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ON THE VALUE DISTRIBUTION OF $f^l(f^{(k)})^n$

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Abstract

Quantitative estimations on the value distribution of $f^l(f^{(k)})^n$ are studied in this paper. As a result of this, some known results are improved.

1. Introduction

Let f denote a transcendental meromorphic function, and the usual symbols: $T(r, f)$, $N(r, f)$, $\bar{N}(r, f)$, $m(r, f)$, $S(r, f)$ of Nevanlinna value distribution theory see, e.g. [7], are used throughout the paper.

A complex value a is said to be a Picard value of f, if and only if, $f(z)-a$ has at most finitely many zeros. W. K. Hayman [8] conjectured that the only possible Picard value of $f^n f'$ is zero, and he himself proved the case when $n\geqq 3$ in [10], and left the cases of $n=1, 2$. Later on, Mues [11] proved the case for $n=2$, and afterwards Clunie [4] proved the case for $n=1$ when f is entire. An affirmative answer to the case when f is meromorphic and $n=1$ is yet to be resolved. Since then a stream of studies on questions of possible Picard values of differential polynomials of f has been launched, and many related results have been obtained, see e.g. [1]-[5] and [10]-[19]. In 1981, Steinmetz [12] proved:

THEOREM A. *Let f be a transcendental meromorphic function in the plane. If $n_0, \cdots, n_k\geqq 0$, $n_0\geqq 2$, $n_1+\cdots+n_k\geqq 1$ and $\psi=f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}-1$, then*

$$\limsup_{r\to\infty}\frac{\bar{N}(r, 1/\psi)}{T(r, \psi)}>0.$$

In 1982, Doeringer [5] proved the following:

THEOREM B. *Let f be a transcendental meromorphic function, $Q(f)$ and $P(f)$ be two non-zero differential polynomials and $\psi=f^nQ(f)+P(f)$. Then for any natural number n with $n\geqq 3+\gamma_p(\gamma_p$: the weight of $P(f))$,*

$$\limsup_{r\to\infty}\frac{\bar{N}(r, 1/\psi)}{T(r, \psi)}>0.$$

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Remark. Particularly, when $n \geqq 3$, $\gamma_p = 0$ (i.e. $P(f)$ is a non-zero small function of f), we can derive from this that

$$\limsup_{r \to \infty} \frac{\bar{N}(r, 1/\psi)}{T(r, f)} \geqq \frac{1}{2}.$$

Thus Theorem B, in some sense, is an improvement and extension of Theorem A. Futhermore, it gives a quantitative estimation of the number of zeros of ψ. Recently, in [17], the following result has been obtained:

THEOREM C. *Let f be a transcendental entire function and n, k be two non-negative integers with $n \geqq 2$, $k \geqq 0$. Then $f(f^{(k)})^n$ assumes every non-zero finite value infinitely many times.*

Hence it is natural to ask:

PROBLEM D. Does Theorem C also hold when f is a transcendental meromorphic function?

In this paper, as an attempt in resolving Problem D, we obtain some quantitative estimations on the zeros of $f^l(f^{(k)})^n - 1$ with $l=1$, 2 and k, $n \geqq 2$ by argument different from that of Theorems B and C, and give an affirmative answer to problem D when $k \geqq 0$ and $n > 9e+1$.

2. Main Result

THEOREM. *Let f be a transcendental meromorphic function in the plane and $F = f^l(f^{(k)})^n - 1$ with l, k and n being three positive integers and $l \leqq 2$.*

(i) If $l=1$ and $n > 9e+1$, then there exists some constant $K > 1$, a set $M(K)$ of upper logarithmic density at most $\delta(K) = \min((2e^{K-1}-1)^{-1}, (1+e(K-1)) \exp(e(1-K)))$, and a set D of finite linear measure such that $n - 9eK - 1 \geqq \varepsilon > 0$ and

$$(\varepsilon - o(1)) T(r, f^{(k)}) \leqq 2\bar{N}\left(r, \frac{1}{F}\right), \qquad r \in E(K)$$

where $E(K) = [0, \infty) \backslash M(K) \cup D$ (note that for $K > 1$, $m(E(K)) = \infty$),

Particularly F assumes zero infinitely often.

(ii) If $l=2$ and k, $n \geqq 2$, then

$$\left(\frac{1}{2} - \eta\right) T(r, f) \leqq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f),$$

for every $0 < \eta < (1/2)$.

Remark. The case of $l \geqq 3$ has been taken care of by Theorem B.

In order to prove the above theorem, we shall make use of the following lemmas.

LEMMA 1 (Frank [6]). *If k is a positive integer and $\varepsilon>0$, then*

$$k\bar{N}(r, f)\leqq N\left(r, \frac{1}{f^{(k)}}\right)+(1+\varepsilon)N(r, f)+S(r, f).$$

LEMMA 2 (Hayman-Miles [9]). *Suppose that $f(z)$ is a transcendental meromorphic function and K, a constant, >1. Then there exists a set $M(K)$ with upper logarithmic density at most*

$$\delta(K)=\min((2e^{K-1}-1)^{-1}, (1+e(K-1)\exp(e(1-K)))),$$

such that for every positive integer k,

$$\limsup_{r\to\infty}\frac{T(r, f)}{T(r, f^{(k)})}\leqq 3eK, \quad r\notin M(K).$$

In particular when r is large and $r\notin M(K)$, then

$$-3eKT(r, f^{(k)})\leqq-(1-o(1))T(r, f). \tag{2.1}$$

LEMMA 3. *If $F=f(f^{(k)})^n-1$; n, $k\geqq 1$ and $g=(F'/F)$, then*

$$m(r, H(z))=S(r, f)+S(r, f^{(k)}),$$

where $H(z)=nf^{(k+1)}+(f'/f)f^{(k)}-gf^{(k)}$.

Proof. From $g=(F'/F)$, we have

$$f(f^{(k)})^{n-1}\left(nf^{(k+1)}+\frac{f'}{f}f^{(k)}-gf^{(k)}\right)=-g. \tag{2.2}$$

Let E_1 be the set of θ in $[0, 2\pi]$ for which $|f(re^{i\theta})|<1$, E_2 be the set of θ in $[0, 2\pi]$ for which $|f(re^{i\theta})|\geqq 1$ and $|f^{(k)}(re^{i\theta})|\geqq 1$, and E_3 be the set $[0, 2\pi]\backslash E_1\cup E_2$. It is easy to see that for $\theta\in E_1$ and $z=re^{i\theta}$,

$$\begin{aligned}\log^+|H(z)|\leqq\log^+|g(z)|+\log^+\left|\frac{f'(z)}{f(z)}\right|+2\log^+\left|\frac{f^{(k)}(z)}{f(z)}\right|\\ +n\log^+\left|\frac{f^{(k+1)}}{f(z)}\right|+O(1),\end{aligned} \tag{2.3}$$

and

$$\log^+|H(z)|\leqq n\log^+\left|\frac{f^{(k+1)}(z)}{f^{(k)}(z)}\right|+\log^+\left|\frac{f'(z)}{f(z)}\right|+\log^+|g(z)|+O(1), \tag{2.4}$$

for $\theta\in E_3$ and $z=re^{i\theta}$. Now, by (2.2), we have

$$\log^+|H(z)|\leqq\log^+|g(z)|+\log^+|f(z)f^{(k)}(z)|^{-1}+O(1),$$

so for $\theta\in E_2$ and $z=re^{i\theta}$,

$$\log^+|H(z)|\leqq\log^+|g(z)|+O(1). \tag{2.5}$$

Hence

$$\int_0^{2\pi}\log^+|H(re^{i\theta})|\,d\theta=\int_{E_1}\log^+|H(re^{i\theta})|\,d\theta+\int_{E_2}\log^+|H(re^{i\theta})|\,d\theta$$
$$+\int_{E_3}\log^+|H(re^{i\theta})|\,d\theta.$$

From this, (2.3), (2.4) and (2.5), and by the well-known lemma on the logarithmic derivative [7], we have,

$$m(r,\ H(z))\leqq O(m(r,\ g))+O\Big(m\Big(r,\ \frac{f^{(k)}}{f}\Big)\Big)+O\Big(m\Big(r,\ \frac{f^{(k+1)}}{f}\Big)\Big)$$
$$+O\Big(m\Big(r,\ \frac{f^{(k+1)}}{f^{(k)}}\Big)\Big)+O(1)$$
$$\leqq S(r,\ f)+S(r,\ f^{(k)}),$$

and this also completes the proof of the lemma.

LEMMA 4. *With F, g and H as defined in Lemma* 3, *we have*

$$(n-1)T(r,\ f^{(k)})\leqq 3T(r,\ f)+2\overline{N}\Big(r,\ \frac{1}{F}\Big)+S(r,\ f)+S(r,\ f^{(k)}).$$

Proof. From the definition of $H(z)$, we see immediately that the possible poles of $H(z)$ occur only at the poles of f and the zeros of F and f. Now note that g can have only simple poles, and hence by (2.2), it is easily verified that any pole of f, say z_0, cannot be a pole of $H(z)$. Consequently

$$N(r,\ H)\leqq\overline{N}\Big(r,\ \frac{1}{F}\Big)+N_{1)}\Big(r,\ \frac{1}{f}\Big).$$

Now combining this fact with lemma 3, (2.2), and by Nevanlinna's first fundemental theorem [7], we have

$$(n-1)T(r,\ f^{(k)})\leqq T(r,\ f)+T(r,\ g)+T(r,\ H).$$

Consequently

$$(n-1)T(r,\ f^{(k)})\leqq T(r,\ f)+\overline{N}(r,\ f)+2\overline{N}\Big(r,\ \frac{1}{F}\Big)+N_{1)}\Big(r,\ \frac{1}{f}\Big)$$
$$+S(r,\ f)+S(r,\ f^{(k)}),$$

where $N_{1)}(r,\ 1/f)$ denotes the counting function corresponding the simple zeros of f. It follows that

$$(n-1)T(r, f^{(k)}) \leqq 3T(r, f)+2\bar{N}\left(r, \frac{1}{F}\right)+S(r, f)+S(r, f^{(k)}).$$

3. The Proof of the Theorem

Proof of (*i*). Choose $K>1$ such that $n-1-9eK>0$. We note that $S(r, f) = o(1)T(r, f)$ outside a finite linear measure A, and $S(r, f^{(k)})=o(1)T(r, f^{(k)})$ outside a finite linear measure B, also by Lemma 2

$$T(r, f)=O(1)T(r, f^{(k)}), \qquad r \notin M(K).$$

Therefore $S(r, f)+S(r, f^{(k)})=o(1)T(r, f^{(k)})$ on $E(K)=[0, \infty)\backslash M(K)\cup A\cup B$. Now from lemma 4,

$$(n-1)T(r, f^{(k)})-3T(r, f) \leqq 2\bar{N}\left(r, \frac{1}{F}\right)+o(1)T(r, f^{(k)}), \quad r\in E(K).$$

Thus, it follows from this and (2.1) of Lemma 2 that

$$(n-1-9eK-o(1))T(r, f^{(k)}) \leqq 2\bar{N}\left(r, \frac{1}{F}\right), \qquad r\in E(K)$$

and, therefore, there exists a positive constant $\varepsilon \leqq n-1-9eK$ such that

$$(\varepsilon-o(1))T(r, f^{(k)}) \leqq 2\bar{N}\left(r, \frac{1}{F}\right), \qquad r\in E(K). \tag{3.1}$$

Now if F assumes zero finite times, then

$$\frac{\bar{N}(r, (1/F))}{T(r f^{(k)})} \longrightarrow 0, \qquad \text{as} \quad r\to\infty,\ r\in E(K)$$

and it follows that $\varepsilon<0$ from (3.1). This is a contradiction and hence, F must have infinitely many zeros.

Proof of (ii). Let $F=f^2(f^{(k)})^n-1$ with $k, n \geqq 2$. Then we have

$$\frac{1}{f^{n+2}}=\left(\frac{f^{(k)}}{f}\right)^n-\frac{F'F}{f^{n+2}F'}.$$

It follows that

$$m\left(r, \frac{1}{f^{n+2}}\right) \leqq m\left(r, \frac{F}{F'}\right)+m\left(r, \frac{F'}{f^{n+2}}\right)+S(r, f).$$

Note that F' is a homogenous differential polynomial in f of degree $n+2$. Hence $m(r, (F'/f^{n+2}))=S(r, f)$ and

$$m\left(r, \frac{1}{f^{n+2}}\right) \leqq m\left(r, \frac{F}{F'}\right)+S(r, f).$$

Thus, again by the first fundemental theorem, the above can be rewritten as

$$m\left(r, \frac{1}{f^{n+2}}\right) \leqq m\left(r, \frac{F'}{F}\right) + N\left(r, \frac{F'}{F}\right) - N\left(r, \frac{F}{F'}\right) + S(r, f).$$

Note $m(r, F'/F) = S(r, F) = S(r, f)$. The above yields

$$m\left(r, \frac{1}{f^{n+2}}\right) \leqq N\left(r, \frac{F'}{F}\right) - N\left(r, \frac{F}{F'}\right) + S(r, f).$$

Consequently

$$m\left(r, \frac{1}{f^{n+2}}\right) \leqq N\left(r, \frac{1}{F}\right) + \bar{N}(r, f) - N\left(r, \frac{1}{F'}\right) + S(r, f). \tag{3.2}$$

On the other hand, from $F' = f(f^{(k)})^{n-1}(nff^{(k+1)} + 2f'f^{(k)})$, we have

$$N\left(r, \frac{1}{f}\right) + (n-1)N\left(r, \frac{1}{f^{(k)}}\right) \leqq N\left(r, \frac{1}{F'}\right). \tag{3.3}$$

Substituting (3.3) into (3.2) and then adding $N(r, (1/f^{n+2}))$ to both sides of (3.2), we get

$$T\left(r, \frac{1}{f^{n+2}}\right) \leqq N\left(r, \frac{1}{F}\right) + \bar{N}(r, f) + N\left(r, \frac{1}{f^{n+2}}\right) - N\left(r, \frac{1}{f}\right)$$
$$-(n-1)N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Hence

$$(n+2)T(r, f) \leqq N\left(r, \frac{1}{F}\right) + \bar{N}(r, f) + (n+1)N\left(r, \frac{1}{f}\right)$$
$$-(n-1)N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Now by combining this and lemma 1, we have for any given $0 < \varepsilon < 1$,

$$(n+2)T(r, f) \leqq N\left(r, \frac{1}{F}\right) + \frac{1}{2}N\left(r, \frac{1}{f^{(k)}}\right) + \frac{1+\varepsilon}{2}N(r, f) + (n+1)N\left(r, \frac{1}{f}\right)$$
$$-(n-1)N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Thus,

$$(n+2)T(r, f) \leqq N\left(r, \frac{1}{F}\right) + \left(n+1+\frac{1+\varepsilon}{2}\right)T(r, f) + S(r, f),$$

which leads to

$$\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)T(r, f) \leqq N\left(r, \frac{1}{F}\right) + S(r, f).$$

This also completes the proof of the theorem.

Remark. It is easily seen the results can be extended to the case when the value 1 for F is replaced by any non-zero small function of f.

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