T. MURAI
KODAI MATH. J.
16 (1993), 355—360

INTERFERENCE OF TWO AEROPLANES

By TAKAFUMI MURAI

1. Introduction

We are concerned with the interaction problem in aerodynamics, and the purpose of this note is to show the interference of two aeroplanes in terms of the aerodynamic force (i.e. the lift). We confine ourselves to the 2-dimensional theory, because the 3-dimensional force is considered as the integration of the 2-dimensional force, and the 2-dimensional results fit sufficiently for actual phenomena. The density of our fluid is denoted by ρ; this is a constant. The 2-dimensional frame is set up by the observer in an aeroplane. Our flow is 2-dimensional, steady, incompressible and irrotational. Navier-Stokes's equation with a constant viscosity is assumed. This is equivalent to assuming Euler's equation of motion, because the viscosity term vanishes from the irrotationality. Let Γ_j $(j=1, 2)$ be two compact sets in the complex plane $\boldsymbol{C}$ such that the boundary $\partial\Gamma_j$ of each Γ_j is a smooth Jordan curve except one sharp edge (i.e. the trailing edge) a_j with intersection angle 0. This is a model of the section of two aeroplanes. An anti-analytic function $\overline{f(w)}=u+iv$ (i.e. $\partial\bar{f}/\partial w=0$) in a fluid domain $\Omega=\boldsymbol{C}\cup\{\infty\}-(\Gamma_1\cup\Gamma_2)$ is regarded as a steady flow obstructed by $\Gamma_1\cup\Gamma_2$ i.e. (u, v) means a 2-dimensional velocity field at this instant. The value $c=\overline{f(\infty)}$ means a uniform flow at infinity, and we consider that the flow $\bar{f}$ is induced by the uniform flow c. We say that $\bar{f}$ satisfies the kinematic boundary condition (KC) if $f(w)dw$ is real-valued on $\partial\Omega-\{a_1, a_2\}$, where the orientation of dw is chosen so that Ω lies to the left. This condition means that the streamlines associated with $\bar{f}$ coincide with the configuration of $\Gamma_1\cup\Gamma_2$ on the boundary. We say that $\bar{f}$ satisfies the Kutta-Joukowski condition (KJ) if the boundary values $\overline{f(a_j)}$ $(j=1, 2)$ exist at the trailing edges a_j $(j=1, 2)$. There exists uniquely a flow $\bar{f}_c$ in the fluid domain Ω satisfying (KC), (KJ) and $\overline{f_c(\infty)}=c$. The aerodynamic force (i.e. the lift) induced by the uniform flow c and $\Gamma_1\cup\Gamma_2$ is defined by

$$\mathcal{L}(\Gamma_1\cup\Gamma_2, c)=-i\int_{\partial\Omega}p_c(w)dw\,,$$

where $p_c(w)$ denotes the so-called static pressure i.e. a real-valued function satisfying

Received November 24, 1992.

$$p_c(w)+\frac{\rho}{2}|f_c(w)|^2=\mathrm{Const} \quad (w\in\Omega).$$

The lift coefficient of $\Gamma_1\cup\Gamma_2$ is defined by

$$\mathcal{L}(\Gamma_1\cup\Gamma_2)=\max_{0\leq\theta\leq 2\pi}|\mathcal{L}(\Gamma_1\cup\Gamma_2, e^{i\theta})|.$$

The lift $\mathcal{L}(\Gamma_j, c)$ and the lift coefficient $\mathcal{L}(\Gamma_j)$ are defined analogously $(j=1, 2)$. We show the following

THEOREM. $\mathcal{L}(\Gamma_1\cup\Gamma_2)<\mathcal{L}(\Gamma_1)+\mathcal{L}(\Gamma_2)$.

This inequality is sharp in the following sense. For two smooth arcs Γ_j $(j=1, 2)$ also, the lift coefficients $\mathcal{L}(\Gamma_1)$, $\mathcal{L}(\Gamma_2)$, $\mathcal{L}(\Gamma_1\cup\Gamma_2)$ are defined analogously once an endpoint of each Γ_j is chosen as the trailing edge. The equality $\mathcal{L}(\Gamma_1\cup\Gamma_2)=\mathcal{L}(\Gamma_1)+\mathcal{L}(\Gamma_2)$ holds, if $\Gamma_j\subset\boldsymbol{R}$ $(j=1, 2)$ and the right endpoint of each segment is chosen as the trailing edge. Here $\boldsymbol{R}$ denotes the real line. We remark that $\mathcal{L}(\Gamma_1\cup\Gamma_2)$, $\Gamma_1\cup\Gamma_2\neq\emptyset$ is not positive in general; take $\Gamma_1=[-2, -1]$ and $\Gamma_2=[1, 2]$ with the trailing edges ± 1, for example. The subadditivity of $|\mathcal{L}(\cdot, c)|$ for a fixed c does not hold. The airfoil data is seen in [AD, G]. This paper is motivated by Suita's subadditivity [S] of analytic capacity, and his method plays an important role.

2. Proof of Theorem

Throughout this section, Γ_j $(j=1, 2)$ are smooth Jordan curves except the trailing edges a_j $(j=1, 2)$, $E=\Gamma_1\cup\Gamma_2$ and $\Omega=\boldsymbol{C}\cup\{\infty\}-E$. We begin by noting some basic facts. The zero lift direction θ_0 is a real number satisfying $\mathcal{L}(E, e^{i\theta_0})=0$, and the maximum lift direction θ_M is a real number satisfying

$$\mathcal{L}(E, e^{i\theta_M})=ie^{i\theta_M}\mathcal{L}(E).$$

The following facts are elementary and interesting in themselves (cf. [M1, pp. 158-162]). Suppose that $\mathcal{L}(E)\neq 0$. Then

(1) The maximum lift direction θ_M is unique (mod 2π).
(2) There exist two zero lift directions and $\theta_0=\theta_M\pm\pi/2$ (mod 2π).
(3) $\mathcal{L}(E, Ue^{i\theta})=U^2 ie^{i\theta}\cos(\theta-\theta_M)\mathcal{L}(E)$.
(4) $\bar{f}_c\neq 0$ in Ω for all $c\in\boldsymbol{C}$.

It is not meaningless to recall that an aeroplane can fly with the aid of the power 2 in (3). Since the proof of these facts is analogous as in [M1], we omit the proof and note only that Blasius's formula [M2, p. 173]

$$\mathcal{L}(E, c)=2\pi\rho c\,\overline{\mathrm{Cir}(f_c)}$$

plays an important role in the proof, where $\mathrm{Cir}(f_c)$ denotes the circulation i.e.

$$\mathrm{Cir}(f_c)=\frac{1}{2\pi i}\int_{\partial\Omega} f_c(w)dw .$$

First Step. We divide the proof into two steps. Let $E_{en}=\{a_1, a_2\}$, $F=[b_1, c_1]\cup[b_2, c_2]\subset\boldsymbol{R}$, $F_{en}=\{b_1, c_1, b_2, c_2\}$. We choose F so that there exists a conformal mapping

$$\phi(\zeta)=e^{-i\alpha}\zeta+d_0+d_1/\zeta+\cdots \quad (|\alpha|\leqq\pi)$$

from F^c onto Ω. In this step, we show that

$$\mathcal{L}(E)\leqq\pi\rho(c_1-b_1+c_2-b_2). \tag{5}$$

From a flow $\bar{g}=\overline{(f_{c_0}\circ\phi)(\cdot)\phi'(\cdot)}$ outside F, where $c_0=e^{i\theta_M}$. Then

$$\overline{g(\infty)}=e^{i(\theta_M+\alpha)} \quad (=e^{-i\theta}, \text{ say}).$$

A simple calculation shows that g is expressed as

$$g(\zeta)=\cos\theta+i\sin\theta\left\{\prod_{j=1}^{2}\sqrt{\frac{\zeta-c_j}{\zeta-b_j}}+(A\zeta+B)\prod_{j=1}^{2}\frac{1}{\sqrt{(\zeta-b_j)(\zeta-c_j)}}\right\}$$

for some $A, B\in\boldsymbol{R}$, where the branch of $\sqrt{\cdot}$ is chosen so that $\sqrt{x}>0$ $(x>0)$. Let $\zeta_j=\phi^{-1}(a_j)$ $(j=1, 2)$. First we prove (5) assuming that

$$\zeta_1, \zeta_2\notin F_{en}. \tag{6}$$

Comparing the configurations of ∂F^c and $\partial\Omega$, we obtain $\phi'(\zeta_j)=0$ $(j=1, 2)$, and hence $g(\zeta_j)=0$ $(j=1, 2)$. Thus

$$\begin{cases}\left\{\prod_{j=1}^{2}\sqrt{\left|\frac{\zeta_1-c_j}{\zeta_1-b_j}\right|}+(A\zeta_1+B)\prod_{j=1}^{2}\frac{1}{\sqrt{|(\zeta_1-b_j)(\zeta_1-c_j)|}}\right\}\sin\theta=\varepsilon_1\cos\theta\\ \left\{\prod_{j=1}^{2}\sqrt{\left|\frac{\zeta_2-c_j}{\zeta_2-b_j}\right|}-(A\zeta_2+B)\prod_{j=1}^{2}\frac{1}{\sqrt{|(\zeta_2-b_j)(\zeta_2-c_j)|}}\right\}\sin\theta=\varepsilon_2\cos\theta,\end{cases} \tag{7}$$

where $\varepsilon_j=1$ if ζ_j is contained in the upper boundary, and $\varepsilon_j=-1$ if ζ_j is contained in the lower boundary. By (7), it follows that

$$\begin{cases}(A\zeta_1+B)\sin\theta=-|(\zeta_1-c_1)(\zeta_1-c_2)|\sin\theta+\varepsilon_1\prod_{j=1}^{2}\sqrt{|(\zeta_1-b_j)(\zeta_1-c_j)|}\cos\theta\\ (A\zeta_2+B)\sin\theta=|(\zeta_2-c_1)(\zeta_2-c_2)|\sin\theta-\varepsilon_2\prod_{j=1}^{2}\sqrt{|(\zeta_2-b_j)(\zeta_2-c_j)|}\cos\theta,\end{cases}$$

and hence

$$A\sin\theta=\frac{|(\zeta_2-c_1)(\zeta_2-c_2)|+|(\zeta_1-c_1)(\zeta_1-c_2)|}{\zeta_2-\zeta_1}\sin\theta$$

$$-\frac{1}{\zeta_2-\zeta_1}\left\{\varepsilon_1\prod_{j=1}^{2}\sqrt{|(\zeta_1-b_j)(\zeta_1-c_j)|}+\varepsilon_2\prod_{j=1}^{2}\sqrt{|(\zeta_2-b_j)(\zeta_2-c_j)|}\right\}\cos\theta$$

$$=(c_1-\zeta_1+c_2-\zeta_2)\sin\theta$$

$$-\frac{1}{\zeta_2-\zeta_1}\left\{\varepsilon_1\prod_{j=1}^{2}\sqrt{|(\zeta_1-b_j)(\zeta_1-c_j)|}+\varepsilon_2\prod_{j=1}^{2}\sqrt{|(\zeta_2-b_j)(\zeta_2-c_j)|}\right\}\cos\theta\,.$$

We have

$$\mathcal{L}(E)=2\pi\rho|\mathrm{Cir}(f_{c_0})|=2\pi\rho|\mathrm{Cir}(g)|$$

$$=2\pi\rho\left|\frac{c_1-b_1+c_2-b_2}{2}\sin\theta-A\sin\theta\right|$$

(8)
$$=2\pi\rho\left|\left\{\frac{c_1-b_1+c_2-b_2}{2}-(c_1-\zeta_1+c_2-\zeta_2)\right\}\sin\theta\right.$$

$$\left.+\frac{1}{\zeta_2-\zeta_1}\left\{\varepsilon_1\prod_{j=1}^{2}\sqrt{|(\zeta_1-b_j)(\zeta_1-c_j)|}+\varepsilon_2\prod_{j=1}^{2}\sqrt{|(\zeta_2-b_j)(\zeta_2-c_j)|}\right\}\cos\theta\right|$$

$$\leqq 2\pi\rho\left(\left\{\frac{c_1-b_1+c_2-b_2}{2}-(c_1-\zeta_1+c_2-\zeta_2)\right\}^2\right.$$

$$\left.+\frac{1}{(\zeta_2-\zeta_1)^2}\left\{\prod_{j=1}^{2}\sqrt{|(\zeta_1-b_j)(\zeta_1-c_j)|}+\prod_{j=1}^{2}\sqrt{|(\zeta_2-b_j)(\zeta_2-c_j)|}\right\}^2\right)^{1/2}.$$

Let

$$l=\zeta_1-b_1,\quad m=c_1-\zeta_1,\quad x=b_2-c_1,\quad L=\zeta_2-b_2,\quad M=c_2-\zeta_2\,.$$

Then the last quantity in (8) is equal to

$$2\pi\rho\left(\left\{\frac{l-m+L-M}{2}\right\}^2\right.$$

$$\left.+\left\{\frac{\sqrt{lm(x+m)(x+m+L+M)}+\sqrt{LM(x+L)(x+l+m+L)}}{x+m+L}\right\}^2\right)^{1/2}$$

$$=2\pi\rho\left\{\left(\frac{l-m+L-M}{2}\right)^2+K\right\}^{1/2},\quad\text{say.}$$

Since

$$K=\frac{1}{(x+m+L)^2}\{lm(x+m)(x+m+L+M)+LM(x+L)(x+l+m+L)$$

$$+2\sqrt{lM(x+m)(x+L)}\sqrt{mL(x+m+L+M)(x+l+m+L)}\}$$

$$\leqq \frac{1}{(x+m+L)^2}\{lm(x+m)(x+m+L+M)+LM(x+L)(x+l+m+L)$$

$$+lM(x+m)(x+L)+mL(x+m+L+M)(x+l+m+L)\}$$

$$=\frac{1}{(x+m+L)^2}\{(l+L)(m+M)x^2+2(l+L)(m+M)(m+L)x$$

$$+(l+L)(m+M)(m+L)^2\}=(l+L)(m+M),$$

we have

$$2\pi\rho\left\{\left(\frac{l-m+L-M}{2}\right)^2+K\right\}^{1/2}\leqq 2\pi\rho\left\{\left(\frac{l-m+L-M}{2}\right)^2+(l+L)(m+M)\right\}^{1/2}$$

$$\leqq \pi\rho(l+m+L+M)=\pi\rho(c_1-b_1+c_2-b_2).$$

Thus (5) holds in the case of (6).

In the case where (6) does not hold, we take a small number $\varepsilon>0$ and modify E so that the complement of the modified set E_ε is conformally equivalent to F^c, the trailing edges of E_ε satisfy (6) and $\mathcal{L}(E)\leqq\mathcal{L}(E_\varepsilon)+\varepsilon$. Then

$$\mathcal{L}(E)\leqq\mathcal{L}(E_\varepsilon)+\varepsilon\leqq\pi\rho(c_1-b_1+c_2-b_2)+\varepsilon\,.$$

Since $\varepsilon>0$ is arbitrary, we have (5).

Second step. It is sufficient to show that

$$\pi\rho(c_j-b_j)<\mathcal{L}(\Gamma_j)\quad (j=1,\,2)\,. \tag{9}$$

This is essentially known; in fact, Suita [S] shows that

$$c_j-b_j<4r_j\quad (j=1,\,2), \tag{10}$$

where r_j denotes the outer radius of Γ_j. A simple calculation shows that $\mathcal{L}([-2r_j,\,2r_j])=4\pi\rho r_j$ $(j=1,\,2)$. Since each Γ_j^c is conformally equivalent to $[-2r_j,\,2r_j]^c$, Blasius's formula yields that $\mathcal{L}(\Gamma_j)=4\pi\rho r_j$. Combining this equality with (10), we obtain (9). Inequalities (5) and (9) immediately yield the required inequality. This completes the proof of Theorem.

It seems not worthless to note here Suita's proof of (10). Without loss of generality, it is sufficient to prove (10) for $j=1$. Let ϕ_1 be a conformal mapping from Γ_1^c onto $\bar{\boldsymbol{D}}_{r_1}^c$ such that $|\phi_1(w)/w|=1+o(1)$ $(w\to\infty)$, where $\boldsymbol{D}_r$ denotes the open disk with center 0 and radius r. The number r_1 is the outer radius of Γ_1. Let κ denote a conformal mapping from $\{\bar{\boldsymbol{D}}_{r_1}\cup\phi_1(\Gamma_2)\}^c$ onto a Grötzsch domain $\{\bar{\boldsymbol{D}}_{r_1'}\cup[b_2',\,c_2']\}^c$ such that $|\kappa(w)/w|=1+o(1)$ $(w\to\infty)$. Then Rengel's inequality [T, p. 409] shows that $r_1'<r_1$. (The equality does not hold, since $|\kappa(w)/w|\not\equiv 1$.) Form a conformal mapping $\phi_2=\kappa\circ\phi_1+r_1'^2/(\kappa\circ\phi_1)$ from Ω onto $\{[-2r_1',\,2r_1']\cup[b_2'+r_1'^2/b_2',\,c_2'+r_1'^2/c_2']\}^c$. Then $|\phi_2(w)/w|=1+o(1)$ $(w\to\infty)$. This shows that $4r_1'=c_1-b_1$, and hence $c_1-b_1=4r_1'<4r_1$. Thus (10) holds.

REFERENCES

[AD] I. H. ABBOTT AND A. E. VON DOENHOFF, Theory of wing sections, Including a summary of airfoil data, Dover, New York, 1959.

[G] E. GARRICK, Potential flow about arbitrary biplane wing sections, Technical Report No. **542** NASA (1936), 47-75.

[M1] L. M. MILNE-THOMSON, Theoretical aerodynamics, Fourth edition, Dover, New York, 1973.

[M2] L. M. MILNE-THOMSON, Theoretical hydrodynamics, Fifth edition, Macmillan, London, 1979.

[S] N. SUITA, On subadditivity of analytic capacity for two continua, Kodai Math. J. **7** (1984), 73-75.

[T] M. TSUJI, Potential theory in modern function theory, Maruzen, Tokyo, 1975.

RHEOLOGY LABORATORY
NAGOYA UNIVERSITY
NAGOYA 464-01
JAPAN