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A GEOMETRICAL APPROACH TO THE JACOBIAN CONJECTURE FOR $n = 2$

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Let us recall the Jacobian conjecture (see [B-C-W] §I p. 288):

Jacobian Conjecture. *Let $F: \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map. Suppose that, for every point $x \in \mathbf{C}^n$, the derivative $F'(x)$ is invertible. Then the map F is invertible.*

In this lecture we describe a geometrical approach to this conjecture when $n = 2$. First we shall need some results on the geometry of complex polynomial functions.

Topology of polynomial functions.

It is known that a complex polynomial function might not be a locally trivial topological fibration over the complement in $\mathbf{C}$ of its critical values. As an example, consider the polynomial $f(X,Y) = X - X^2Y$ (see [B]): this complex polynomial function has no critical point, but it is not a locally trivial topological fibration on $\mathbf{C}$. In fact the fiber $f = 0$ has two connected components, but, for any $\lambda \neq 0$, $f = \lambda$ has only one component. However there is a general theorem of R. Thom:

THEOREM (THOM). *Let $f: \mathbf{C}^n \to \mathbf{C}$ be a complex polynomial function. There is a minimal finite set $A(f)$, such that f induces a locally trivial topological fibration over the complement of $A(f)$ in $\mathbf{C}$.*

One can check that the finite set $A(f)$ always contains the set of critical values $D(f)$. Let us write:

$$A(f) = D(f) \cup I(f)$$

where $D(f)$ and $I(f)$ might not be disjoint. By definition $I(f) \neq \emptyset$ depends on "accidents" at ∞. We shall call $A(f)$ the set of *atypical values* of f.

Polynomials which are locally trivial topological fibrations are well understood because of the following observation:

PROPOSITION. *If the complex polynomial function*

$$f: \mathbf{C}^2 \to \mathbf{C}$$

is a locally trivial fibration on $\mathbf{C}$, there is an algebraic automorphism σ of the complex plane, such that $f \circ \sigma = X$.

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This proposition is consequence of the Embedding Theorem of S. Abhyankar and T.T. Moh ([A-M]):

THEOREM (S. ABHYANKAR - T.T. MOH). *Let C be a complex algebraic affine curve embedded in the complex plane $\mathbf{C}^2$ and isomorphic to the complex line $\mathbf{C}$. Let $f = 0$ be its (reduced) equation. Then, there is an algebraic automorphism σ of the complex plane, such that $f \circ \sigma = X$, where X is one of the coordinates of $\mathbf{C}^2$.*

The methods developed in this note are the basic tool for a geometrical proof of the Abhyankar-Moh Theorem (see [A]).

The proposition above has the following consequence:

COROLLARY. *Let $F = (f, g)\colon \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map. Suppose that, for every point $x \in \mathbf{C}^2$, the derivative $F'(x)$ is invertible and furthermore, suppose that f is a locally trivial topological fibration on $\mathbf{C}$. Then F is invertible.*

Proof. If we assume that f is a locally trivial topological fibration on $\mathbf{C}$, the general fibers of f are isomorphic to the complex line. Therefore the Embedding Theorem of S. Abhyankar and T.T. Moh shows that there is an algebraic automorphism σ of the complex plane such that $f \circ \sigma = X$. This proves the proposition above. Now the Jacobian $J(F \circ \sigma)$ of $F \circ \sigma$ is the product of the Jacobians $J(F)$ and $J(\sigma)$ and therefore is equal to a non-zero constant k. But $F \circ \sigma = (X, g \circ \sigma)$, so that, by integration

$$g \circ \sigma = kY + h(X)$$

where h is a complex polynomial function. Hence $F \circ \sigma$ is an automorphism of the complex plane, which implies that F is also an automorphism of the complex plane, as announced.

As a consequence, the Jacobian conjecture will be proved, if one can show that:

CONJECTURE. *Suppose that $f\colon \mathbf{C}^2 \to \mathbf{C}$ is not a locally trivial topological fibration on $\mathbf{C}$. Then, for any complex polynomial function g, the Jacobian $J(f, g)$ cannot be a non-zero constant.*

We have seen that $f\colon \mathbf{C}^2 \to \mathbf{C}$ is not locally trivial topological fibration on $\mathbf{C}$, if and only if $A(f)$ is non-empty. If the set of critical values is non-empty, our conjecture is obviously true. Therefore we shall assume that the polynomial f has no critical point. We are led to understand the meaning of $I(f) \neq \emptyset$. In [H-L], we give a way to calculate $I(f)$. Now we describe how to do it.

Let $f(x, y)$ be the complex polynomial of degree d which defines f:

$$f(x, y) = \sum_{\alpha+\beta\leq d} c_{\alpha,\beta} x^\alpha y^\beta .$$

The homogeneization of $f(X, Y)$ is the homogeneous complex polynomial $F(X, Y, T)$ of degree d:

$$F(X, Y, T) = \sum_{\alpha+\beta\leq d} c_{\alpha,\beta} X^\alpha Y^\beta T^{d-\alpha-\beta}$$

The compactification of the fibers $f = \lambda$ in the complex projective plane $\mathbf{P}^2$ are the projective curves C_λ with projective equations $F - \lambda T^d = 0$. These curves pass through the same points on the line at infinity $T = 0$, namely through:

$$\{x_1, \dots, x_r\} = \{F = T^d = 0\}$$

which corresponds to the asymptotic directions of the level curves $f = \lambda$.

For each x_i, $1 \leqq i \leqq r$, there is an integer

$$\mu_i := \inf_{\{\lambda \in \mathbf{C}\}} \mu(C_\lambda, x_i)$$

where $\mu(C_\lambda, x_i)$ is the Milnor number of the curve C_λ at the point x_i. Now we can state:

THEOREM (HÀ-LÊ). *We have*

$$I(f) = \{\lambda \in \mathbf{C} \mid \textit{there is } 1 \leqq i \leqq r \textit{ such that, } \mu(C_\lambda, x_i) \neq \mu_i\}.$$

For example, consider $f(X, Y) = X - X^2 Y$. Then

$$F(X, Y, T) = XT^2 - X^2 Y$$

and the asymptotic directions of the level curves $f = \lambda$ are the points $x_1 := (1 : 0 : 0)$ and $x_2 := (0 : 1 : 0)$. At x_1, we have $\mu_1 = 0$ and the Milnor number of all the curves C_λ is 0, for all $\lambda \in \mathbf{C}$. At x_2, we have $\mu_2 = 2$ and the Milnor number of C_0 at x_2 is 3. In this case $A(f) = I(f) = \{0\}$.

Compactification of polynomial functions.

In general it is easier to deal with proper maps. In the case of complex polynomial functions in two variables, there is a natural way to compactify the function. We consider the rational function F/T^d on the complex projective plane $\mathbf{P}^2$. The set where $F = T^d = 0$ is the set of indeterminacy of the rational function F/T^d. By blowing-up points, one can remove the indeterminacy of the rational function F/T^d. We shall describe a minimal way to do it.

. First fix $\lambda_0 \notin A(f)$. Let $q: \mathcal{Y} \to \mathbf{P}^2$ be the embedded resolution of the projective curve C_{λ_0}. Let L be a linear form which does not vanish at the points $x_1, \dots, x_r$. For each component D of the divisor

$$D_\infty(\mathcal{Y}) := q^{-1}(T = 0)$$

we denote by $\nu_D(F/L^d)$ the multiplicity of $F/L^d \circ q$ along D. Now consider the points of indeterminacy of $F/T^d \circ q$ which might remain on $\mathcal{Y}$. Each of these points ξ belongs to a component D_ξ of the divisor $D_\infty(\mathcal{Y})$. Now at every point ξ of indeterminacy of $F/T^d \circ q$ on $\mathcal{Y}$, perform point blowing-ups so that to separate non-singular branches having an intersection number equal to $\nu_{D_\xi}(T^d/L^d) - \nu_{D_\xi}(F/L^d)$ at ξ. In this way we obtain a map $p: \mathcal{X} \to \mathcal{Y}$. Denote $D_\infty(\mathcal{X}) := (q \circ p)^{-1}(T = 0)$. We have:

THEOREM. *On $\mathcal{X}$, $F/T^d \circ q \circ p$ defines a rational map onto $\mathbf{P}^1$. The restriction of this map to $\mathcal{X} - D_\infty(\mathcal{X})$ induces a map into $\mathbf{C}$ isomorphic to f by $\pi := q \circ p$.*

The map π is minimal in the following sense:

OBSERVATION. *The only components of the divisor* $D_\infty(\mathcal{X}) := \pi^{-1}(T=0)$ *which may have self-intersection* -1 *are some of the components* D *such that the restriction of* $F/T^d \circ \pi$ *to* D *is not constant and, possibly, the strict transform of* $T=0$ *by* π *in* $\mathcal{X}$.

Note that Vitushkin (see [V] Introduction) has used the existence of a non-singular compactification of f in relation with the Jacobian conjecture when $n=2$.

For convenience, we shall call π the *minimal compactification* of f. In fact the modification π has interesting properties. First the intersection graph of the divisor $D_\infty(\mathcal{X})$ is a tree $\mathcal{A}$. Furthermore a careful study (using results of [L-M-W]) leads to the following key result:

CONNECTEDNESS THEOREM. *The space* $(F/T^d \circ \pi)^{-1}(\infty)$ *is connected.*

In other words the divisor $(F/T^d \circ \pi)^{-1}(\infty)$ defines a strict connected subtree $\mathcal{A}_\infty$ of $\mathcal{A}$. If the general fiber of $F/T^d \circ \pi$ is connected, this theorem is actually an immediate consequence of Zariski Main Theorem (cf [Mu] Theorem 3.24 and the footnote p. 52).

Let $\varphi := F/T^d \circ \pi$. We shall call the divisor $D_\infty(\mathcal{X})$, the divisor at infinity of $\mathcal{X}$, and a component of $D_\infty(\mathcal{X})$ on which φ is not constant is called *dicritical*.

The proof of the existence theorem for the minimal embedded resolution of a projective plane curve implies the following theorem:

THEOREM. *Each connected component of* $\mathcal{A} - \mathcal{A}_\infty$ *is a bamboo which contains a unique dicritical component of* φ *and this dicritical component is the only irreducible component of the bamboo which meets* $\mathcal{A}_\infty$.

According to Orevkov (see [O] Lemma 2.1), this last theorem was already observed by Vitushkin.

Remark. Let $\mathcal{B}$ be a bamboo of $\mathcal{A}-\mathcal{A}_\infty$ and let $D_\mathcal{B}$ be its dicritical component. If $\mathcal{B}$ has more than one component, the components of $\mathcal{B}$ other than $D_\mathcal{B}$ define a sub-bamboo $\mathcal{B}'$ of $\mathcal{B}$. The restricition to $\mathcal{B}'$ of the function φ is a finite constant. This value will be called the **atypical value of the bamboo** $\mathcal{B}$.

Results.

Consider a polynomial function f of degree d defined on $\mathbf{C}^2$. We assume that f has no critical point. We shall call *Jacobian pair* a pair (f,g) of polynomial functions defined on $\mathbf{C}^2$ for which the Jacobian is a non-zero constant. We shall say that f is a *Jacobian polynomial*, if there is a polynomial g such that (f,g) is a Jacobian pair.

The conjecture stated above can be reformulated as follows. It is equivalent to the Jacobian conjecture:

CONJECTURE. *If* f *is not a locally trivial topological fibration (on* $\mathbf{C}$*), it cannot be a Jacobian polynomial.*

The Theorem of Hà and Lê can be translated in the following way:

$$I(f) = \{\text{critical values } \neq \infty \text{ of the restriction of } \varphi \text{ to dicritical components } \} \cup$$
$$\{\text{atypical values of } \varphi \text{ on each bamboos of } \mathcal{A} - \mathcal{A}_\infty \text{ with at least two vertices}\}$$

This new formulation of the Theorem of Hà and Lê implies the following theorem:

THEOREM. *Let f be a complex polynomial function on $\mathbf{C}^2$ and let φ be the minimal compactification of f. Assume that f has no critical point. Then, f is not a locally trivial fibration on $\mathbf{C}$ if and only if, either there is a bamboo of $\mathcal{A} - \mathcal{A}_\infty$ with at least two components or all bamboos in $\mathcal{A} - \mathcal{A}_\infty$ only contain one component and the restriction of φ to at least one of them has critical points.*

Therefore in an attempt to prove our conjecture above, we shall consider two cases:

CASE 1. In the graph $\mathcal{A} - \mathcal{A}_\infty$ there is at least one connected component with two vertices;

CASE 2. In the graph $\mathcal{A} - \mathcal{A}_\infty$ all the connected components have only one vertex (which is a dicritical component of φ) and the restriction of φ on at least one dicritical component has degree strictly greater than one.

Examples. The function $X - X^2Y$ corresponds to the case 1. The function $X - X^4Y^4$ belongs to the case 2.

Now let g be another complex polynomial function of degree d'. Denote by G the homogenized polynomial associated to g. Then, in order to prove our conjecture, the cases 1 and 2 subdivide in subcases, depending on the behaviour of the rational function $G/T^{d'}$.

CASE 1. Let us call $D_0, D_1, \ldots, D_r$ the ordered components in a bamboo of $\mathcal{A} - \mathcal{A}_\infty$ with at least two vertices ($r \geqq 1$), D_0 being the dicritical component.

The case 1 subdivides in the following subcases:

a) The rational function $G/T^{d'}$ is defined in some neighbourhood $\mathcal{U}$ of $D_1 \cup \ldots \cup D_r$:
   i) The rational function $G/T^{d'}$ is constant on $D_1, \ldots, D_r$ and $D \cap \mathcal{U}$;
   ii) There is a D_i, with $1 \leqq i \leqq r-1$, which is dicritical for $G/T^{d'}$;
   iii) The component D_r is dicritical for $G/T^{d'}$;
   iv) The component D_0 is dicritical for $G/T^{d'}$;

b) There are points $x_1, \ldots, x_s$ in $D_1 \cup \ldots \cup D_r$ where $G/T^{d'}$ is not defined:
   i) $s \geqq 2$;
   ii) $s = 1$ and x_1 is not on $D_r - \cup_{j=1}^{r-1} D_j$;
   iii) $s = 1$ and x_1 is on $D_r - \cup_{j=1}^{r-1} D_j$;

CASE 2. We first have the following result:

PROPOSITION. *Assume that f has no critical point and that f is not a locally trivial fibration. If all the connected components of $\mathcal{A} - \mathcal{A}_\infty$ contain only one vertex, there is at least one dicritical component D of φ on which φ has a critical point.*

Proof. If f is not a locally trivial fibration, the translation of the Theorem of Hà and Lê given above shows that under the assumptions of the proposition, the restriction of φ to one of the dicritical components of φ is critical at some point of this dicritical component. The proposition asserts that there is at least one dicritical component of φ which carries a critical point of φ itself and not its restriction.

Assume that φ has no critical point. Then, Ehresmann Lemma implies that φ is a locally trivial fibration on $\mathbf{C}$. This would imply that the Euler characteristic of the general fiber of φ equals $1 + \ell$, where ℓ is the number of dicritical components of φ. Therefore necessarily $\ell = 1$. A result by T.T. Moh ([Mo]) interpreted in our setting implies that the restriction of φ to this unique component must be of degree 1, which would contradict the fact that this restriction has a critical point on this component.

Indeed, a direct proof of this theorem of T.T. Moh is possible using the connectedness theorem and a little topological argument.

Now consider a dicritical component D_0 of φ on which φ has a critical point x.

The case 2 subdivides in the following subcases:

a) The rational function $G/T^{d'}$ is defined in some neighbourhood $\mathcal{U}$ of D_0:
   i) The rational function $G/T^{d'}$ is constant on D_0;
   ii) The component D_0 is dicritical for $G/T^{d'}$;

b) There are points $x_1, \ldots, x_s$ in D_0 where $G/T^{d'}$ is not defined:
   i) none of these points are equal to x;
   ii) one of them, say x_1 is actually x.

We can prove that, if f has no critical point and is not a locally trivial fibration, for any polynomial function g on $\mathbf{C}^2$, the pair (f, g) is not a Jacobian pair in all the preceeding cases except the cases, case 1 a iv) and case 2 a ii). In the unsolved case 1 a iv), we can conclude if we know that one of the restrictions of φ or $\psi := G/T^{d'}$ on the dicritical component D_0 has degree one.

We now express our results differently. Let us say that a dicritical component of φ is *non-equisingular* if D_0 belongs to a bamboo in $\mathcal{A} - \mathcal{A}_\infty$ of length at least two or if the restriction of φ to D_0 has at least one critical point. Let us say that D_0 is *strongly non-equisingular* if D_0 belongs to a bamboo in $\mathcal{A} - \mathcal{A}_\infty$ of length at least two or if φ has a critical point on D_0.

The above proposition says that if $C(f) = \emptyset$ and $I(f) \neq \emptyset$, then φ has at least one strongly non-equisingular component.

The results we obtained so far in our attempt to prove the Jacobian Conjecture can be summarize as follows:

MAIN THEOREM. *Let f be a polynomial such that $C(f) = \emptyset$ and $I(f) \neq \emptyset$. Let g be a polynomial with $C(g) = \emptyset$. Then the pair (f, g) cannot be a Jacobian pair if at least one of the following conditions is fulfilled:*

1. *There exists a strongly non-equisingular component of φ which is not dicritical for ψ;*
2. *There exists a strongly non-equisingular dicritical component D_0 for φ for which the restriction $\varphi|D_0$ or $\psi|D_0$ has degree one.*

The following proposition is not difficult but very useful ·

PROPOSITION. *Let (f, g) be a Jacobian pair. Let $\varpi: \mathcal{Z} \to \mathbf{P}^2$ be the composition of a finite sequence of point blowing-ups centered above the line at infinity $T = 0$, such that $F/T^d \circ \varpi$ and $G/T^{d'} \circ \varpi$ define morphisms φ and ψ from $\mathcal{Z}$ to $\mathbf{P}^1$. Then, the support of the divisor of the 2-form $d\varphi \wedge d\psi$ is contained in $D_\infty(\mathcal{Z}) = \varpi^{-1}(T = 0)$.*

In other words, the divisor of $d\varphi \wedge d\psi$ is a canonical divisor for $\mathcal{Z}$ which is confined at infinity (i.e. on $D_\infty(\mathcal{Z})$). The multiplicities of such a canonical divisor are well defined and can be computed from the sequence of blowing-ups. They do not depend on φ and ψ.

For instance, this proposition implies that, in the case 1 above, if the dicritical component D_0 of φ in $\mathcal{X}$ has a negative multiplicity in the canonical divisor of $\mathcal{X}$ concentrated at infinity, its strict transform D_0' in $\mathcal{Z}$ has the same negative multiplicity in the canonical divisor of $\mathcal{Z}$ concentrated at infinity. This implies that ψ has a pole along D_0'. Therefore D_0 cannot be a common dicritical component of φ and ψ, which means that we are in one of the cases where our conjecture is true.

As an example, these observations applied to the case of $f(X, Y) = X - X^2Y$ show that it is not a Jacobian polynomial. The same argument works to show that many polynomials with no critical points cannot be Jacobian polynomials.

This leads to the following corollary of the Main Theorem:

COROLLARY. *Let $f: \mathbf{C}^2 \to \mathbf{C}$ be a polynomial function without critical point. Then f cannot be a Jacobian polynomial if there exists a dicritical component D_0 of φ in $\mathcal{X}$ which is strongly non-equisingular and such that the multiplicity of D_0 in the canonical divisor of $\mathcal{X}$ confined at infinity either* 1) *is strictly negative or* 2) *is positive and the restriction of $\varphi|D_0$ is of degree one.*

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