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ON AN ULTRAHYPERELLIPTIC SURFACE WITH PICARD CONSTANT THREE

Dedicated to Professor Nobuyuki Suita on his 60th birthday

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§1. Introduction.

Let R be an open Riemann surface. Let $M(R)$ be the family of non-constant meromorphic functions on R. Let $P(f)$ be the number of values which are not taken by $f\in M(R)$. The Picard constant $P(R)$ of R is defined by

$$P(R)=\sup\{P(f)\,;\ f\in M(R)\}$$

Then we have $P(R)\geqq 2$. The significance of the Picard constant is in the following fact: *If* $P(R)<P(S)$, *then there is no non-constant analytic mapping of* R *into* S (Ozawa [5]).

Let R be the ultrahyperelliptic surface defined by

$$y^2=G(z), \tag{1.1}$$

where G is an entire function having an infinite number of simple zeros and no other zeros. For the class of this surfaces we have $P(R)\leqq 4$ from the value distribution theory of two-valued algebroid functions.

We now consider a characterization of ultrahyperelliptic surfaces in terms of the Picard constant. We first have

THEOREM A (Ozawa [6]). $P(R)=4$, *if and only if there is a non-constant entire function* H ($H(0)=0$), *an entire function* F *and constants* γ *and* δ *such that* G *in* (1.1) *satisfies*

$$F(z)^2G(z)=(e^{H(z)}-\gamma)(e^{H(z)}-\delta),\qquad \gamma\delta(\gamma-\delta)\neq 0\,. \tag{1.2}$$

When $P(R)=3$ we have

THEOREM B (Hiromi-Ozawa [1]). *If* $P(R)=3$, *then there are two non-constant entire functions* H *and* L ($H(0)=L(0)=0$), *an entire function* F *and non-zero constants* β_1 *and* β_2 *such that* G *in* (1.1) *satisfies*

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$$(1.3)\qquad F(z)^2G(z)$$
$$=1-2\beta_1 e^{H(z)}-2\beta_2 e^{L(z)}+\beta_1^2 e^{2H(z)}-2\beta_1\beta_2 e^{H(z)+L(z)}+\beta_2^2 e^{2L(z)}.$$

Conversely, if G satisfies (1.3), *then the ultrahyperelliptic surface R defined by* (1.1) *is of $P(R)\geqq 3$.*

It is quite difficult to determine all ultrahyperelliptic surfaces with $P(R)=3$ and we have not perfectly succeed in it yet. However we have following:

THEOREM C (Ozawa [7]). *Let R be the ultrahyperelliptic surface defined by* (1.1) *with G satisfying* (1.3). *If H and L are polynomials, then $P(R)=3$ with following four exceptional cases*: (i) $H=L$; (ii) $H=2L$, $16\beta_1=\beta_2^2$; (iii) $2H=L$, $\beta_1^2=16\beta_2$; (iv) $H=-L$, $16\beta_1\beta_2=1$. *In these exceptional cases we have $P(R)=4$.*

From this theorem we conjecture that Theorem C is also true when H and L in (1.3) are transcendental.

In this paper we shall prove the following:

THEOREM. *Let R be the ultrahyperelliptic surfaces defined by* (1.1) *with G satisfying* (1.3). *If H and L in* (1.3) *are transcendental entire functions such that $H(0)=L(0)=0$ and*

$$(1.4)\qquad L(z)=\lambda H(z)+K(z),$$

where λ is a rational number and K is an entire function satisfying

$$(1.5)\qquad m(r, e^K)=o(m(r, e^H)),\qquad r\to\infty,$$

outside a set of finite measure, then $P(R)=3$ with following four exceptional cases: (i) $H=L$; (ii) $H=2L$, $16\beta_1=\beta_2^2$; (iii) $2H=L$, $\beta_1^2=16\beta_2$; (iv) $H=-L$, $16\beta_1\beta_2=1$. *In these exceptional cases we have $P(R)=4$.*

Remark. Hiromi-Ozawa [1] proved this theorem when $\lambda=0$ in (1.4).

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and the usual notations such as $T(r, f)$, $N(r, a, f)$, $N_1(r, a, f)$, $m(r, f)$ etc. (see e. g. [2]).

§ 2. Lemmas.

Here we state Lemmas needed in the proofs of our Theorem and Proposition.

LEMMA A ([1]). *Let $a_1, \cdots, a_n$ be meromorphic functions and H a non-constant entire function. Suppose that*

$$T(r, a_\mu)=o(m(r, e^H))\qquad (r\to\infty)\qquad \mu=1, \cdots, n$$

hold outside a set of finite measure. Then the equation

$$\sum_{\mu=1}^{n} a_{\mu}(z)e^{\mu H(z)}=0$$

cannot hold unless $a_1\equiv\cdots\equiv a_n\equiv 0$.

LEMMA B ([4]). *Let* $a_{\mu,\nu}$ $(\mu, \nu=0, 1, \cdots, m)$ *be meromorphic functions and* H *and* M *two non-constant entire functions such that*

$$m(r, e^H)\sim m(r, e^M) \quad \text{and} \quad T(r, a_{\mu,\nu})=o(m(r, e^H)) \qquad (r\to\infty)$$

hold for $\mu, \nu=0, 1, \cdots, m$, *outside a set of* r *of finite measure. Further suppose that*

$$\sum_{\mu,\nu=0}^{m} a_{\mu,\nu}(z)e^{\mu H(z)+\nu M(z)}=0.$$

(I) *If* $a_{m,m}(z)\not\equiv 0$, *then* $a_{m,0}(z)\equiv a_{0,m}(z)\equiv 0$ *and* $m(r, e^{H+M})=o(m(r, e^H))$ $(r\to\infty)$ *outside a set of* r *of finite measure.*

(II) *If* $a_{\nu,\nu}(z)\equiv 0$ $(\nu=0, 1, \cdots, m-1)$ *and* $a_{m,0}\not\equiv 0$ *or* $a_{0,m}\not\equiv 0$, *then* $a_{m,m}(z)\equiv 0$ *and* $m(r, e^{H-M})=o(m(r, e^H))$ $(r\to\infty)$ *outside a set of* r *of finite measure.*

Let $N_2(r, 0, f)$ be the counting function of simple zeros of the indicated function f. We can deduce from Nevanlinna's second fundamental theorem that

LEMMA C (cf. [6]). *Let* H *be a non-constant entire function and* a $(\not\equiv 0)$ *a meromorphic function satisfying*

$$T(r, a)=o(m(r, e^H)) \qquad (r\to\infty)$$

outside a set of r *of finite measure. Then we have*

$$N_2(r, 0, e^H-a)\sim m(r, e^H) \quad \text{and} \quad N_1(r, 0, e^H-a)=o(m(r, e^H)) \quad (r\to\infty)$$

outside a set of finite measure.

From Lemma 4 and Lemma 5 in [3] and Lemma C we can deduce

LEMMA D. *Let* H *and* a_j $(j=1, \cdots, \mu)$ *be entire functions satisfying*

$$m(r, a_j)=o(m(r, e^H)) \qquad (r\to\infty) \qquad j=1, \cdots, \mu$$

outside a set of finite measure. If the discriminant of the equation

$$Q_{\mu}(x):=x^{\mu}+a_1(z)x^{\mu-1}+\cdots+a_{\mu}(z)=0$$

is not identically zero, then we have

$$N_2(r, 0, Q_{\mu}(e^H))\sim \mu m(r, e^H) \quad \text{and} \quad N_1(r, 0, Q_{\mu}(e^H))=o(m(r, e^H)) \quad (r\to\infty) \tag{2.1}$$

outside a set of finite measure.

We can easily deduce

LEMMA E. *The discriminant D of the algebraic equation $P(x):=x^{p}+ax+b=0$, where a and b are non-zero constants, is*

$$D=(-1)^{p(p-1)/2+(p-1)}\{(p-1)^{p-1}a^{p}+(-1)^{p-1}p^{p}b^{p-1}\}.$$

Further, if D is zero, then the discriminant of the equation $Q(x)=0$ is not zero, where

$$P(x)=\{x+pb/(p-1)a\}^{2}Q(x).$$

LEMMA F. *The discriminant D of the algebraic equation $P(x):=x^{q}+ax^{q-1}+b=0$, where a and b are non-zero constants, is*

$$D=(-1)^{q(q-1)/2+(q-1)}b^{q-2}\{(q-1)^{q-1}a^{q}+(-1)^{q-1}q^{q}b\}.$$

Further, if D is zero, then the discriminant of the equation $Q(x)=0$ is not zero, where

$$P(x)=\{x+(q-1)a/q\}^{2}Q(x).$$

LEMMA G. *Let p and q be coprime integers satisfying $p>q>1$. The algebraic equation*

$$x^{p}+ax^{q}+b=0 \qquad (ab\neq 0) \tag{2.2}$$

has a multiple root α, if and only if

$$D_0:=(p-q)^{p-q}q^{q}a^{p}+(-1)^{p-1}p^{p}b^{p-q}=0.$$

Then α is only one double root and satisfies

$$\alpha^{p-q}=-aq/p \quad \text{and} \quad \alpha^{q}=-bp/a(p-q).$$

We here note that D_0 is equal to the discriminant of the algebraic equation (2.2) modulo a non-zero.

§ 3. Proposition.

PROPOSITION. *Let H and M be non-constant entire functions with $H(0)=M(0)=0$ and a_μ $(\mu=0, 1, \cdots, 2p)$ entire functions satisfying $a_0\not\equiv 0$, $a_{2p}\not\equiv 0$ and*

$$m(r, a_\mu)=o(m(r, e^{H})) \qquad (r\to\infty) \qquad j=0, 1, \cdots, 2p, \tag{3.1}$$

outside a set of finite measure, where p is a positive integer. Further we assume that

$$g(z):=\sum_{\mu=0}^{2p} a_\mu(z)e^{\mu H(z)} \tag{3.2}$$

satisfies

$$N_2(r, 0, g) \sim 2pm(r, e^H) \quad \textit{and} \quad N_1(r, 0, g) = o(m(r, e^H)) \qquad (r \to \infty) \tag{3.3}$$

outside a set of finite measure. If the identity

$$f(z)^2(e^{pM(z)} - \gamma)\,(e^{pM(z)} - \delta) = g(z), \tag{3.4}$$

holds with a suitable meromorphic function f and two suitable constants γ and δ satisfying $\gamma\delta(\gamma-\delta) \neq 0$, then we have

$$\begin{aligned} &a_p = -a_{2p}(\gamma+\delta)e^{p(H+M)}/\gamma\delta, \qquad a_0 = a_{2p}e^{2p(H+M)}/\gamma\delta, \\ &a_{2p-1} \equiv \cdots \equiv a_{p+1} \equiv a_{p-1} \equiv \cdots \equiv a_1 \equiv 0, \qquad f^2 = a_{2p}e^{2pH}/\gamma\delta, \end{aligned} \tag{3.5}$$

or

$$\begin{aligned} &a_p = -a_{2p}(\gamma+\delta)e^{p(H-M)}, \qquad a_0 = a_{2p}\gamma\delta e^{2p(H-M)}, \\ &a_{2p-1} \equiv \cdots \equiv a_{p+1} \equiv a_{p-1} \equiv \cdots \equiv a_1 \equiv 0, \qquad f^2 = a_{2p}e^{2p(H-M)}. \end{aligned} \tag{3.6}$$

Proof. It follows from Lemma C that

$$N_2(r, 0, (e^{pM}-\gamma)(e^{pM}-\delta)) \sim 2pm(r, e^M),$$

$$N_1(r, 0, (e^{pM}-\gamma)(e^{pM}-\delta)) = o(m(r, e^M)), \qquad (r \to \infty)$$

outside a set of finite measure. Hence, considering simple zeros and multiple zeros of the both sides of (3.4), from (3.1), (3.2), (3.3), (3.4) and the reasoning of [4, p. 298] we can deduce that

$$\begin{aligned} &m(r, e^M) \sim m(r, e^H), \\ &T(r, f) = O(m(r, e^H)), \qquad T(r, f'/f) = o(m(r, e^H)) \end{aligned} \qquad (r \to \infty) \tag{3.7}$$

outside a set of finite measure. Differentiating both sides of (3.4) and using (3.2) and (3.4) we obtain

$$\sum_{\mu,\nu=0}^{2p} b_{\mu,\nu}(z)e^{\mu H(z)+\nu M(z)} = 0, \tag{3.8}$$

where $b_{2p,2p} = (2f'/f + 2pM')a_{2p} - (a_{2p}' + 2pa_{2p}H')$, $b_{2p,0} = \gamma\delta(2a_{2p}f'/f - a_{2p}' - 2pa_{2p}H')$, $b_{0,2p} = (2f'/f + 2pM')a_0 - a_0'$ and the others $b_{\mu,\nu}$ are meromorphic functions. It is clear from (3.7) and our assumption (3.1) that

$$T(r, b_{\mu,\nu}) = o(m(r, e^H)) \qquad (r \to \infty) \qquad \text{for } \mu, \nu = 0, 1, \cdots, 2p$$

outside a set of finite measure. We apply Lemma B to the identity (3.8).

Suppose that $b_{2p,2p} \not\equiv 0$. Then (I) of Lemma B yields $b_{2p,0} \equiv b_{0,2p} \equiv 0$ and

$$m(r, e^{H+M}) = o(m(r, e^H)) \qquad (r \to \infty) \tag{3.9}$$

outside a set of finite measure. Hence we have $f(z)^2 = ca_{2p}(z)e^{2pH(z)} =$

$da_0(z)e^{-2pM(z)}$ with non-zero constants c and d. Substituting these relations into (3.4) and using (3.2) we have

$$(1-c\gamma\delta)a_{2p}e^{2pH}+a_{2p-1}e^{(2p-1)H}+\cdots+a_{p+1}e^{(p+1)H}$$

$$+(a_p+c(\gamma+\delta)a_{2p}e^{p(H+M)})e^{pH}+a_{p-1}e^{(p-1)H}+\cdots+a_1e^H+a_0-ca_{2p}e^{2p(H+M)}=0.$$

Hence it follows from Lemma A and (3.9) that

$$(1-c\gamma\delta)a_{2p}\equiv 0,\quad a_{2p-1}\equiv\cdots\equiv a_{p+1}\equiv 0,\quad a_p+c(\gamma+\delta)a_{2p}e^{p(H+M)}\equiv 0,$$

$$a_{p-1}\equiv\cdots\equiv a_1\equiv 0,\quad a_0-ca_{2p}e^{2p(H+M)}\equiv 0.$$

Thus we have (3.5) because of $a_{2p}\not\equiv 0$.

Next suppose that $b_{2p,2p}\equiv 0$. Then we have $f(z)^2=ca_{2p}(z)e^{2p(H(z)-M(z))}$ with a non-zero constant c and so (3.4) reduces to

$$\sum_{\mu,\nu=0}^{2p} c_{\mu,\nu}(z)e^{\mu H(z)+\nu M(z)}=0, \tag{3.10}$$

where $c_{2p,2p}=(1-c)a_{2p}$, $c_{2p-1,2p-1}=\cdots=c_{1,1}=c_{0,0}=0$, $c_{2p,0}=-c\gamma\delta a_{2p}$, $c_{0,2p}=b_0$ and the others $c_{\mu,\nu}$ are entire functions satisfying

$$T(r,c_{\mu,\nu})=o(m(r,e^H))\qquad (r\to\infty)$$

outside a set of finite measure. Since $c_{2p,0}\not\equiv 0$ and $c_{0,2p}\not\equiv 0$, (II) of Lemma B implies $(1-c)a_{2p}\equiv 0$ and

$$m(r,e^{H-M})=o(m(r,e^H))\qquad (r\to\infty) \tag{3.11}$$

outside a set of finite measure. Hence, since $a_{2p}\not\equiv 0$ so $c=1$, the identity (3.10) further reduces to

$$a_{2p-1}e^{(2p-1)(H-M)}e^{(2p-1)M}+\cdots+a_{p+1}e^{(p+1)(H-M)}e^{(p+1)M}$$

$$+(a_pe^{p(H-M)}+a_{2p}(\gamma+\delta)e^{2p(H-M)})e^{pM}$$

$$+a_{p-1}e^{(p-1)(H-M)}e^{(p-1)M}+\cdots+a_1e^{H-M}e^M+a_0-a_{2p}\gamma\delta e^{2p(H-M)}=0.$$

Therefore it follows from (3.11) and Lemma A that

$$a_{2p-1}e^{(2p-1)(H-M)}\equiv\cdots\equiv a_{p+1}e^{(p+1)(H-M)}\equiv 0,$$

$$a_pe^{p(H-M)}+a_{2p}(\gamma+\delta)e^{2p(H-M)}\equiv 0,$$

$$a_{p-1}e^{(p-1)(H-M)}\equiv\cdots\equiv a_1e^{H-M}\equiv 0,\quad a_0-a_{2p}\gamma\delta e^{2p(H-M)}\equiv 0.$$

Hence we have (3.6).

Thus the proof of our Proposition is complete.

§ 4. Proof of Theorem.

It is sufficient to consider only the cases only the cases $\lambda=0$ and $|\lambda|\geqq 1$ in (1.4).

Eight main cases are to be considered.

(A) If $\lambda=0$ in (1.4), then our assumption (1.5) and Theorem C in Hiromi-Ozawa [1] imply $P(R)=3$.

(B) Assume that $\lambda=1$ in (1.4). Then (1.3) reduces to

$$g(z):=F(z)^2G(z)=(\beta_1-\beta_2e^{K(z)})^2e^{2H(z)}+2(\beta_1+\beta_2e^{K(z)})e^{H(z)}+1. \tag{4.1}$$

If $K\equiv$const., then we have $K\equiv 0$ and so $H\equiv L$, because of $H(0)=L(0)=0$ in (1.4), and (4.1) reduces to

$$g=F^2G=4\beta_1(e^{H/2}-(2\beta_1{}^{1/2})^{-1})(e^{H/2}+(2\beta_1{}^{1/2})^{-1}), \qquad \text{if} \quad \beta_1=\beta_2,$$

and

$$g=F^2G=(\beta_1-\beta_2)^2(e^H-\gamma_1)(e^H-\gamma_2), \qquad \text{if} \quad \beta_1\neq\beta_2,$$

where $\gamma_1=-(\beta_1{}^{1/2}+\beta_2{}^{1/2})^2/(\beta_1-\beta_2)^2$ and $\gamma_2=-(\beta_1{}^{1/2}-\beta_2{}^{1/2})^2/(\beta_1-\beta_2)^2$. Hence it follows from Theorem A that $P(R)=4$. So this case corresponds the exceptional case (i).

If $K\not\equiv$ const., then $\beta_1-\beta_2e^K\not\equiv 0$. The discriminant of the equation $Q(x):=(\beta_1-\beta_2e^{K(z)})^2x^2+2(\beta_1+\beta_2e^{K(z)})x+1=0$ is $16\beta_1\beta_2e^{K(z)}\not\equiv 0$ and $g(z)=Q(e^{H(z)})$ by (4.1). Hence Lemma D yields

$$N_2(r, 0, g)\sim 2m(r, e^H) \quad \text{and} \quad N_1(r, 0, g)=o(m(r, e^H)) \qquad (r\to\infty) \tag{4.2}$$

outside a set of finite measure. Now assume $P(R)=4$. Then it follows from Theorem A that there is a non-constant entire function M, a meromorphic function f and constants γ and δ such that

$$f(z)^2(e^{M(z)}-\gamma)(e^{M(z)}-\delta)=g(z), \qquad M(0)=0, \qquad \gamma\delta(\gamma-\delta)\neq 0. \tag{4.3}$$

Here, since (1.5) and (4.2) hold, we can apply our Proposition to the identity (4.3) with (4.1). Put (4.1) into $g(z)=a_2(z)e^{2H(z)}+a_1(z)e^{H(z)}+a_0(z)$. Then Proposition implies that

$$a_1(z)=-a_2(z)(\gamma+\delta)e^{H(z)+M(z)}/\gamma\delta, \qquad a_0(z)=a_2(z)e^{2(H(z)+M(z))}/\gamma\delta$$

or

$$a_1(z)=-a_2(z)(\gamma+\delta)e^{H(z)-M(z)}, \qquad a_0(z)=a_2(z)\gamma\delta e^{2(H(z)-M(z))},$$

that is,

$$2(\beta_1+\beta_2e^{K(z)})=-(\beta_1-\beta_2e^{K(z)})^2(\gamma+\delta)e^{H(z)+M(z)}/\gamma\delta,$$

$$1=(\beta_1-\beta_2e^{K(z)})^2e^{2(H(z)+M(z))}/\gamma\delta$$

or

$$2(\beta_1+\beta_2 e^{K(z)})=-(\beta_1-\beta_2 e^{K(z)})^2(\gamma+\delta)e^{H(z)-M(z)},$$

$$1=(\beta_1-\beta_2 e^{K(z)})^2\gamma\delta e^{2(H(z)-M(z))}.$$

This is a contradiction, because the function $\beta_1-\beta_2 e^{K(z)}$ has simple zeros since $K\not\equiv$ const..

Thus we have showed that if $K\not\equiv$ const., then $P(R)\neq 4$ and hence, by Theorem B, $P(R)=3$.

(C) Assume that $\lambda=2$ in (1.4). Then (1.3) is

$$g:=F^2G=\beta_2^2e^{2K}e^{4H}-2\beta_1\beta_2e^Ke^{3H}+(\beta_1^2-2\beta_2e^K)e^{2H}-2\beta_1e^H+1. \tag{4.4}$$

If $K\equiv$ const. and $\beta_1^2=16\beta_2$, then $K\equiv 0$, so $L\equiv 2H$ and (4.4) reduces to

$$g=F^2G=(\beta_1^4/16^2)(e^H-4\beta_1^{-1})^2(e^H-4(1+\sqrt{2})^2\beta_1^{-1})(e^H-4(1-\sqrt{2})^2\beta_1^{-1}).$$

Hence we have

$$\{F(16/\beta_1^{-2})(e^H-4\beta_1^{-1})^{-1}\}^2G=(e^H-4(1+\sqrt{2})^2\beta_1^{-1})(e^H-4(1-\sqrt{2})^2\beta_1^{-1})$$

and so theorem A implies $P(R)=4$. This case corresponds the exceptional case (iii).

If $K\not\equiv$ const. or $\beta_1^2\neq 16\beta_2$, then (4.4) is rewritten as follows:

$$g(z)=\beta_2^2e^{2K(z)}g_1(e^{H(z)/2})g_2(e^{H(z)/2})g_3(e^{H(z)/2})g_4(e^{H(z)/2}), \tag{4.5}$$

where

$$g_1(x)=x^2+(\beta_1/\beta_2)^{1/2}e^{-K/2}x-\beta_2^{-1/2}e^{-K/2},$$

$$g_2(x)=x^2-(\beta_1/\beta_2)^{1/2}e^{-K/2}x+\beta_2^{-1/2}e^{-K/2},$$

$$g_3(x)=x^2-(\beta_1/\beta_2)^{1/2}e^{-K/2}x-\beta_2^{-1/2}e^{-K/2},$$

$$g_4(x)=x^2+(\beta_1/\beta_2)^{1/2}e^{-K/2}x+\beta_2^{-1/2}e^{-K/2}.$$

Since $K\not\equiv$ const. or $\beta_1^2\neq 16\beta_2$, none of the discriminants of equations $g_j(x)=0$ $(j=1,\cdots,4)$ vanish and these equations have no common algebroid solution. Hence the discriminant of the equation $g^*(x)=0$ is not identically zero, where $g^*(x)=g_1(x)\cdots g_4(x)$, and so (4.5) and Lemma D imply

$$N_2(r,0,g)\sim 4m(r,e^H) \quad\text{and}\quad N_1(r,0,g)=o(m(r,e^H)) \qquad (r\to\infty) \tag{4.6}$$

outside a set of finite measure. Now assume $P(R)=4$. Then it follows from Theorem A that there is a non-constant entire function M, a meromorphic function f and two constants γ and δ such that

$$f(z)^2(e^{2M(z)}-\gamma)(e^{2M(z)}-\delta)=g(z), \qquad M(0)=0, \qquad \gamma\delta(\gamma-\delta)\neq 0.$$

Then it follows from (1.5), (4.5), (4.6) and our Proposition that the coefficients of e^{3H} and e^H in (4.4) must be identically zero, that is,

$$2\beta_1\beta_2 e^{K(z)} \equiv -2\beta_1 \equiv 0,$$

which contradicts $\beta_1 \neq 0$.

Thus we have showed that if $K \not\equiv$ const. or $\beta_1^2 \neq 16\beta_2$, then $P(R) \neq 4$.

(D) Assume that $\lambda = -1$ in (1.5). Then (1.3) is

$$(4.7) \qquad g := F^2 G e^{2H} = \beta_1^2 e^{4H} - 2\beta_1 e^{3H} + (1 - 2\beta_1\beta_2 e^K) e^{2H} - 2\beta_2 e^K e^H + \beta_2^2 e^{2K}.$$

If $K \equiv$ const. and $16\beta_1\beta_2 = 1$, then $K \equiv 0$ and so $L \equiv -H$, and if follows from (4.7) that

$$\{F e^H \beta_1^{-1}(e^H - 1/4\beta_1)^{-1}\}^2 G = (e^H - (1+\sqrt{2})^2/4\beta_1)(e^H - (1-\sqrt{2})^2/4\beta_1).$$

Hence Theorem A implies $P(R) = 4$. This case corresponds the exceptional case (iv).

If $K \not\equiv$ const. or $16\beta_1\beta_2 \neq 1$, then (4.7) is rewritten as follows:

$$(4.8) \qquad g(z) = \beta_1^2 g_1(e^{H(z)/2}) g_2(e^{H(z)/2}) g_3(e^{H(z)/2}) g_4(e^{H(z)/2}),$$

where

$$g_1(x) = x^2 + \beta_1^{-1/2} x - (\beta_2/\beta_1)^{1/2} e^{K/2},$$

$$g_2(x) = x^2 - \beta_1^{-1/2} x + (\beta_2/\beta_1)^{1/2} e^{K/2},$$

$$g_3(x) = x^2 - \beta_1^{-1/2} x - (\beta_2/\beta_1)^{1/2} e^{K/2},$$

$$g_4(x) = x^2 + \beta_1^{-1/2} x + (\beta_2/\beta_1)^{1/2} e^{K/2}.$$

Since $K \not\equiv$ const. or $16\beta_1\beta_2 \neq 1$, none of the discriminants of equations $g_j(x) = 0$ ($j = 1, \cdots, 4$) vanish and these equations have no common algebroid solution. Hence the discriminant of the equation $g^*(x) = 0$ is not identically zero, where $g^*(x) = g_1(x) \cdots g_4(x)$, and so (4.8) and Lemma D imply

$$(4.9) \qquad N_2(r, 0, g) \sim 4m(r, e^H) \quad \text{and} \quad N_1(r, 0, g) = o(m(r, e^H)) \qquad (r \to \infty)$$

outside a set of finite measure. Now assume $P(R) = 4$. Then it follows from Theorem A that there is a non-constant entire function M, a meromorphic function f and two constants γ and δ such that

$$f(z)^2 (e^{2M(z)} - \gamma)(e^{2M(z)} - \delta) = g(z), \qquad M(0) = 0, \qquad \gamma\delta(\gamma - \delta) \neq 0.$$

Then it follows from (1.5), (4.8), (4.9) and our Proposition that the coefficient $-2\beta_1$ of e^{3H} in (4.7) must be identically zero, which contradicts $\beta_1 \neq 0$.

Thus we have showed that if $K \not\equiv$ const. or $16\beta_1\beta_2 \neq 1$, then $P(R) \neq 4$.

(E) Assume that $\lambda = p$ is an integer and $p > 2$ in (1.4). Then (1.3) is

$$(4.10) \qquad \begin{aligned} g := F^2 G &= \beta_2^2 e^{2K} e^{2pH} - 2\beta_1\beta_2 e^K e^{(p+1)H} - 2\beta_2 e^K e^{pH} + \beta_1^2 e^{2H} - 2\beta_1 e^H + 1 \\ &= \beta_2^2 e^{2K} g_1(e^{H/2}) g_2(e^{H/2}) g_3(e^{H/2}) g_4(e^{H/2}), \end{aligned}$$

where

$$g_1(x)=x^p+(\beta_1/\beta_2)^{1/2}e^{-K/2}x-\beta_2^{-1/2}e^{-K/2},$$

$$g_2(x)=x^p-(\beta_1/\beta_2)^{1/2}e^{-K/2}x+\beta_2^{-1/2}e^{-K/2},$$

$$g_3(x)=x^p-(\beta_1/\beta_2)^{1/2}e^{-K/2}x-\beta_2^{-1/2}e^{-K/2},$$

$$g_4(x)=x^p+(\beta_1/\beta_2)^{1/2}e^{-K/2}x+\beta_2^{-1/2}e^{-K/2}.$$

Suppose that $K\not\equiv$ const. or $(p-1)^{2p-2}\beta_1{}^p\neq p^{2p}\beta_2$. Then it follows from Lemma E that none of the discriminants of equations $g_j(x)=0$ $(j=1, \cdots, 4)$ vanish. Hence, since these equations have no common algebroid solution, (4.10) and Lemma D imply

$$(4.11)\qquad N_2(r, 0, g)\sim 2pm(r, e^H) \quad\text{and}\quad N_1(r, 0, g)=o(m(r, e^H)) \qquad (r\to\infty)$$

outside a set of finite measure. Since the coefficient of $e^{(p+1)H(z)}$ in (4.10) is $-2\beta_1\beta_2e^{K(z)}$, which is not identically zero, our Proposition and Theorem A imply that $P(R)\neq 4$ if $K\not\equiv$const. or $(p-1)^{2p-2}\beta_1{}^p\neq p^{2p}\beta_2$.

Next we suppose that $K\equiv$const. and $(p-1)^{2p-2}\beta_1{}^p=p^{2p}\beta_2$. Then it follows from Lemma E that two of discriminants D_j of equations $g_j(x)=0$ $(j=1, \cdots, 4)$ are zero and the others are not zero. For example, $D_1=D_3=0$ and $D_2=\pm D_4\neq 0$ if $(p-1)^{p-1}\beta_1{}^{p/2}=p^p\beta_2{}^{1/2}$ and p is even. Then (4.10) reduces to

$$g=F^2G=\beta_2{}^2(e^H-\alpha)^2g_0(e^H)$$

and so we have

$$(4.12)\qquad \{F(z)\beta_2{}^{-1}(e^{H(z)}-\alpha)^{-1}\}^2G(z)=g_0(e^{H(z)}),$$

where $\alpha=p^2/\beta_1(p-1)^2$ and

$$g_0(y):=y^{2p-2}+2\alpha y^{2p-3}+\cdots+(p-1)\alpha^{p-2}y^p$$
$$+a_{p-1}y^{P-1}+\cdots+a_1y+(p-1)^4\beta_1{}^2/p^4\beta_2{}^2,$$

where $a_1, \cdots, a_{p-1}$ are suitable constants. It follows from Lemma E that the discriminant of equation $g_0(y)=0$ is not zero. So Lemma D implies

$$N_2(r, 0, g_0(e^H))\sim(2p-2)m(r, e^H),$$

$$N_1(r, 0, g_0(e^H))=o(m(r, e^H)) \qquad (r\to\infty)$$

outside a set of finite measure. Since $2p-3>p-1$ and the coefficient of $e^{(2p-3)H(z)}$ in $g_0(e^{H(z)})$ is 2α, which is not zero, it follows from (4.12), Theorem A and our Proposition that $P(R)\neq 4$ if $K\equiv$const. and $(p-1)^{2p-2}\beta_1{}^p=p^{2p}\beta_2$.

Thus we have proved that $P(R)\neq 4$ if λ is an integer greater than 2.

(F) Assume that $\lambda=-q+1$ is an integer and $\lambda<-1$, that is, $q>2$ in (1.4). Then from (1.3) we have

(4.13) $$\begin{aligned} g &:= (Fe^{(q-1)H})^2 G \\ &= \beta_1^2 e^{2qH} - 2\beta_1 e^{(2q-1)H} + e^{(2q-2)H} - 2\beta_1\beta_2 e^K e^{qH} - 2\beta_2 e^K e^{(q-1)H} + \beta_2^2 e^{2K} \\ &= \beta_1^2 g_1(e^{H/2}) g_2(e^{H/2}) g_3(e^{H/2}) g_4(e^{H/2}), \end{aligned}$$

where

$$g_1(x) = x^q + \beta_1^{-1/2} x^{q-1} - (\beta_2/\beta_1)^{1/2} e^{K/2},$$

$$g_2(x) = x^q - \beta_1^{-1/2} x^{q-1} + (\beta_2/\beta_1)^{1/2} e^{K/2},$$

$$g_3(x) = x^q - \beta_1^{-1/2} x^{q-1} - (\beta_2/\beta_1)^{1/2} e^{K/2},$$

$$g_4(x) = x^q + \beta_1^{-1/2} x^{q-1} + (\beta_2/\beta_1)^{1/2} e^{K/2}.$$

Suppose that $K \not\equiv$ const. or $q^{2q}\beta_1^{q-1}\beta_2 \neq (q-1)^{2q-2}$. Then it follows from Lemma F that none of the discriminants of equations $g_j(x)=0$ $(j=1, \cdots, 4)$ vanish. Hence, since these equations have no common algebroid solution, from Lemma D we have

$$N_2(r, 0, g) \sim 2qm(r, e^H) \quad \text{and} \quad N_1(r, 0, g) = o(m(r, e^H)) \qquad (r \to \infty)$$

outside a set of finite measure. Further the coefficient of $e^{(2q-1)H(z)}$ in (4.13) is $-2\beta_1$, which is not zero. Hence we have $P(R) \neq 4$ from our Proposotion and Theorem A.

Next we suppose that $K \equiv$const. and $q^{2q}\beta_1^{q-1}\beta_2 = (q-1)^{2q-2}$. Then it follows from Lemma F that two of discriminants D_j of equations $g_j(x)=0$ $(j=1, \cdots, 4)$ are zero and the others are not zero. For example, $D_2 = D_4 = 0$ and $D_3 = \pm D_1 \neq 0$ if $q^q \beta_1^{(q-1)/2} \beta_2^{1/2} = (q-1)^{q-1}$ and q is even. Then (4.13) reduces to

$$g = (Fe^{(q-1)H})^2 G = \beta_1^2 (e^H - \beta)^2 g_0(e^H)$$

and so we have

(4.14) $$\{F(z) e^{(q-1)H(z)} \beta_1^{-1} (e^{H(z)} - \beta)^{-1}\}^2 G(z) = g_0(e^{H(z)})$$

where $\beta = (q-1)^2/q^2\beta_1$ and

$$g_0(y) := y^{2q-2} - (2(2q-1)/\beta_1 q^2) y^{2q-3} + a_{2q-4} y^{2q-4} + \cdots + a_1 y + \beta_2 \beta^{q-3}/q^2 \beta_1^2,$$

where $a_1, \cdots, a_{2q-4}$ are suitable constants. It follows from Lemma F that the discriminant of equation $g_0(y)=0$ is not zero. So Lemma D implies

$$N_2(r, 0, g_0(e^H)) \sim (2q-2) m(r, e^H),$$

$$N_1(r, 0, g_0(e^H)) = o(m(r, e^H)) \qquad (r \to \infty)$$

outside a set of finite measure. Further $2q-3 > q-1$ and the coefficient of $e^{(2q-3)H(z)}$ in $g_0(e^{H(z)})$ is $-2(2q-1)/\beta_1 q^2$, which is not zero. Hence we have $P(R) \neq 4$.

Thus we have showed that $P(R)\neq 4$ if λ is an integer less than -1.

(G) Assume that λ is a rational number greater than one in (1.4) and is not an integer. We put $\lambda=p/q$, where p and $q(q>1)$ are coprime integer Then (1.3) is

$$
\begin{aligned}
(4.15)\qquad g:=F^2G&=\beta_2{}^2e^{2K}e^{4pS}-2\beta_1\beta_2e^Ke^{2(p+q)S}+\beta_1{}^2e^{4qS}\\
&\quad-2\beta_2e^Ke^{2pS}-2\beta_1e^{2qS}+1\\
&=\beta_2{}^2e^{2K}g_1(e^S)g_2(e^S)g_3(e^S)g_4(e^S),
\end{aligned}
$$

where $S(z)=H(z)/2q$ and

$$
\begin{aligned}
g_1(x)&=x^p+(\beta_1/\beta_2)^{1/2}e^{-K/2}x^q-\beta_2{}^{-1/2}e^{-K/2},\\
g_2(x)&=x^p-(\beta_1/\beta_2)^{1/2}e^{-K/2}x^q+\beta_2{}^{-1/2}e^{-K/2},\\
g_3(x)&=x^p-(\beta_1/\beta_2)^{1/2}e^{-K/2}x^q-\beta_2{}^{-1/2}e^{-K/2},\\
g_4(x)&=x^p+(\beta_1/\beta_2)^{1/2}e^{-K/2}x^q+\beta_2{}^{-1/2}e^{-K/2}.
\end{aligned}
$$

Suppose that $K\not\equiv$const. or $(p-q)^{2(p-q)}q^{2q}\beta_1{}^p\neq p^{2p}\beta_2{}^q$. Then it follows from Lemma G that none of the discriminants of equations $g_j(x)=0\,(j=1,\cdots,4)$ vanish. Hence, since these equations have no common algebroid solution, (4.15) and Lemma D imply

$$(4.16)\qquad N_2(r,0,g)\sim 4pm(r,e^S)\quad\text{and}\quad N_1(r,0,g)=o(m(r,e^S))\qquad(r\to\infty)$$

outside a set of finite measure. Hence it follows from our Proposition and Theorem A that $P(R)\neq 4$, because the coefficient of $e^{2(p+q)S(z)}$ in (4.15) is $-2\beta_1\beta_2e^{K(z)}$, which is not identically zero.

Next we suppose that $K\equiv$const. and $(p-q)^{2(p-q)}q^{2q}\beta_1{}^p=p^{2p}\beta_2{}^q$.

Further, if $q=p-1$, that is, $p=q+1>2$, then the argument in the case (F) is applicable to this case. So we can deduce that $P(R)\neq 4$ in this case.

If $p>q+1$, then $4p-2>2p+2q$ and it follows from Lemma G that two of discriminants D_j of equations $g_j(x)=0\,(j=1,\cdots,4)$ are identically zero and the others are not identically zero. (4.15) reduces to

$$g=F^2G=\beta_1{}^2\{e^{2S}-(\alpha+\beta)e^S+\alpha\beta\}^2g_0(e^S)$$

and so we have

$$(4.17)\qquad \{F(z)\beta_1{}^{-1}(e^{2S(z)}-(\alpha+\beta)e^{S(z)}+\alpha\beta)^{-1}\}^2G(z)=g_0(e^{S(z)}),$$

where α and β satisfy one of the following six cases:

(a) if p is even, q is odd and $D_1=D_3=0$, then

$$
\begin{aligned}
\alpha^{p-q}&=-pq^{-1}(\beta_1/\beta_2)^{1/2}, &\alpha^q&=p(p-q)^{-1}\beta_1{}^{1/2},\\
\beta^{p-q}&=pq^{-1}(\beta_1/\beta_2)^{1/2}, &\beta^q&=-p(p-q)^{-1}\beta_1{}^{1/2},
\end{aligned}
$$

(b) if p is even, q is odd and $D_2=D_4=0$, then

$$\alpha^{p-q}=pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \alpha^q=p(p-q)^{-1}\beta_1^{1/2},$$
$$\beta^{p-q}=-pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \beta^q=-p(p-q)^{-1}\beta_1^{1/2},$$

(c) if p is odd, q is even and $D_1=D_2=0$, then

$$\alpha^{p-q}=-pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \alpha^q=p(p-q)^{-1}\beta_1^{1/2},$$
$$\beta^{p-q}=pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \beta^q=p(p-q)^{-1}\beta_1^{1/2},$$

(d) if p is odd, q is even and $D_3=D_4=0$, then

$$\alpha^{p-q}=pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \alpha^q=-p(p-q)^{-1}\beta_1^{1/2},$$
$$\beta^{p-q}=-pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \beta^q=-p(p-q)^{-1}\beta_1^{1/2},$$

(e) if p is odd, q is odd and $D_1=D_4=0$, then

$$\alpha^{p-q}=-pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \alpha^q=p(p-q)^{-1}\beta_1^{1/2},$$
$$\beta^{p-q}=-pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \beta^q=-p(p-q)^{-1}\beta_1^{1/2},$$

(f) if p is odd, q is odd and $D_2=D_3=0$, then

$$\alpha^{p-q}=pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \alpha^q=p(p-q)^{-1}\beta_1^{1/2},$$
$$\beta^{p-q}=pq^{-1}(\beta_1/\beta_2)^{1/2}, \qquad \beta^q=-p(p-q)^{-1}\beta_1^{1/2}$$

and

$$g_0(y):=y^{4p-4}-2(\alpha+\beta)y^{4p-5}+(\alpha-\beta)^2y^{4p-6}+a_{4p-7}y^{4p-7}+\cdots+a_1y+a_0,$$

where $a_{4p-7}, \cdots, a_0$ are suitable constants. It follows from Lemma G that the discriminant of equation $g_0(y)=0$ is not identically zero. So Lemma D implies

$$N_2(r, 0, g_0(e^S))\sim(4p-4)m(r, e^S),$$
$$N_1(r, 0, g_0(e^S))=o(m(r, e^S)) \qquad (r\to\infty)$$

outside a set of finite measure. Further $4p-6>2p-2$ and the coefficients of $e^{(4p-5)S(z)}$ and $e^{(4p-6)S(z)}$ in $g_0(e^{S(z)})$ are $-2(\alpha+\beta)$ and $(\alpha-\beta)^2$, respectively, which are not simultaneously zero, because $\alpha\neq0$ and $\beta\neq0$. Hence we have also $P(R)\neq4$.

(H) Assume that λ is a rational number less than -1 in (1.4) and is not an integer. We can put $\lambda=-q/(p-q)$, where p and q are coprime integer such that $2q>p>q+1>1$. Then from (1.3) we have

$$(4.18)\quad g:=(Fe^{2qS})^2G=\beta_1^2e^{4pS}-2\beta_1e^{2(p+q)S}+e^{4qS}-2\beta_1\beta_2e^Ke^{2pS}-2\beta_2e^Ke^{2qS}+\beta_2^2e^{2K}$$
$$=\beta_1^2g_1(e^S)g_2(e^S)g_3(e^S)g_4(e^S),$$

where $S(z)=H(z)/2(p-q)$ and

$$g_1(x)=x^p+\beta_1^{-1/2}x^q-(\beta_2/\beta_1)^{1/2}e^{K/2},$$

$$g_2(x)=x^p-\beta_1^{-1/2}x^q+(\beta_2/\beta_1)^{1/2}e^{K/2},$$

$$g_3(x)=x^p-\beta_1^{-1/2}x^q-(\beta_2/\beta_1)^{1/2}e^{K/2},$$

$$g_4(x)=x^p+\beta_1^{-1/2}x^q+(\beta_2/\beta_1)^{1/2}e^{K/2}.$$

Therefore the same argument in the case (G) leads to $P(R)\neq 4$ in this case.

Thus the proof of our Theorem is complete.

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