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TIMELIKE SURFACES WITH MEAN CURVATURE ONE IN ANTI-DE SITTER 3-SPACE

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1. Introduction

It is well-known that the classical Weierstrass-Enneper representation formula describes minimal surfaces in Euclidean 3-space R^3 in terms of their Gauss maps and auxiliary holomorphic functions [7]. This representation formula plays a very important role in constructing and studying minimal surfaces in R^3. Later, D. A. Hoffman and R. Osserman obtained the higher dimensional version of the classical Weierstrass-Enneper representation formula for minimal surfaces in Euclidean n-space R^n [3]. A natural question is how to generalize the above results to the surfaces in space forms of other constant curvature. In 1987, R. L. Bryant gave a representation formula for surfaces of mean curvature one in hyperbolic 3-space H^3 [1]. In his paper, he also pointed out that the surfaces of constant mean curvature in S^3 have no representation in terms of holomorphic data.

While considering the surfaces in Lorentz space forms, O. Kobayashi represented spacelike maximal surfaces in Lorentz-Minkowski 3-space R_1^3 in terms of holomorphic data [5]. Also C. H. Gu obtained the representation formula for the timelike and mixed type extremal surfaces in R_1^3 [2].

Motivated by these results, in this paper we obtain a reprsentation formula for timelike surfaces with mean curvature one in 3-dimensional anti-de Sitter H_1^3. By this formula, we get some timelike surfaces with mean curvature one in H_1^3.

This paper is organized as follows. In section 2, we introduce the standard model of H_1^3, and set up another model of H_1^3 which is quite useful for computation. In section 3, we will prove the main theorems (Theorem 3.1 and Theorem 3.3) which describe the timelike surfaces of mean curvature one in H_1^3 in terms of two simple mappings. At last, in section 4, after writing the representation formula into a suitable form, we will give some examples.

2. Models for H_1^3

On the 4-dimensional real vecter space E^4, we consider the symmetric form

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$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4,$$

$$x=(x_1, \cdots, x_4)\in E^4, \qquad y=(y_1, \cdots, y_4)\in E^4.$$

The pair $(E^4, \langle\cdot, \cdot\rangle)$ will be denoted by R_2^4. Define H_1^3 as follows:

$$H_1^3 = \{x\in R_2^4 ;\ \langle x, x\rangle = -1\}. \tag{2.1}$$

When we consider H_1^3 with the induced pseudo-metric from R_2^4, it is easily shown that H_1^3 is a complete 3-dimensional, pseudo-Riemannian manifold of constant sectional curvature -1 and has signature $(+, +, -)$. One may refer to [8] for more detail to understand the completeness and other properties of anti-de Sitter 3-space.

Besides the above standard model for H_1^3, there is another way of describing H_1^3 which will be quite useful in our calculations. We identify R_2^4 with the space of 2×2 real matrices by identifying (x_1, x_2, x_3, x_4) with the matrix

$$\begin{pmatrix} x_1+x_4 & x_2-x_3 \\ x_2+x_3 & -x_1+x_4 \end{pmatrix}. \tag{2.2}$$

The real Lie group, $SL(2, R)\times SL(2, R)$, two copies of 2×2 real matrices with determinant 1, acts naturally on R_2^4 by the representation

$$(g_1, g_2)\cdot v = g_1 v g_2^t \tag{2.3}$$

where we regard v as a 2×2 real matrix by (2.2). Under this identification, we clearly have $\langle v, v\rangle = -\det v$. Thus $SL(2, R)\times SL(2, R)$ preserves $\langle\cdot, \cdot\rangle$ and H_1^3 can be recognized as the space $SL(2, R)$

$$H_1^3 = \{g\in \mathrm{gl}(2, R) : \det g = 1\}. \tag{2.4}$$

Let $\mathcal{F}$ be the oriented orthonormal frame bundle of R_2^4 which consists of the bases (e_1, e_2, e_3, e_4) of R_2^4 satisfying conditions:

$$e_1\wedge e_2\wedge e_3\wedge e_4 > 0,$$

$$\langle e_\alpha, e_\beta\rangle = \varepsilon_\alpha \delta_{\alpha\beta}$$

where $\varepsilon_1=\varepsilon_2=1$ and $\varepsilon_3=\varepsilon_4=-1$. We can use $SL(2, R)\times SL(2, R)$ to parametrize $\mathcal{F}$ as follows.

Assume that

$$\underline{e}_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underline{e}_2=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{e}_3=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \underline{e}_4=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

and let $e_\alpha(g_1, g_2) = (g_1, g_2)\cdot \underline{e}_\alpha = g_1 \underline{e}_\alpha g_2^t$. Then the map $(g_1, g_2)\mapsto (e_1(g_1, g_2), \cdots, e_4(g_1, g_2))$ is a 2-1 covering map of $SL(2, R)\times SL(2, R)$ onto $\mathcal{F}$.

By submersion $e_4\colon \mathcal{F}\to H_1^3$, we may regard $\mathcal{F}$ as the oriented orthonormal frame bundle of H_1^3 such that $e_1, e_2, e_3 \in T_{e_4}H_1^3$ is an orthonormal frame of $T_{e_4}H_1^3$. Denoting its dual frame fields by $\{\omega^1, \omega^2, \omega^3\}$. Then there exist unique 1-forms on H_1^3, $\{\omega_j^i \mid i, j=1, 2, 3\}$ so that

$$(2.6)\qquad \begin{aligned} de_4 &= \sum \omega^i e_i, \\ de_i &= \sum \omega_i^j e_j + \omega^i e_4, \\ \omega_i^j \varepsilon_j &+ \omega_j^i \varepsilon_i = 0. \end{aligned}$$

Denoting the metric on H_1^3 by ds^2, we have

$$(2.7)\qquad ds^2 = \langle de_4, de_4 \rangle = (\omega^1)^2 + (\omega^2)^2 - (\omega^3)^2.$$

For an orthonormal frame on H_1^3 given by $\{e_i(g_1, g_2) \mid i=1, 2, 3\}$. The canonical forms $\{\omega^i, \omega_j^i \mid i, j=1, 2, 3\}$ can be expressed by Maurer-Cartan forms of g_1 and g_2.

LEMMA 2.1. *Let* $\{\omega^i, \omega_j^i \mid i, j=1, 2, 3\}$ *be the canonical forms associated with the frame* $\{e_i = g_1 \underline{e}_i g_2^t \mid i=1, 2, 3\}$. *Then we have*

$$(2.8a)\qquad g_1^{-1} dg_1 = \frac{1}{2}\begin{pmatrix} \omega^1 - \omega_2^3 & \omega_2^1 + \omega_3^1 + \omega^2 - \omega^3 \\ -\omega_2^1 + \omega_3^1 + \omega^2 + \omega^3 & -\omega^1 + \omega_2^3 \end{pmatrix},$$

$$(2.8b)\qquad g_2^{-1} dg_2 = \frac{1}{2}\begin{pmatrix} \omega^1 + \omega_2^3 & \omega_2^1 - \omega_3^1 + \omega^2 + \omega^3 \\ -\omega_2^1 - \omega_3^1 + \omega^2 - \omega^3 & -\omega^1 - \omega_2^3 \end{pmatrix}.$$

Proof. From $e_\alpha = g_1 \underline{e}_\alpha g_2^t$, we get

$$de_\alpha = g_1 [g_1^{-1} dg_1 \underline{e}_\alpha + \underline{e}_\alpha (g_2^{-1} dg_2)^t] g_2^t.$$

On the other hand, by (2.6) we have

$$de_\alpha = \sum \omega_\alpha^\beta e_\beta = \sum g_1 (\omega_\alpha^\beta \underline{e}_\beta) g_2^t.$$

From these two equations and noting that $\omega_4^i = \omega^i = \omega_i^4 \varepsilon_i$, we can easily verify the lemma by direct computation. ∎

3. Timelike surface theory in H_1^3 and the case $H=1$

Throughout this section, M will denote an oriented connected smooth 2-dimensional manifold, and $f: M \to H_1^3$ will be a timelike smooth immersion.

We let $\mathcal{F}_f^{(1)} \subset M \times \mathcal{F}$ denote the first order frame bundle of f. Thus $(m; e_1, e_2, e_3, e_4) \in \mathcal{F}_f^{(1)}$ if $e_4 = f(m)$ and $e_2 \wedge e_3 = f_*(T_m M)$ as oriented 2-plane. We restrict all forms and maps to $\mathcal{F}_f^{(1)}$. It follows that $e_1 \in T_{f(m)} H_1^3$ is the oriented united normal to $f_*(T_m M)$ and hence, we may regard e_1 as well-defined as a map $e_1: M \to R_2^4$.

We have $\langle e_1, df \rangle = \omega^1 = 0$, so the induced metric by f on M is $ds_f^2 = (\omega^2)^2 - (\omega^3)^2$, and the structure equations for immersion f are given as follows:

$$
\begin{aligned}
&d\omega^2=-\omega_3^2\wedge\omega^3,\\
(3.1)\qquad &d\omega^3=-\omega_2^3\wedge\omega^2,\\
&d\omega_3^2=\omega^2\wedge\omega^3+\omega_2^1\wedge\omega_3^1.
\end{aligned}
$$

Since $d\omega^1=\omega^2\wedge\omega_2^1+\omega^3\wedge\omega_3^1=0$, it follows that there exist smooth functions $h_{ij}=h_{ji}(i,\ j=2,\ 3)$ so that

$$
(3.2)\qquad
\begin{aligned}
&\omega_2^1=h_{22}\omega^2-h_{23}\omega^3,\\
&\omega_3^1=-h_{32}\omega^2+h_{33}\omega^3.
\end{aligned}
$$

One easily checks that $II=h_{22}(\omega^2)^2-2h_{23}\omega^2\omega^3+h_{33}(\omega^3)^2$ is a well-defined smooth quadratic form on M, which is called the second fundamental form. Its trace with respect to ds_f^2 denoted by $H=(h_{22}-h_{33})/2$ is defined as mean curvature of immersion f. It's easily checked that the function H is a well-defined smooth function on M.

After the above preparation, we now set up to establish our main theorems.

THEOREM 3.1. *Let $U\subseteqq R^{1,1}$ be a domain in 2-dimentional Lorentz-Minkowski space $R^{1,1}$ and $\{\eta,\ \xi\}$ be the global oriented null coordinates on $R^{1,1}$. Let $g_1,\ g_2$: $U\to SL(2,\ R)$ be two maps satisfying the following three conditions*:

(1) $\dfrac{\partial g_1}{\partial\xi}=\dfrac{\partial g_2}{\partial\eta}=0$,

(2) $\det(g_1^{-1}dg_1)=\det(g_2^{-1}dg_2)=0$,

(3) $\det(g_1^{-1}dg_1+(g_2^{-1}dg_2)^t)\neq 0$.

Then the map $f=g_1g_2^t$: $U\subseteqq R^{1,1}\to H_1^3$ is a conformal (timelike) immersion with the mean curvature one.

Proof. Let $(e_i=g_1\underline{e}_ig_2^t)$ be the orthonormal frame associated to $g_1,\ g_2$. Under this frame, we have canonical 1-forms $\{\omega^i,\ \omega_j^i\,|\,i,\ j=1,\ 2,\ 3\}$.

Denote by

$$
\begin{aligned}
&\pi_1=\omega_3^1-\omega_2^1+\omega^2+\omega^3,\\
&\pi_2=\omega_3^1+\omega_2^1-\omega^2+\omega^3,\\
&\omega^+=\omega^2+\omega^3,\\
&\omega^-=\omega^2-\omega^3.
\end{aligned}
$$

By Lemma 2.1, we have

$$
g_1^{-1}dg_1=\frac{1}{2}\begin{pmatrix}\omega^1-\omega_2^3 & \pi_2+2\omega^- \\ \pi_1 & -\omega^1+\omega_2^3\end{pmatrix},
$$

and

$$
g_2^{-1}dg_2=\frac{1}{2}\begin{pmatrix}\omega^1+\omega_2^3 & -\pi_1+2\omega^+ \\ -\pi_2 & -\omega^1-\omega_2^3\end{pmatrix}.
$$

By condition (1), we assume that

$$\omega_1^4-\omega_2^3=2\alpha_1=2A_1(\eta)d\eta, \qquad \omega_1^4+\omega_2^3=2\alpha_2=2A_2(\xi)d\xi,$$
$$\pi_1=2\gamma_1=2C_1(\eta)d\eta, \qquad -\pi_2=2\gamma_2=2C_2(\xi)d\xi,$$
$$\pi_2+2\omega^-=2\beta_1=2B_1(\eta)d\eta, \qquad -\pi_1+2\omega^+=2\beta_2=2B_2(\xi)d\xi.$$

Condition (2) means

$$\alpha_1^2+\beta_1\gamma_1=\alpha_2^2+\beta_2\gamma_2=0. \tag{3.3}$$

So the induced metric ds_f^2 on U is

$$ds_f^2=\langle df, df\rangle=\langle d(g_1g_2^t), d(g_1g_2^t)\rangle=-\det(g_1^{-1}dg_1+(g_2^{-1}dg_2)^t)\neq 0.$$

More precisely,

$$ds_f^2=2\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=(2A_1A_2+B_1B_2+C_1C_2)d\eta d\xi. \tag{3.4}$$

It then follows that $f=g_1g_2^t\colon U\to H_1^3$ is a conformal (timelike) immersion.

We will now show that for this immersion $H=1$ by computing H in a first order adapted frame. Without lose its generity, we assume that $2A_1A_2+B_1B_2+C_1C_2>0$, then we can write down that $ds_f^2=\lambda^2d\eta d\xi$ on U, for some smooth function $\lambda>0$ on U. From (3.3) and (3.4), we have

$$A_i^2+B_iC_i=0, \tag{3.5}$$

for $i=1, 2$ and

$$2A_1A_2+B_1B_2+C_1C_2=\lambda^2. \tag{3.6}$$

First we will prove an assertion.

ASSERTION. *For any $p\in U$, if there exist a neighborhood V of p in U and some smooth functions p_i, $q_i (i=1, 2)$ on V so that*

$$\begin{aligned} A_i&=\lambda p_iq_i\\ B_i&=\mathrm{sign}(B_i)\lambda p_i^2\\ C_i&=\mathrm{sign}(C_i)\lambda q_i^2\end{aligned} \tag{3.7}$$

for $i=1, 2$, then $H(p)=1$.

Note that $B_iC_i=-A_i^2\leqq 0$, and hence $\mathrm{sign}(B_i)\,\mathrm{sign}(C_i)=-1$. For simplicity, we assume that $\mathrm{sign}(B_i)=1$, $\mathrm{sign}(C_i)=-1$ and $p_1p_2+q_1q_2=1$ from (3.6) and (3.7).

Let $h\colon V\to SL(2, R)$ be defined by

$$h=\begin{pmatrix} p_1 & q_2\\ -q_1 & p_2\end{pmatrix},$$

then $e_4(g_1h, g_2(h^t)^{-1})=e_4(g_1, g_2)=g_1g_2^t$. Moreover we compute that

$$(g_1h)^{-1}d(g_1h)=h^{-1}(g_1^{-1}dg_1)h+h^{-1}dh$$
$$=\begin{bmatrix} q_2dq_1+p_2dp_1 & -q_2dp_2+p_2dq_2+\lambda d\eta \\ -p_1dq_1+q_1dp_1 & p_1dp_2+q_1dq_2 \end{bmatrix},$$

and

$$(g_2(h^{-1})^t)^{-1}d(g_2(h^{-1})^t)=h^t(g_2^{-1}dg_2)(h^t)^{-1}-(h^{-1}dh)^t$$
$$=-\begin{bmatrix} q_2dq_1+p_2dp_1 & -p_1dq_1+q_1dp_1-\lambda d\xi \\ -q_2dp_2+p_2dq_2 & p_1dp_2+q_1dq_2 \end{bmatrix}.$$

Also denote the 1-forms $\{\omega^i,\ \omega^i_j\,|\,i,\ j=1,\ 2,\ 3\}$ be the canonical 1-forms associated to the frame $\{e_\alpha(g_1h,\ g_2(h^{-1})^t)\,|\,\alpha=1,\ \cdots,\ 4\}$. By Lemma 2.1, it follows that

$$\omega^1=0,\quad \omega^-=\omega^2-\omega^3=\lambda d\eta,\quad \omega^+=\omega^2+\omega^3=\lambda d\xi$$

and

$$\pi_1=-\omega^1_2+\omega^1_3+\omega^2+\omega^3=2(-p_1dq_1+q_1dp_1).$$

Thus $\{e_\alpha(g_1h,\ g_2(h^{-1})^t)\,|\,\alpha=1,\ \cdots,\ 4\}$ is an oriented adapted frame field on V for immersion $f=g_1g_2^t$. The 1-form $-p_1dq_1+q_1dp_1$ must have the form $\Phi d\eta$ for some function Φ on V, since we have the representations

$$-p_1dq_1+q_1dp_1=\begin{cases} -p_1^2d\dfrac{q_1}{p_1}=-p_1^2d\Big(\dfrac{A_1(\eta)}{B_1(\eta)}\Big) & \text{where } p_1\neq 0, \\ q_1^2d\dfrac{p_1}{q_1}=-q_1^2d\Big(\dfrac{A_1(\eta)}{C_1(\eta)}\Big) & \text{where } q_1\neq 0. \end{cases}$$

By the following Lemma 3.2, we conclude that *assertion* holds.

Now we continue to prove our theorem. Let

$$U_i=\{p\in U\,|\,B_i(p)\neq 0\}$$

and

$$V_i=\{p\in U\,|\,C_i(p)\neq 0\}$$

for $i=1,\ 2$.

For $p\in U_1\cap U_2\cap V_1\cap V_2$, we can choose a neighborhood V of p such that $B_i|_V\neq 0$ and $C_i|_V\neq 0$, for $i=1,\ 2$. Let $p_i=\mathrm{sign}(A_i)\sqrt{|B_i|/\lambda}$ and $q_i=\sqrt{|C_i|/\lambda}$ by *assertion* we conclude that $H=1$ holds on $U_1\cap U_2\cap V_1\cap V_2$.

Next by the continuity of the mean curvature function H, we have that $H=1$ holds on $\bar{U}_1\cap U_2\cap V_1\cap V_2\subseteq\overline{U_1\cap U_2\cap V_1\cap V_2}$. On the other hand, for $p\in(U\backslash\bar{U}_1)\cap U_2\cap V_1\cap V_2$, we can choose a neighborhood V of p such that $B_1|_V=0$, $B_2|_V\neq 0$ and $C_i|_V\neq 0$ for $i=1,\ 2$. Let $p_1=0$, $p_2=\mathrm{sign}(A_2)\sqrt{|B_2|/\lambda}$ and $q_i=\sqrt{|C_i|/\lambda}$, by *assertion* we conclude that $H=1$ holds on $(U\backslash\bar{U}_1)\cap U_2\cap V_1\cap V_2$. So we see that $H=1$ holds on $U_2\cap V_1\cap V_2$.

Repeat the above discussion, we conclude that $H=1$ holds on $V_1\cap V_2$, on V_2 and finally on U. ■

LEMMA 3.2. *Let $f:\ U\subseteq R^{1,1}\to H^3_1$ be an conformal (timelike) immession and*

$\{\eta, \xi\}$ be the global oriented null coordinates on $R^{1,1}$. Let $\{\omega^i, \omega^i_j \mid i, j=1, 2, 3\}$ be the canonical 1-forms associated with an oriented adapted frame field $\{e_i \mid i=1, 2, 3\}$. Then the immersion f has mean curvature one if and only if $-\omega^1_2+\omega^1_3+\omega^2+\omega^3$ has the form $\Phi d\eta$ for some function Φ on U.

Proof. Let $ds_f=\lambda^2 d\eta d\xi$ denote the induced metric on U, so we have $\omega^2-\omega^3=\lambda d\eta$ and $\omega^2+\omega^3=\lambda d\xi$. We compute that

$$\begin{aligned}-\omega^1_2+\omega^1_3&=-(h_{22}\omega^2-h_{23}\omega^3)+(-h_{23}\omega^2+h_{33}\omega^3)\\&=-2H\omega^3-(h_{23}+h_{33})(\omega^2-\omega^3),\end{aligned}$$

and

$$\begin{aligned}-\omega^1_2+\omega^1_3+\omega^2+\omega^3&=-2(H-1)\omega^3-(-1+h_{33}+h_{23})(\omega^2-\omega^3)\\&=-2(H-1)\omega^3-\lambda(-1+h_{33}+h_{23})d\eta\,.\end{aligned}$$

So it then follows that lemma holds. ■

To complete the representation for the timelike surfaces with mean curvature one in H^3_1, we shall prove the following theorem.

THEOREM 3.3. *Let $U\subseteq R^{1,1}$ be a simply connected domain and $f: U\to H^3_1$ be a conformal (timelike) immersion with mean curvature one. Then there exist two maps $F_1, F_2: U\to SL(2, R)$ satisfying condition (1), (2) and (3) such that*

$$f=F_1F_2^t\,.$$

Proof. Let $ds_f^2=\lambda^2 d\eta d\xi$ be the induced metric on U, and e_1, e_2, e_3 be the adapted frame fields on U such that e_1 is the unit normal vecter field of f in H^3_1. Then $\{e_1, e_2, e_3, e_4=f\}$ is a frame field of R^4_2. By the fact that U is simply connected, we have the lifting maps $g_1, g_2: U\to SL(2, R)$ such that $e_i(g_1, g_2)=e_i$ for $i=1, 2, 3$ and $f=g_1g_2^t$. Again let $\{\omega^i, \omega^i_j \mid i, j=1, 2, 3\}$ be the canonical 1-forms associated to the frame field $\{e_i \mid i=1, 2, 3\}$. By Lemma 2.1 and $\omega^1=0$, we have

$$g_1^{-1}dg_1=\frac{1}{2}\begin{pmatrix}-\omega^3_2 & \omega^1_2+\omega^1_3+\omega^2-\omega^3\\ -\omega^1_2+\omega^1_3+\omega^2+\omega^3 & \omega^3_2\end{pmatrix},$$

$$g_2^{-1}dg_2=\frac{1}{2}\begin{pmatrix}\omega^3_2 & \omega^1_2-\omega^1_3+\omega^2+\omega^3\\ -\omega^1_2-\omega^1_3+\omega^2-\omega^3 & -\omega^3_2\end{pmatrix}.$$

Consider the $\mathfrak{sl}(2, R)$-valued 1-form μ on U:

$$\mu=\frac{1}{2}\begin{pmatrix}-\omega^3_2 & \omega^1_2+\omega^1_3-\omega^2+\omega^3\\ -\omega^1_2+\omega^1_3+\omega^2+\omega^3 & \omega^3_2\end{pmatrix}.$$

It is easy to see that μ satisfies $d\mu=-\mu\wedge\mu$ (since f has mean curvature one). It follows by the Frobenius theorem that there exists a smooth map $h: U\to SL(2, R)$ so that $\mu=h^{-1}dh$.

Let us write

$$h=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for smooth functions a, b, c and d on U. Then if we set $F_1=g_1h^{-1}$ and $F_2=g_2h^t$, by the fact that $\omega^2-\omega^3=\lambda d\eta$ and $\omega^2+\omega^3=\lambda d\xi$, we easily compute

$$F_1^{-1}dF_1=\begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix}\lambda d\eta,$$

$$F_2^{-1}dF_2=\begin{pmatrix} bd & d^2 \\ -b^2 & -bd \end{pmatrix}\lambda d\xi.$$

Since dF_1 (resp. dF_2) has the form $\Phi d\eta$ (resp. $\Psi d\xi$) for $\mathfrak{gl}(2, R)$-valued function Φ (resp. Ψ), we must have that F_1 and F_2 satisfy the condition (1). Clearly, F_1 and F_2 also satisfy condition (2), (3) and

$$F_1F_2^t=g_1g_2^t=f.$$

This completes our proof. ■

4. Representation formula and examples

Let $U\subseteqq R^{1,1}$ be a simply connected domain and $\{\eta, \xi\}$ be the global null coordinates on $R^{1,1}$. For given smooth functions $\alpha_i(\eta)$, $\beta_i(\xi)$ $(i=1, 2, 3)$ on U, satisfying

(i) $$\alpha_1\alpha_2+\alpha_3^2=0$$

(ii) $$\beta_1\beta_2+\beta_3^2=0,$$

by Frobenius theorem, there exist two maps

$$A(\eta): U \longrightarrow SL(2, R),$$

$$B(\xi): U \longrightarrow SL(2, R),$$

such that

$$A(\eta)^{-1}dA(\eta)=\begin{pmatrix} \alpha_3 & \alpha_1 \\ \alpha_2 & -\alpha_3 \end{pmatrix}d\eta \tag{4.1}$$

$$B(\xi)^{-1}dB(\xi)=\begin{pmatrix} \beta_3 & \beta_1 \\ \beta_2 & -\beta_3 \end{pmatrix}d\xi. \tag{4.2}$$

Then we obtain that a mapping given by

$$f(\xi, \eta)=A(\eta)B(\xi)^t: U \longrightarrow H_1^3$$

is a branched conformal (timelike) immersion with mean curvature one.

The mapping f is an immersion if the functions $\alpha_i(\eta)$ and $\beta_i(\xi)$ $(i=1, 2, 3)$

satisfy

(iii) $$\alpha_1\beta_1+\alpha_2\beta_2+2\alpha_3\beta_3\neq 0.$$

At last we shall give here some examples.

Examples. 1) Let $\alpha_1=\beta_1=-1$, $\alpha_2=\alpha_3=\beta_2=\beta_3=1$. By solving (4.1) and (4.2), it follows that

$$A(\eta)=\begin{pmatrix} \eta+1 & -\eta \\ \eta & -\eta+1 \end{pmatrix},$$

$$B(\xi)=\begin{pmatrix} \xi+1 & -\xi \\ \xi & -\xi+1 \end{pmatrix}.$$

Hence an entire timelike immersion $f: R^{1,1}\to H_1^3$ with mean curvature one is given by

$$f(\xi, \eta)=(\xi+\eta, 2\xi\eta, \eta-\xi, 2\xi\eta+1).$$

2) Let $\alpha_1=\beta_1=-1$, $\alpha_3=\eta$, $\beta_3=\xi$ and $\alpha_2=\eta^2$, $\beta_2=\xi^2$. We have by (4.1) and (4.2):

$$A(\eta)=\begin{pmatrix} \sin\eta-\eta\cos\eta & \cos\eta \\ -\cos\eta-\eta\sin\eta & \sin\eta \end{pmatrix},$$

$$B(\xi)=\begin{pmatrix} \sin\xi-\xi\cos\xi & \cos\xi \\ -\cos\xi-\xi\sin\xi & \sin\xi \end{pmatrix}.$$

Then we have a branched immersion $f: R^{1,1}\to H_1^3$ with mean curvature one:

$$f(u, v)=\frac{1}{2}\Big(-u\sin u+\frac{u^2-v^2}{4}\cos u,\ u\cos u+\frac{u^2-v^2}{4}\sin u,$$
$$2\sin v-v\cos v+\frac{u^2-v^2}{4}\sin v,\ 2\cos v+v\sin v+\frac{u^2-v^2}{4}\cos v\Big)$$

where $u=\eta+\xi$, $v=\eta-\xi$. And f is an immersion on domain

$$U=\{(u, v)\in R^{1,1}: u^2-v^2+4\neq 0\}.$$

3) Let $\alpha_1=\beta_1=-1$, $\beta_2=\beta_3=1$ and $\alpha_2=\eta^2$, $\alpha_3=\eta$. By (4.1) and (4.2), we get that

$$A(\eta)=\begin{pmatrix} \sin\eta-\eta\cos\eta & \cos\eta \\ -\cos\eta-\eta\sin\eta & \sin\eta \end{pmatrix},$$

$$B(\xi)=\begin{pmatrix} \xi+1 & -\xi \\ \xi & -\xi+1 \end{pmatrix}.$$

Then we have a branched immersion $f: R^{1,1}\to H_1^3$ with mean curvature one given by $f(\eta, \xi)=(1/2)(x_1, \cdots, x_4)$, where

$$x_1=2\xi\sin\eta-\eta\cos\eta+\xi\eta(\sin\eta-\cos\eta),$$

$$x_2=-2\xi\cos\eta-\eta\sin\eta-\xi\eta(\sin\eta+\cos\eta),$$

$$x_3=-2\cos\eta-(2\xi+\eta)\sin\eta+\xi\eta(\cos\eta-\sin\eta),$$

$$x_4=2\sin\eta-(2\xi+\eta)\cos\eta-\xi\eta(\sin\eta+\cos\eta).$$

And f is an immersion on domain $U=\{(\xi,\eta)\in R^{1,1};\ \eta\neq-1\}$.

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