note that $d(Q_\gamma; f) \leq 0$. By (i) and Lemma (2.1), we have to check the stability at $A_{i,j}$ with $\nu_{i,j} \geq 2$.

3. Toric estimation.

Let $f(x,y)$ be as before. We will generalize Theorem (2.4) using the toric embedding theory. We assume for brevity that $\dim \Delta(f) = 2$ but every argument works even in the case $\dim \Delta(f) = 1$. Let $D_1, \dots, D_m$ be the faces of $\Delta(f)$ in the clockwise orientation so that D_1, D_m contain the origin. Let $P_i = {}^t(p_i, q_i)$ be the corresponding primitive integral covector of D_i. To get a better estimation, we first introduce the reduced polynomial $\widetilde{f}(x,y) := f(x,y) - f(0,0)$. Note that $\Delta(\widetilde{f}) \subset \Delta(f)$ but $O \notin \Delta(\widetilde{f})$. We factorize $\widetilde{f}_{P_i}(x,y)$ as follows.

$$\widetilde{f}_{P_i}(x,y) = \delta_i x^{r_i} y^{s_i} \prod_{j=1}^{\ell_i} (y^{p_i} - \xi_{i,j} x^{q_i})^{\nu_{i,j}} \tag{3.1}$$

Note that $f_{P_i}(x,y) = \widetilde{f}_{P_i}(x,y)$ for $i = 2, \dots, m-1$. We define the following integers

$$\nu(D_i) = \sum_{i=1}^{\ell_i} (\nu_{i,j} - 1), \quad \eta(D_i) = \sum_{i=1}^{\ell_i} \nu_{i,j} \tag{3.2.1}$$

$$\eta(D_1)' = \begin{cases} \eta(D_1), & p_1 < 0 \\ 0, & p_1 = 0 \end{cases}, \quad \eta(D_m)' = \begin{cases} \eta(D_m), & q_m < 0 \\ 0, & q_m = 0 \end{cases} \tag{3.2.2}$$

$$\varepsilon_x(f) = s_1 + p_1 \sum_{j=1}^{\ell_1} \nu_{1,j}, \qquad \varepsilon_y(f) = x_m + q_m \sum_{j=1}^{\ell_m} \nu_{m,j} \tag{3.2.3}$$

$$\varepsilon(f) = \begin{cases} 0, & \max(\varepsilon_x(f), \varepsilon_y(f)) \leq 1 \\ 1, & \max(\varepsilon_x(f), \varepsilon_y(f)) \geq 2 \end{cases} \tag{3.2.4}$$

Note that $\varepsilon_x(f)$ (respectively $\varepsilon_y(f)$) is the y-coordinate (resp. x-coordinate) of the left side edge of $\widetilde{\Delta}_1 = \Delta(P_1; \widetilde{f})$ (resp. $\widetilde{\Delta}_m = \Delta(P_m; \widetilde{f})$).

Let us define define *the toric degeneracy* $\nu_\infty^{tor}(f)$ by

$$\nu_\infty^{tor}(f) = \sum_{i=2}^{m-1} \nu(D_i) + \eta(D_1)' + \eta(D_m)' + \varepsilon(f) \tag{3.3}$$

The toric degeneracy $\nu_\infty^{tor}(f)$ is smaller than the projective degeneracy $\nu_\infty^{pr}(f)$ in general. Now we are ready to state the main theorem.

MAIN THEOREM (3.4). *The number of critical values from infinity of the function f is less than or equal to $\nu_\infty^{tor}(f)$.*

We say that $f(x,y)$ is *non-degenerate on the outside boundary* if $\nu(D_i) = 0$ for any $2 \leq i \leq m-1$. Recall that $\widetilde{f}$ is convenient iff $\widetilde{f}(x,0) \not\equiv 0$ and $\widetilde{f}(0,y) \not\equiv 0$.

COROLLARY (3.4.1) ([20]). *Assume that $\widetilde{f}(x,y)$ is a convenient polynomial. Then $\nu_\infty^{tor}(f) = \sum_{i=2}^{m-1} \nu(D_i)$. In particular, if $\widetilde{f}(x,y)$ has non-degenerate outside Newton boundaries, f has no critical value from the infinity.*

We give an outline of the Main theorem. For the detail, we refer to [12].

Let γ be an unstable irreducible component of $\overline{\Gamma_\ell(f)}$ at infinity and let $\Phi_\gamma(t)$ be a parametrization of γ in the coordinates (x,y) where $x_\gamma(t)$ and $y_\gamma(t)$ are Laurent series in the variable t. We assume first that

$$x_\gamma(t), y_\gamma(t) \not\equiv 0 \quad \text{and} \quad |x_\gamma(t)|^2 + |y_\gamma(t)|^2 \to \infty \ (t \to 0)$$

and we expand them in Laurent series as

$$\begin{cases} x_\gamma(t) = a_\gamma t^{p_\gamma} + \quad \text{(higher terms)} \\ y_\gamma(t) = b_\gamma t^{q_\gamma} + \quad \text{(higher terms)} \end{cases} \tag{3.5}$$

The case $x_\gamma(t)y_\gamma(t) \equiv 0$ will be treated later. Let $Q_\gamma := {}^t(p_\gamma, q_\gamma) \in N$ and $A_\gamma := (a_\gamma, b_\gamma)$. By the assumption we have that

$$A_\gamma \in \mathbf{C}^{*2}, \quad \min(p_\gamma, q_\gamma) < 0 \tag{3.6}$$

First we have the following Proposition:

PROPOSITION (3.7). *We have* $\mathrm{val}_t f(\ell(x_\gamma(t), y_\gamma(t))) \geq d(Q_\gamma; f)$ *and the inequality holds if and on if*

$$Q_\gamma = cP_i, \quad f_{P_i}(A_\gamma) = 0 \quad \textit{for some } c > 0 \quad \textit{and } 1 \leq i \leq m.$$

Recall that $\Gamma_\ell(f)$ is defined by $\Gamma_\ell(f) = \{(x,y) \in \mathbf{C}^2; J(x,y) = 0\}$ where

$$J(x,y) = \alpha\frac{\partial f}{\partial x}(x,y) + \beta\frac{\partial f}{\partial y}(x,y) = \alpha\frac{\partial \tilde{f}}{\partial x}(x,y) + \beta\frac{\partial \tilde{f}}{\partial y}(x,y) = 0$$

First we observe that the Newton boundary $\Delta(J)$ is slightly different from $\Delta(\tilde{f})$ but the following is enough for our purpose.

$$J_{Q_\gamma}(x,y) = \begin{cases} \alpha\dfrac{\partial \tilde{f}_{Q_\gamma}}{\partial x}(x,y) + \beta\dfrac{\partial \tilde{f}_{Q_\gamma}}{\partial y}(x,y) & p_\gamma = q_\gamma \\ \alpha\dfrac{\partial \tilde{f}_{Q_\gamma}}{\partial x}(x,y) & p_\gamma > q_\gamma,\ \tilde{f}_{Q_\gamma}(x,y) \neq \tilde{f}_{Q_\gamma}(0,y) \\ \beta\dfrac{\partial \tilde{f}_{Q_\gamma}}{\partial y}(x,y) & p_\gamma < q_\gamma,\ \tilde{f}_{Q_\gamma}(x,y) \neq \tilde{f}_{Q_\gamma}(x,0) \end{cases} \tag{3.8}$$

We divide the situation in two cases.

CASE I. $d(Q_\gamma; f) < 0$. CASE II. $d(Q_\gamma; f) = 0$.

We first consider the case:

CASE I. $d(Q_\gamma; f) < 0$. We assume that γ is an unstable irreducible component of $\overline{\Gamma_\ell(f)}$ at infinity. Then by Proposition (3.7), we must have $Q_\gamma = cP_i$ with $2 \leq i \leq m-1$. We call the face $D_2, \ldots, D_{m-1}$ the *outside faces* of $\Delta(f)$. We ask how many such components are possible for a fixed i. By an easy computation, we can see that the multiplicity of $y^{p_i} - \xi_{i,j}x^{q_i}$ in the factorization of $J_{P_i}(x,y)$ is exactly $\nu_{i,j} - 1$. Thus by the argument in the previous section, the local equation of $\overline{\Gamma_\ell(f)}$ in the toric coordinate chart $\mathbf{C}^2_{\sigma_{\nu_i}}$ is of the form

$$\delta_i \eta(v_{\nu_i}) \left\{ \prod_{j=1}^{\ell_i} (v_{\nu_i} - \xi_{i,j})^{\nu_{i,j}-1} + u_{\nu_i} g(u_{\nu_i}, v_{\nu_i}) \right\} = 0$$

where $\delta_i \neq 0$, $\eta(v_{\nu_i})$ is a polynomial with $\eta(\xi_{i,j}) \neq 0$ for any $j = 1, \ldots, \ell_i$. (Recall that $P_i = R_{\nu_i}$ and $\sigma_{\nu_i} = \mathrm{Cone}(R_{\nu_i}, R_{\nu_i+1})$.) Let $A_{i,j} = (0, \xi_{i,j}) \in \mathbf{C}^2_{\sigma_{\nu_i}}$. Thus by an easy argument, we have

PROPOSITION (3.9). *The number of local irreducible components γ at infinity of $\overline{\Gamma_\ell(f)}$ such that $\lim_{t\to 0}(x_\gamma(t), y(\gamma(t))) = A_{i,j}$ is at most $\nu_{i,j} - 1$ for any $2 \le i \le m-1$. Thus the number of the unstable irreducible components γ such that the limit $\lim_{t\to 0}(x_\gamma(t), y(\gamma(t)))$ intersect with the divisor $\widehat{E}(P_i)$ is bounded by $\nu(D_i)$.*

Now we consider the second case: $d(Q_\gamma; f) = 0$. Then it is clear that $d(Q_\gamma; \widetilde{f}) \ge 0$. We divide Case II into two subcases.

CASE II-1. $d(Q_\gamma; \widetilde{f}) = 0$. CASE II-2. $d(Q_\gamma; \widetilde{f}) > 0$.

Recall that D_1 and D_m are the face which contains the origin O. Let $\widetilde{D_1} = \Delta(P_1; \widetilde{f})$ and $\widetilde{D_m} = \Delta(P_m; \widetilde{f})$. We call D_1 and D_m (respectively $\widetilde{D_1}$ and $\widetilde{D_m}$) *the right and left conical faces* of $f(x,y)$ (respectively of $\widetilde{f}(x,y)$). Note that $\widetilde{D_i} \subset D_i$ and $\widetilde{D_i}$ might be a vertex for $i = 1, m$. $\widetilde{D_1}, \widetilde{D_m}$ are called bad faces in [14]. It is more convenient to consider the factorization of $\widetilde{f}_{P_1}(x,y)$:

$$\widetilde{f}_{P_1}(x,y) = \delta_1 (x^{q_1} y^{-p_1})^{e_1} \prod_{j=1}^{\ell_1} (1 - \xi_{1,j} x^{q_1} y^{-p_1})^{\nu_{1,j}}, \qquad e_1 > 0 \tag{3.10.1}$$

$$\widetilde{f}_{P_m}(x,y) = \delta_m (x^{-q_m} y^{p_m})^{e_m} \prod_{j=1}^{\ell_m} (x^{-q_m} y^{p_m} - \xi_{m,j})^{\nu_{m,j}}, \qquad e_m > 0 \tag{3.10.2}$$

Comparing with (3.1), we have

$$r_1 = q_1 e_1, \quad s_1 + p_1 \sum_{j=1}^{\ell_1} \nu_{1,j} = -p_1 e_1$$

$$r_m + q_m \sum_{j=1}^{\ell_m} \nu_{m,j} = -q_m e_m, \quad s_m = p_m e_m$$

Now we consider Case II-1 first. In this case, we must have either $Q_\gamma = cP_1$ or $Q_\gamma = cP_m$ for some $c > 0$. Let us consider the case $Q_\gamma = cP_1$ for instance. By the assumption $\min(p_i, q_i) < 0$ and $\dim(\Delta(f)) = 2$, we must have $p_1 < 0 < q_1$ if such a γ exists. Now we assert

LEMMA (3.11). *The number of the local irreducible components of $\overline{\Gamma_\ell(f)}$ of type Case II-1 with $Q_\gamma = cP_1$, $c > 0$ (respectively $Q_\gamma = cP_m$, $c > 0$) is less than or equal to $\eta(D_1)'$ (respectively $\eta(D_m)'$). They are all unstable.*

See [12] for the proof. Now we consider the last case:

CASE II-2. $d(Q_\gamma; f) = 0$ and $d(Q_\gamma; \tilde{f}) > 0$.

This is the case if and only if $\Delta(Q_\gamma; f) = \{O\}$ and $p_\gamma q_\gamma < 0$. So we assume for example

$$p_\gamma < 0 < q_\gamma \tag{3.12}$$

By the assumption $d(Q_\gamma; \tilde{f}) > 0$, we have that $\tilde{f}_{Q_\gamma}(x, 0) \equiv 0$. (If $\tilde{f}_{Q_\gamma}(x, 0) \not\equiv 0$, we get a contradiction: $d(Q_\gamma; \tilde{f}) < 0$.) and $\tilde{f}_{Q_\gamma}(x, y) \neq \tilde{f}_{Q_\gamma}(x, 0)$. Thus by (3.8) $J_{Q_\gamma}(x, y) = \dfrac{\partial f_{Q_\gamma}}{\partial y}(x, y)$. The assumption $\Delta(Q_\gamma; f) = \{O\}$ implies that $p_1 < 0 < q_1$ and

$$\det(Q_\gamma, P_1) > 0 \tag{3.13}$$

We consider the equality $J(x_\gamma(t), y_\gamma(t)) \equiv 0$. The leading term of this equality gives the following necessary condition is that

$$J_{Q_\gamma}(A_\gamma) = \frac{\partial f_{Q_\gamma}}{\partial y}(A_\gamma) = 0 \tag{3.14}$$

By (3.14), we must have

$$\dim \Delta(Q_\gamma; \tilde{f}) = 1 \tag{3.15}$$

Such a face $\Delta(Q_\gamma; \tilde{f})$ is called *an inside face with mixed weight vector* of $\tilde{f}(x, y)$. Geometrically the supporting line of such a face separates the Newton polygon $\Delta(\tilde{f})$ and the origin O. We consider the right conical face $\widetilde{D_1}$. By the expression (3.1) or (3.10), the left edge of $\widetilde{D_1}$ is $R := (r_1, s_1 + p_1 \sum_{j=1}^{\ell_1} \nu_{1,j}) = (q_1 e_1, -p_1 e_1)$. This gives a vertex $(q_1 e_1, -p_1 e_1 - 1)$ of the Newton polygon $\Delta(J)$ by the differential in y.

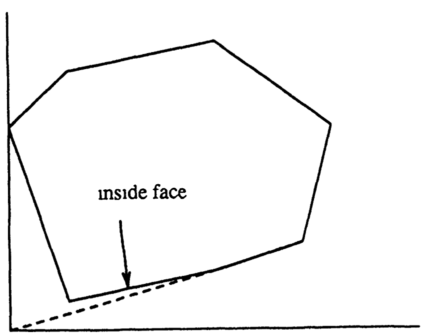

Figure (3.17.A)

If $-p_1 e_1 = 1$, it is easy to see that there exists no inside face of mixed weight Q_γ with $p_\gamma < 0 < q_\gamma$. Therefore we assume

$$-p_1 e_1 \geq 2 \tag{3.16}$$

In this case, it is not necessary to count the number of such local irreducible components. In fact, we have

PROPOSITION (3.17). *Each local irreducible components γ of $\overline{\Gamma_\ell(f)}$ of Case II-2 gives the limit critical value $-f(0,0)$.*

Proof. By the assumption, we have

$$f(x_\gamma(t), y_\gamma(t)) = f(0,0) + (\text{higher terms})$$

Thus the assertion is trivial. □

Until now, we have assumed that $x_\gamma(t), y_\gamma(t) \not\equiv 0$. Now we consider the exceptional case that $x_\gamma(t) \equiv 0$ or $y_\gamma(t) \equiv 0$. Assume for example

$$\gamma:\ x_\gamma(t) = 1/t, \qquad y_\gamma(t) \equiv 0$$

This implies that y divides $J(x,y)$. By the above argument, it is necessary that $-p_1 e_1 \geq 2$. In this case, we can see that $f(x,0) \equiv f(0,0)$. Thus if this is the case, $\mathrm{val}_t f(x_\gamma(t), y_\gamma(t)) = 0$ and γ is unstable and the corresponding limit critical value is again $-f(0,0)$. Now summerizing the above argument, we have

PROPOSITION (3.18). *Assume that $-p_1 e_1 \geq 2$ in* (3.10.1) *(respectively $-q_m e_m \geq 2$ in* (3.10.2)*). Then either there exists an unstable local irreducible component of $\overline{\Gamma_\ell(f)}$ of type Case II-2 with $p_\gamma < 0 < q_\gamma$ (resp. $q_\gamma < 0 < p_\gamma$), or $y = 0$ (resp. $x = 0$) is a (global) component of $\overline{\Gamma_\ell(f)}$. In any case, the possible limit critical value is $-f(0,0)$.*

Now we give several examples.

Example (3.19). **(A)** Let $\widetilde{f}(x,y) = y^{2n} + x^{3n}y^n(x+y)^n + x^4 y$. Then $\Delta(\widetilde{f})$ has four faces. In this example, $d = 5n$ and $\widetilde{f}_{5n} = x^{3n}y^n(x+y)^n$, and the projective degeneracy at infinity $\nu_\infty^{pr}(f) = 5n - 3$. On the other hand, $\eta(D_l)' = n-1$, $\nu(\Delta_2) = n-1$ and $\nu(D_3) = 0$, $\eta(D_4)' = 0$ and $\varepsilon(f) = 0$. Thus we have $\nu_\infty^{tor}(f) = 2n-2$.

(B) Let $\widetilde{f}(x,y) = x^4y^4 + xy^3 + x^3y^2 + xy$. In this example, we have $\nu(\Delta_2) = \nu(\Delta_3) = 0$, $\eta(D_1)' = \eta(D_4)' = 0$ and $\varepsilon(f) = 1$ and $\nu_\infty^{tor}(f) = 1$. In fact, 0 is the only critical point of $\widetilde{f}$ from the infinity.

(C) Let $\widetilde{f}(x,y) = x + c_2x^2 + \cdots + c_nx^n + x^m y$. Then $\Delta(\widetilde{f})$ has three faces and $\nu_\infty^{tor}(\widetilde{f}) = 1$. In fact, $\widetilde{f}$ has one critical value 0 from the infinity. This polynomial has no critical point ([19]).

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