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THE IDENTITY MAP AS A HARMONIC MAP OF A $(4r+3)$-SPHERE WITH DEFORMED METRICS

Dedicated to Professor Yoji Hatakeyama on his sixtieth birthday

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§ 1. Introduction.

It is known that the identity map Id_M of a compact Riemannian manifold (M, g) is a harmonic map. (M, g) is said to be unstable, if the Jacobi operator J defined by the second variation of the energy functional at Id_M has negative eigenvalues. The standard m-dimensional sphere (S^m, g_0) of constant curvature 1 is unstable for $m \geqq 3$. More generally, unstable, simply connected compact (irreducible) symmetric spaces were determined (Smith [5], Nagano [3], Ohnita [4], Urakawa [11]).

In [9] the author studied instability of spheres $(S^m, g(t))$ with $m=2n+1$, as a class of homogeneous Riemannian manifolds which are not symmetric nor Einstein (cf. also Urakawa [12]), and gave the expression of some eigenvector of the Jacobi operator corresponding to a negative eigenvalue. The Riemannian metrics $g(t)$ considered in [9] or [12] is related to the Hopf fibration $(S^{2n+1}, g_0) \to (CP^n, h_0)$, where (CP^n, h_0) denotes the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4.

As the next step, we study the Riemannian metrics related to the Hopf fibration $(S^{4r+3}, g_0) \to (QP^r, h_0)$, where QP^r denotes the quaternion projective space. (S^{4r+3}, g_0) admits a Sasakian 3-structure $\{\eta_{(1)}, \eta_{(2)}, \eta_{(3)}\}$. The dual vector fields $\{\xi_{(1)}, \xi_{(2)}, \xi_{(3)}\}$ define the 3-dimensional distribution on S^{4r+3} whose integral submanifolds are fibers of the Hopf fibration. We define a 1-parameter family of Riemannian metrics $g(t)$ on S^{4r+3} by

$$g(t)=t^{-1}g_0+t^{-1}(t^{m/3}-1)\textstyle\sum_\alpha \eta_{(\alpha)} \otimes \eta_{(\alpha)}\,, \tag{1.1}$$

where $m=4r+3$, and $0<t<\infty$ (cf. Tanno [7]). The volume form for $g(t)$ is unchanged for all t. The purpose of this paper is to show the following:

THEOREM. (i) *For* $m=4r+3=7$, 11, *the sphere* $(S^{4r+3}, g(t))$ *is unstable.*

(ii) *For* $m=4r+3 \geqq 15$, *and for* $t \in (0, t_0(m))$ *or* $t \in (t_1(m), \infty)$, *the sphere* $(S^{4r+3}, g(t))$ *is unstable, where* $t_0(m)$ *and* $t_1(m)$ *are given in* § 4.

(iii) *For each eigenfunction* f *corresponding to the* (*non-zero*) *first eigenvalue* $\lambda_1=4r+3$ *of the Laplacian acting on functions on* (S^{4r+3}, g_0),

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(1.2) $$f\xi_{(1)}+t^{m/3}k(t)\nabla\xi_{\mathrm{grad}f}\xi_{(1)}+(1-t^{m/3}k(t))((\xi_{(3)}f)\xi_{(2)}-(\xi_{(2)}f)\xi_{(3)})$$

is an eigenvector corresponding to the negative eigenvalue $\mu(t)$ of the Jacobi operator $J_{(t)}$. $k(t)$ and $\mu(t)$ are given in § 4. The multiplicity of $\mu(t)$ is $m+1$ for almost all t in $(0, t_0(m))$ or $(t_1(m), \infty)$.

It is an open problem if the positivity of $\mu(t)$ (for $t_0(m)<t<t_1(m)$) is related to some geometric property, and if $(S^{4r+3}, g(t))$ is stable for $t_0(m)<t<t_1(m)$.

§ 2. Preliminaries.

Let (S^{4r+3}, g_0) be the unit sphere in the $4(r+1)$-dimensional Euclidean space $E^{4(r+1)}$, where $E^{4(r+1)}$ is considered as a product space $Q\times\cdots\times Q$ of $r+1$ copies of the space Q of quaternions with the canonical metric. Let $\{x^\sigma, y^\sigma, z^\sigma, w^\sigma;\ \sigma=1, \cdots, r+1\}$ be the natural coordinate system of $E^{4(r+1)}$. Let $\{I, J, K\}$ be the quaternion structure of $E^{4(r+1)}$. If one considers a point $x=(x^\sigma, y^\sigma, z^\sigma, w^\sigma)$ of S^{4r+3} as a unit vector in $E^{4(r+1)}$ and

$$Ix=(y^\sigma, -x^\sigma, w^\sigma, -z^\sigma),$$

$$Jx=(z^\sigma, -w^\sigma, -x^\sigma, y^\sigma),$$

$$Kx=(w^\sigma, z^\sigma, -y^\sigma, -x^\sigma),$$

as tangent vectors at x to S^{4r+3}, we get a field of orthonormal vectors $\{\xi_{(1)}, \xi_{(2)}, \xi_{(3)}\}$ on S^{4r+3}. Each $\xi_{(\alpha)}$ is a Killing vector field on (S^{4r+3}, g_0), and we have the following:

(2.1) $$[\xi_{(1)}, \xi_{(2)}]=2\xi_{(3)}, \qquad [\xi_{(2)}, \xi_{(3)}]=2\xi_{(1)}, \qquad [\xi_{(3)}, \xi_{(1)}]=2\xi_{(2)}\,.$$

The 3-dimensional distribution defined by $\{\xi_{(1)}, \xi_{(2)}, \xi_{(3)}\}$ on S^{4r+3} is integrable and each integral submanifold is isometric to a unit 3-sphere in E^4. This gives the Hopf fibration $S^{4r+3}\to QP^r$. Now let $\{\eta_{(1)}, \eta_{(2)}, \eta_{(3)}\}$ be the dual of $\{\xi_{(1)}, \xi_{(2)}, \xi_{(3)}\}$ with respect to g_0. Then each $\eta_{(\alpha)}$ defines a contact structure on S^{4r+3}, and $\{\eta_{(\alpha)}, g_0\}$ is a Sasakian structure. Furthermore, $\{\eta_{(1)}, \eta_{(2)}, \eta_{(3)}; g_0\}$ is called the canonical Sasakian 3-structure of (S^{4r+3}, g_0). For each α ($\alpha=1, 2, 3$), we define a (1, 1)-tensor field $\phi_{(\alpha)}$ by

(2.2) $$\phi_{(\alpha)}=-\nabla\xi_{(\alpha)}\,.$$

$\phi_{(1)}$, $\phi_{(2)}$, and $\phi_{(3)}$ are canonically related to I, J, and K.

We have the following relations:

(2.3) $$\phi_{(\alpha)}\xi_{(\alpha)}=0, \qquad \eta_{(\alpha)}\phi_{(\alpha)}=0,$$

(2.4) $$\phi^2_{(\alpha)}X=-X+\eta_{(\alpha)}(X)\xi_{(\alpha)},$$

(2.5) $$g_0(X, Y)=g_0(\phi_{(\alpha)}X, \phi_{(\alpha)}Y)+\eta_{(\alpha)}(X)\eta_{(\alpha)}(Y),$$

$$\nabla_{\xi(\alpha)}\xi_{(\alpha)}=\nabla_{\xi(\alpha)}\eta_{(\alpha)}=0, \qquad L_{\xi(\alpha)}\phi_{(\alpha)}=L_{\xi(\alpha)}\eta_{(\alpha)}=0, \tag{2.6}$$

$$(\nabla_X\phi_{(\alpha)})(Y)=g_0(X, Y)\xi_{(\alpha)}-\eta_{(\alpha)}(Y)X, \tag{2.7}$$

$$\phi_{(\alpha)}\xi_{(\beta)}=-\phi_{(\beta)}\xi_{(\alpha)}=\xi_{(\gamma)}, \qquad [\alpha, \beta, \gamma: cyclic] \tag{2.8}$$

$$\phi_{(\alpha)}\phi_{(\beta)}-\xi_{(\alpha)}\eta_{(\beta)}=-\phi_{(\beta)}\phi_{(\alpha)}+\xi_{(\beta)}\eta_{(\alpha)}=\phi_{(\gamma)}, \qquad [\alpha, \beta, \gamma: cyclic] \tag{2.9}$$

where X, Y denote tangent vectors or vector fields, and $[\alpha, \beta, \gamma: cyclic]$ means that $\{\alpha, \beta, \gamma\}$ in (2.8) and (2.9) is a cyclic permutation of (1, 2, 3). This convention is used also in the following. Also, we have the following:

$$\eta_{(\alpha)}\phi_{(\beta)}=-\eta_{(\beta)}\phi_{(\alpha)}=\eta_{(\gamma)}, \qquad [\alpha, \beta, \gamma: cyclic] \tag{2.10}$$

$$\begin{aligned} &\nabla_{\xi(\alpha)}\eta_{(\beta)}=-\nabla_{\xi(\beta)}\eta_{(\alpha)}=\eta_{(\gamma)}, \\ &\nabla_{\xi(\alpha)}\xi_{(\beta)}=-\nabla_{\xi(\beta)}\xi_{(\alpha)}=\xi_{(\gamma)}, \qquad [\alpha, \beta, \gamma: cyclic] \end{aligned} \tag{2.11}$$

$$\begin{aligned} &L_{\xi(\alpha)}\phi_{(\beta)}=-L_{\xi(\beta)}\phi_{(\alpha)}=2\phi_{(\gamma)}, \\ &L_{\xi(\alpha)}\eta_{(\beta)}=-L_{\xi(\beta)}\eta_{(\alpha)}=2\eta_{(\gamma)}, \qquad [\alpha, \beta, \gamma: cyclic] \end{aligned} \tag{2.12}$$

where L_X denotes the Lie derivation by X. Next, we define an operator L by

$$L=\sum_\alpha L_{\xi(\alpha)}L_{\xi(\alpha)}\,.$$

The restriction of L (for functions) to each integral submanifold (which is isometric to the 3-sphere) is identical with the usual Laplacian Δ acting on functions on (S^3, g_0). Thus, we have the following (cf. Tanno [7]):

PROPOSITION 2.1. *For a non-negative integer k, the eigenspace V_k corresponding to the k-th eigenvalue of the Laplacian Δ acting on functions on (S^{4r+3}, g_0) has the orthogonal decomposition;*

$$V_k=W_{k,k}+W_{k,k-2}+\cdots+W_{k,k-2[k/2]}$$

such that $f\in W_{k,\theta}$ satisfies

$$Lf=\theta(\theta+2)f\,.$$

§ 3. Riemannian metrics $g(t)$.

We define $\{g(t); 0<t<\infty\}$ such that $g(1)=g_0$ on S^{4r+3} by

$$g(t)=t^{-1}g_0+t^{-1}(t^{m/3}-1)\sum_\alpha \eta_{(\alpha)}\otimes\eta_{(\alpha)}\,. \tag{3.1}$$

For simplicity we denote $g(t)$ by $\tilde{g}$, and g_0 by g in the following calculation. So, in the local coordinate expression we have

$$\tilde{g}_{jk}=t^{-1}g_{jk}+t^{-1}(t^{m/3}-1)\sum_\alpha \eta_{(\alpha)j}\eta_{(\alpha)k}\,, \tag{3.1}$$

$$\tilde{g}^{ij}=tg^{ij}-t(1-t^{-m/3})\textstyle\sum_\alpha \xi_{(\alpha)}{}^i\xi_{(\alpha)}{}^j . \tag{3.2}$$

LEMMA 3.1. *The difference W^i_{jk} of the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ and Γ^i_{jk} with respect to $g(t)$ and g_0 is given by*

$$W^i_{jk}=-(t^{m/3}-1)\textstyle\sum_\alpha (\phi_{(\alpha)}{}^i_j\eta_{(\alpha)}{}_k+\phi_{(\alpha)}{}^i_k\eta_{(\alpha)}{}_j) . \tag{3.3}$$

Proof. By a classical formula we have

$$\begin{aligned} W^i_{jk}&=\tilde{\Gamma}^i_{jk}-\Gamma^i_{jk} \\ &=(1/2)\tilde{g}^{ir}(\nabla_j\tilde{g}_{rk}+\nabla_k\tilde{g}_{rj}-\nabla_r\tilde{g}_{jk}) . \end{aligned}$$

Substituting (3.1) and (3.2) into the above, and using (2.2), etc., we have (3.3).

LEMMA 3.2. *The Riemannian curvature tensor $\tilde{R}=R_{(t)}$ of $\tilde{g}=g(t)$ is given by*

$$\begin{aligned} \tilde{R}^i_{jkl}=R^i_{jkl}&+(1-t^{m/3})\textstyle\sum_\alpha\{2\phi_{(\alpha)}{}^i_j\phi_{(\alpha)}{}_{kl}-\phi_{(\alpha)}{}^i_l\phi_{(\alpha)}{}_{jk}+\phi_{(\alpha)}{}^i_k\phi_{(\alpha)}{}_{jl} \\ &+\xi_{(\alpha)}{}^i(g_{jk}\eta_{(\alpha)}{}_l-g_{jl}\eta_{(\alpha)}{}_k)-2\eta_{(\alpha)}{}_j(\delta^i_k\eta_{(\alpha)}{}_l-\delta^i_l\eta_{(\alpha)}{}_k)\} \\ &+(1-t^{m/3})^2\textstyle\sum_\alpha \eta_{(\alpha)}{}_j(\delta^i_k\eta_{(\alpha)}{}_l-\delta^i_l\eta_{(\alpha)}{}_k) \\ &+(1-t^{m/3})^2\textstyle\sum_{[cyclic]}\{2\phi_{(\alpha)}{}^i_j(\eta_{(\beta)}{}_k\eta_{(\gamma)}{}_l-\eta_{(\gamma)}{}_k\eta_{(\beta)}{}_l) \\ &+\phi_{(\alpha)}{}^i_l(\eta_{(\beta)}{}_k\eta_{(\gamma)}{}_j-\eta_{(\gamma)}{}_k\eta_{(\beta)}{}_j)-\phi_{(\alpha)}{}^i_k(\eta_{(\beta)}{}_l\eta_{(\gamma)}{}_j-\eta_{(\gamma)}{}_l\eta_{(\beta)}{}_j) \\ &+2(\xi_{(\alpha)}{}^i\eta_{(\beta)}{}_j-\xi_{(\beta)}{}^i\eta_{(\alpha)}{}_j)(\eta_{(\alpha)}{}_k\eta_{(\beta)}{}_l-\eta_{(\beta)}{}_k\eta_{(\alpha)}{}_l)\} . \end{aligned} \tag{3.4}$$

Proof. We calculate the following:

$$\tilde{R}^i_{jkl}=R^i_{jkl}+\nabla_kW^i_{lj}-\nabla_lW^i_{kj}+W^s_{lj}W^i_{ks}-W^s_{kj}W^i_{ls} .$$

By (2.2), (2.7), etc., and (3.3) we have

$$\begin{aligned} \nabla_kW^i_{lj}-\nabla_lW^i_{kj}=(1-t^{m/3})\textstyle\sum_\alpha&\{2\phi_{(\alpha)}{}^i_j\phi_{(\alpha)}{}_{kl}-\phi_{(\alpha)}{}^i_l\phi_{(\alpha)}{}_{jk}+\phi_{(\alpha)}{}^i_k\phi_{(\alpha)}{}_{jl} \\ &+\xi_{(\alpha)}{}^i(g_{jk}\eta_{(\alpha)}{}_l-g_{jl}\eta_{(\alpha)}{}_k)-2\eta_{(\alpha)}{}_j(\delta^i_k\eta_{(\alpha)}{}_l-\delta^i_l\eta_{(\alpha)}{}_k)\} . \end{aligned}$$

As for $W^s_{lj}W^i_{ks}$, we calculate it directly as

$$W^s_{lj}W^i_{ks}=(1-t^{m/3})^2\{\textstyle\sum_\alpha(\phi_{(\alpha)}{}^s_l\eta_{(\alpha)}{}_j+\phi_{(\alpha)}{}^s_j\eta_{(\alpha)}{}_l)\}\sum_\beta(\phi_{(\beta)}{}^i_k\eta_{(\beta)}{}_s+\phi_{(\beta)}{}^i_s\eta_{(\beta)}{}_k) .$$

Summing up the results, proof is completed.

LEMMA 3.3. *The Ricci curvature tensor $(\tilde{R}_{jl})$ and the scalar curvature $\tilde{S}=S_{(t)}$ of $(S^{4r+3}, g(t))$ are given by*

$$\begin{aligned} \tilde{R}_{jl}&=R_{jl}+6(1-t^{m/3})g_{jl} \\ &\quad+(1-t^{m/3})[-2m+(m-3)(1-t^{m/3})]\textstyle\sum_\alpha \eta_{(\alpha)}{}_j\eta_{(\alpha)}{}_l , \end{aligned} \tag{3.5}$$

$$\tilde{S}=tS-3t(1-t^{-m/3})[2+(m-3)t^{m/3}]\,. \tag{3.6}$$

The Ricci operator $\tilde{Q}=Q_{(t)}$ *is given by*

$$\tilde{R}_l^h=t[m-1+6(1-t^{m/3})]\delta_l^h \tag{3.7}$$
$$-t(1-t^{-m/3})[2-(m+3)t^{m/3}]\textstyle\sum_\alpha \xi_{(\alpha)}{}^h\eta_{(\alpha)l}\,.$$

Proof. By contraction $\tilde{R}_{jil}^i$ we have $\tilde{R}_{jl}$. $\tilde{S}$ is given by $\tilde{g}^{jl}\tilde{R}_{jl}$. And, $\tilde{R}_l^h=\tilde{g}^{hj}\tilde{R}_{jl}$.

LEMMA 3.4. *Each* $\xi_{(\alpha)}$ *is a Killing vector field with respect to* $g(t)$.

Proof. It suffices to show that $\xi_{(1)}$ is a Killing vector field with respect to $g(t)$. $L_{\xi_{(1)}}g=L_{\xi_{(1)}}\eta_{(1)}=0$ is trivial. By (2.12) we have

$$L_{\xi_{(1)}}(\eta_{(2)}\otimes\eta_{(2)})=2(\eta_{(3)}\otimes\eta_{(2)}+\eta_{(2)}\otimes\eta_{(3)}),$$
$$L_{\xi_{(1)}}(\eta_{(3)}\otimes\eta_{(3)})=-2(\eta_{(2)}\otimes\eta_{(3)}+\eta_{(3)}\otimes\eta_{(2)})\,.$$

Since $g(t)$ is given by (3.1), proof is completed.

We define two operators Φ and Ψ acting on 1-forms by the following:

$$\Phi(w)=\textstyle\sum_\alpha \phi_{(\alpha)}{}^{rs}\nabla_r w_s\cdot\eta_{(\alpha)}\,,$$
$$\Psi(w)=\textstyle\sum_{(\alpha,\beta,\gamma)} sgn\begin{pmatrix}1 & 2 & 3\\ \alpha & \beta & \gamma\end{pmatrix}(\nabla_{\xi_{(\alpha)}}w)(\xi_{(\beta)})\eta_{(\gamma)}\,,$$

where sgn$(\cdots)$ denotes the signature of permutation.

PROPOSITION 3.5. *The Laplacian* $\tilde{\Delta}=\Delta_{(t)}$ *for functions and 1-forms are given by*

$$\Delta_{(t)}f=t\Delta f-t(1-t^{-m/3})Lf\,, \tag{3.8}$$

$$\Delta_{(t)}w=t\Delta w-t(1-t^{-m/3})Lw+2t(1-t^{m/3})\Phi(w) \tag{3.9}$$
$$+2t(1-t^{-m/3})(1-t^{m/3})\Psi(w)\,.$$

Proof. (3.8) was proved in [7]. To show (3.9), we calculate the following:

$$\begin{aligned}\tilde{\Delta}w_i&=\tilde{g}^{rs}\tilde{\nabla}_r\tilde{\nabla}_s w_i-\tilde{R}_i^s w_s\\ &=\tilde{g}^{rs}\tilde{\nabla}_r(\nabla_s w_i-W_{si}^h w_h)-\tilde{R}_i^s w_s\\ &=\tilde{g}^{rs}(\nabla_r\nabla_s w_i-\nabla_r W_{si}^h w_h-W_{si}^h\nabla_r w_h-W_{rs}^q\nabla_q w_i\\ &\quad+W_{rs}^q W_{qi}^h w_h-W_{ri}^q\nabla_s w_q+W_{ri}^q W_{sq}^h w_h)-\tilde{R}_i^s w_s\,.\end{aligned}$$

First we check the following relation for each α;

$$L_{\xi(\alpha)}L_{\xi(\alpha)}w=\nabla_{\xi(\alpha)}\nabla_{\xi(\alpha)}w-2\nabla_{\xi(\alpha)}w\cdot\phi_{(\alpha)}-w+w(\xi_{(\alpha)})\eta_{(\alpha)}\,. \tag{3.10}$$

Then we have

$$\tilde{g}^{rs}\nabla_r\nabla_s w_i=t\nabla_r\nabla^r w_i-t(1-t^{-m/3})\{3w_i+(Lw)_i$$

$$+\textstyle\sum_\alpha(2\nabla_{\xi(\alpha)}w_h\cdot\phi_{(\alpha)}{}^h_i-w(\xi_{(\alpha)})\eta_{(\alpha)i})\}\,,$$

$$\tilde{g}^{rs}\nabla_r W^h_{si}w_h=t(1-t^{m/3})\{-3w_i+m\textstyle\sum_\alpha w(\xi_{(\alpha)})\eta_{(\alpha)i}\}\,,$$

$$\tilde{g}^{rs}W^h_{si}\nabla_r w_h=t(1-t^{m/3})\{\textstyle\sum_\alpha\nabla_{\xi(\alpha)}w_h\cdot\phi_{(\alpha)}{}^h_i-(\Phi(w))_i\}$$

$$-t(1-t^{m/3})(1-t^{-m/3})\{\textstyle\sum_\alpha\nabla_{\xi(\alpha)}w_h\cdot\phi_{(\alpha)}{}^h_i+(\Psi(w))_i\}\,,$$

$$\tilde{g}^{rs}W^q_{rs}\nabla_q w_i=\tilde{g}^{rs}W^q_{rs}W^h_{qi}w_h=0\,,$$

$$\tilde{g}^{rs}W^q_{ri}W^h_{sq}w_h=-3t(1-t^{m/3})^2t^{-m/3}(w_i-\textstyle\sum_\alpha w(\xi_{(\alpha)})\eta_{(\alpha)i})\,.$$

Summing up the above and using (3.7), we obtain (3.9).

LEMMA 3.6. *For each α, $\eta_{(\alpha)}$ is an eigenform of $\Delta_{(t)}$;*

$$\Delta_{(t)}\eta_{(\alpha)}=-2t[(m-1)t^{m/3}-2(1-t^{-m/3})(1+t^{m/3})]\eta_{(\alpha)}\,. \tag{3.11}$$

Proof. It is known that $\Delta\eta_{(\alpha)}=-2(m-1)\eta_{(\alpha)}$ holds. Furthermore we have

$$L\eta_{(\alpha)}=-8\eta_{(\alpha)}\,,$$

$$\Phi(\eta_{(\alpha)})=(m-1)\eta_{(\alpha)}\,,$$

$$\Psi(\eta_{(\alpha)})=-2\eta_{(\alpha)}\,.$$

By (3.9) we obtain (3.11).

Remark. If one wants to obtain geometric expressions of eigen 1-forms of the Laplacian $\Delta_{(t)}$ of $(S^{4r+3}, g(t))$, then the decomposition of V_k given by Proposition 2.1 and the expression (3.9) of $\Delta_{(t)}$ are helpful (cf. Tanno [8]).

§4. The Jacobi operator $J_{(t)}$.

The Jacobi operator $J_{(t)}$ acting on 1-forms on $(S^{4r+3}, g(t))$ is given by

$$J_{(t)}=-\Delta_{(t)}-2Q_{(t)} \tag{4.1}$$

and the local coordinate expression is $(J_{(t)}w)_i=-(\Delta_{(t)}w)_i-2\tilde{R}^h_iw_h$ (cf. Smith [5]). The Jacobi operator $J_{(t)}$ for vector fields is understood by the natural correspondence between the space of 1-forms and the space of vector fields. In the following we use $J_{(t)}$ for 1-forms.

Putting $P(w)=\sum_\alpha w(\xi_{(\alpha)})\eta_{(\alpha)}$, we have

$$J_{(t)}w=-t\Delta w+t(1-t^{-m/3})Lw-2t(1-t^{m/3})\Phi(w) \\ -2t(1-t^{-m/3})(1-t^{m/3})\Psi(w)-2t[m-1+6(1-t^{m/3})]w \\ +2t(1-t^{-m/3})[2-(m+3)t^{m/3}]P(w)\,. \tag{4.2}$$

PROPOSITION 4.1. *For each α, we have $J_{(t)}\eta_{(\alpha)}=0$.*

Proof. This follows from the fact that $\xi_{(\alpha)}$ is a Killing vector field and Killing vector fields belong to the null eigenspace of $J_{(t)}$. Also, the direct calculation using (3.7) and (3.11) is easy. q.e.d.

Let V_1 be the eigenspace corresponding to the first eigenvalue $\lambda_1=4r+3$ of the Laplacian acting on functions on $(S^{4r+3}, g(t))$, and let $f\in V_1$. Then f satisfies $\nabla_j\nabla_i f=-fg_{ij}$. Therefore, we have

$$\xi_{(\alpha)}\xi_{(\alpha)}f=-f\,, \tag{4.3}$$

$$\xi_{(\alpha)}\xi_{(\beta)}f=\xi_{(\gamma)}f\,. \tag{4.4}$$

LEMMA 4.2. *Let $f\in V_1$. Then we have the following:*

(1-i) $\Delta(f\eta_{(1)})=-(3m-2)f\eta_{(1)}+2df\cdot\phi_{(1)}\,,$

(1-ii) $L(f\eta_{(1)})=-11f\eta_{(1)}+4((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(1-iii) $\Phi(f\eta_{(1)})=(m-1)f\eta_{(1)}-((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(1-iv) $\Psi(f\eta_{(1)})=-2f\eta_{(1)}+((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(2-i) $\Delta(df\cdot\phi_{(1)})=2(m-1)f\eta_{(1)}-(m+2)df\cdot\phi_{(1)},$

(2-ii) $L(df\cdot\phi_{(1)})=8f\eta_{(1)}-3df\cdot\phi_{(1)}-4((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(2-iii) $\Phi(df\cdot\phi_{(1)})=-(m-1)f\eta_{(1)}+((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(2-iv) $\Psi(df\cdot\phi_{(1)})=2f\eta_{(1)}-((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(3-i) $\Delta((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})=4f\eta_{(1)}+4df\cdot\phi_{(1)}-3m((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(3-ii) $L((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})=8f\eta_{(1)}-7((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(3-iii) $\Phi((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})=-2f\eta_{(1)}+(m-2)((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,,$

(3-iv) $\Psi((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})=2f\eta_{(1)}-((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})\,.$

Proof. Verification is done by a direct calculation using (2.2)~(2.12).

LEMMA 4.3. *With respect to the projection P, we have*

$$P(df\cdot\phi_{(1)})=(\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)},$$

$P(f\eta_{(1)})=f\eta_{(1)}$ *and* $P((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)})=(\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)}$.

Proof. The first identity is verified by (2.8).

PROPOSITION 4.4. *For* $f\in V_1$, $w_{[f,t]}$ *is an eigenform of* $J_{(t)}$ *corresponding to* $\mu(t)$ *for each* t $(0<t<\infty)$, *where* $w_{[f,t]}$ *is defined by*

$$w_{[f,t]}=f\eta_{(1)}+k(t)df\cdot\phi_{(1)}+(1-k(t))((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)}), \tag{4.5}$$

$$k(t)=[2(m-3)t^{m/3}]^{-1}\{m+2-2t^{-m/3}-6t^{m/3} \tag{4.6}$$
$$+[(m+2-2t^{-m/3}-6t^{m/3})^2+12(m-3)t^{m/3}]^{1/2}\},$$

and $\mu(t)$ *is given by*

$$\mu(t)=t[-5+t^{-m/3}+6t^{m/3}] \tag{4.7}$$
$$-t[(m+2-2t^{-m/3}-6t^{m/3})^2+12(m-3)t^{m/3}]^{1/2}.$$

For $m=4r+3=7$, 11, $\mu(t)$ *is negative for all* t $(0<t<\infty)$. *For* $m\geqq 15$, $\mu(t)$ *is negative, if* $0<t<t_0(m)$ *or* $t_1(m)<t$, *where*

$$3(t_0(m))^{-m/3}=2m-1+[(m-2)(m-14)]^{1/2}, \tag{4.8}$$

$$3(t_1(m))^{-m/3}=2m-1-[(m-2)(m-14)]^{1/2}. \tag{4.9}$$

Proof. We define $w_{[f,t]}$ by (4.5) with undetermined $k(t)$. By (4.2), Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}J_{(t)}w_{[f,t]}=&\,t\{m-3-t^{-m/3}-2(m-3)t^{m/3}k(t)\}f\eta_{(1)}\\&+t\{-6-(m+7-3t^{-m/3}-12t^{m/3})k(t)\}df\cdot\phi_{(1)}\\&+t\{m+3-t^{-m/3}\\&+(m+7-2(m+3)t^{m/3}-3t^{-m/3})k(t)\}((\xi_{(3)}f)\eta_{(2)}-(\xi_{(2)}f)\eta_{(3)}).\end{aligned}$$

Therefore, $J_{(t)}w_{[f,t]}=\mu(t)w_{[f,t]}$ holds, if $k(t)$ is a solution of

$$(m-3)t^{m/3}k(t)^2-(m+2-2t^{-m/3}-6t^{m/3})k(t)-3=0,$$

and $\mu(t)$ is given by

$$\mu(t)=t[m-3-t^{-m/3}-2(m-3)t^{m/3}k(t)].$$

Because we are interested in the case where $\mu(t)$ is negative, we choose $k(t)$ as the positive solution of the above equation. So, $k(t)$ is given by (4.6), and consequently, $\mu(t)$ is equal to (4.7). t satisfies $\mu(t)=0$, if and only if

$$3t^{-2m/3}-2(2m-1)t^{-m/3}+m^2+4m-9=0 .$$

Therefore, two solutions $t_0(m)$ and $t_1(m)$ of the above equation are given by (4.8) and (4.9).

Remark. The values $t_0(m)$ and $t_1(m)$ for $m=15$, 19 are given by

$$t_0(15)=0.6205\cdots, \qquad t_1(15)=0.6523\cdots,$$

$$t_0(19)=0.6493\cdots, \qquad t_1(19)=0.7036\cdots.$$

PROPOSITION 4.5. *Let $\Omega_{(t)}$ denote the eigenspace of $J_{(t)}$ corresponding to $\mu(t)$, and let $\Omega'_{(t)}=\{w_{[f,t]}; f\in V_1\}$. Then $\Omega_{(t)}=\Omega'_{(t)}$ and $\dim \Omega_{(t)}=m+1$ hold except for at most countably many values of t in $(0, \infty)$.*

Proof. Let $\{f_1, f_2, \cdots, f_{m+1}\}$ be a basis of V_1 and define $w_{[f_\rho, t]}$ by (4.5) with $f=f_\rho$. For each t, the set

$$\{w_{[f_\rho, t]}; \rho=1, 2, \cdots, m+1\}$$

is linearly independent. In fact, it suffices to see that the set $\{w_{[f_\rho, t]}(\eta_{(1)})\}=\{f_\rho\}$ is linearly independent. So, we have $\dim \Omega'_{(t)}=m+1$. For all t, $\Omega_{(t)}\supset\Omega'_{(t)}$ is trivial. At $t=1$, we see that $\dim \Omega_{(1)}=m+1=\dim \Omega'_{(1)}$. Next, if t is near 1, then $\Omega_{(t)}=\Omega'_{(t)}$, and $\dim \Omega_{(t)}=m+1$. Since $J_{(t)}$ depends on t analytically, the case $\dim \Omega_{(t)}>m+1$ happens only for at most countably many values of t.

PROPOSITION 4.6. *For each α, $L_{\xi_{(\alpha)}}$ defines an isomorphism of $\Omega'_{(t)}$.*

Proof. Since $\xi_{(\alpha)}$ is a Killing vector field with respect to $g(t)$, two operators $L_{\xi_{(\alpha)}}$ and $J_{(t)}$ are commutative. Thus, $L_{\xi_{(\alpha)}}$ preserves $\Omega_{(t)}$. Since $\Omega_{(t)}=\Omega'_{(t)}$ except for at most countably many values of t, $L_{\xi_{(\alpha)}}$ preserves $\Omega'_{(t)}$.

Remark. The expression (4.5) of $w_{[f,t]}$ is based on $\xi_{(1)}$ and $\phi_{(1)}$. Contrary to this, we define $w'_{[f,t]}$ and $w''_{[f,t]}$ by

$$w'_{[f,t]}=f\eta_{(2)}+k(t)df\cdot\phi_{(2)}+(1-k(t))((\xi_{(1)}f)\eta_{(3)}-(\xi_{(3)}f)\eta_{(1)}), \tag{4.10}$$

$$w''_{[f,t]}=f\eta_{(3)}+k(t)df\cdot\phi_{(3)}+(1-k(t))((\xi_{(2)}f)\eta_{(1)}-(\xi_{(1)}f)\eta_{(2)}). \tag{4.11}$$

Then we have the following relations:

$$L_{\xi_{(1)}}w_{[f,t]}=w_{[\xi_{(1)}f,t]},$$

$$L_{\xi_{(2)}}w_{[f,t]}=w_{[\xi_{(2)}f,t]}-2w''_{[f,t]},$$

$$L_{\xi_{(3)}}w_{[f,t]}=w_{[\xi_{(3)}f,t]}+2w'_{[f,t]}.$$

Since $L_{\xi_{(\alpha)}}w_{[f,t]}$ and $w_{[\xi_{(\alpha)}f,t]}$ belong to $\Omega'_{(t)}$, we see that $w'_{[f,t]}$ and $w''_{[f,t]}$ belong to $\Omega'_{(t)}$. This means that the expression (4.5) based on $\xi_{(1)}$ and $\phi_{(1)}$ is enough for our purpose.

Remark. Let X be a unit tangent vector at a point x of $(S^{4r+3}, g(t))$ satisfying $\eta_{(\alpha)}(X)=0$ for $\alpha=1, 2, 3$, then the sectional curvature $K_{(t)}(X, \phi_{(1)}X)$ is given by

$$K_{(t)}(X, \phi_{(1)}X)=t(4-3t^{m/3}).$$

So, it takes a negative value for $4<3t^{m/3}$.

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