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ON THE MINIMAL SUBMANIFOLDS IN $CP^m(c)$ AND $S^N(1)$

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Abstract

Let M be an n-dimensional compact totally real submanifold minimally immersed in $CP^m(c)$. Let σ be the second fundamental form of M. A known result states that if $m=n$ and $|\sigma|^2 \leqq (n(n+1)c)/(4(2n-1))$, then M is either totally geodesic or a finite Riemannian covering of the unique flat torus minimally imbedded in $CP^2(c)$. In this paper, we improve the above pinching constant to $(n+1)c/6$ and prove a pinching theorem for $|\sigma|^2$ without the assumption on the codimension. We have also some pinching theorems for $\delta(u) := |\sigma(u, u)|^2$, $u \in UM$, $M \to CP^m(c)$ and the Ricci curvature of a minimal submanifold in a sphere. In particular, a simple proof of a Gauchman's result is given.

1. Introduction.

Let M be an n-dimensional compact submanifold minimally immersed in a complex projective space $CP^m(c)$ of holomorphic sectional curvature c and of complex dimension m. Denote by σ the second fundamental form of M. Chen and Ogiue ([1]), Naitoh and Takeuchi ([7]), and Yau ([13]) proved that if M is totally real, $m=n$ and $|\sigma|^2 \leqq (n(n+1)c)/(4(2n-1))$, then M is either totally geodesic or a finite Riemannian covering of the unique flat torus minimally imbadded in $CP^2(c)$ with parallel second fundamental form. In this paper, by using a method different from those in [1], [7] and [13], we improve the above result and prove a pinching theorem for $|\sigma|^2$ without the assumption on the codimension of M. Namely, we have

THEOREM 1. *Let M be an n-dimensional compact totally real minimal submanifold in $CP^n(c)$. Let σ be the second fundamental form of M. If $|\sigma|^2 \leqq (n+1)c/6$, then M is either totally geodesic or a finite Riemannian covering of the unique flat torus embedded in $CP^2(c)$ with parallel second fundamental form.*

THEOREM 2. *Let M be an n-dimensional compact totally real minimal submanifold immersed in $CP^m(c)$. If $|\sigma|^2 \leqq nc/6$, then either M is totally geodesic or the immersion of M into $CP^m(c)$ is one of the following immersions*

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$\varphi_{1,p}: RP^2(c/12)\to CP^{4+p}(c)$; $\varphi_{2,p}: S^2(c/12)\to CP^{4+p}(c)$ $(p=0, 1, 2, \cdots)$.

A. Ros in [10] showed that if M is a compact Kaehler submanifold of $CP^m(c)$ and if $\delta(n)=:|\sigma(u, u)|^2<c/4$ for any $u\in UM$, then M is totally geodesic. Moreover, in [9], Ros gave a complete list of Kaehler submanifolds of $CP^m(c)$ satisfying the condition $\max_{u\in UM}\delta(u)=c/4$. In [3], H. Gauchman obtained the following analogous result for totally real minimal submanifold in $CP^m(c)$.

THEOREM 3. *Let M be an n-dimensional compact totally real minimal submanifold immersed in $CP^m(c)$. Then, $\delta(u)=:|\sigma(u, u)|^2\leqq c/12$ for any $u\in UM$ if and only if one of the following conditions is satisfied:*

i) $\delta\equiv 0$ *(i.e., M is totally geodesic).*

ii) $\delta\equiv c/12$ *and the immersion of M into $CP^m(c)$ is one of the following immersions*: $\varphi_{1,p}: RP^2(c/12)\to CP^{4+p}(c)$; $\varphi_{2,p}: S^2(c/12)\to CP^{4+p}(c)$; $\varphi_{3,p}: CP^2(c/3)\to CP^{7+p}(c)$; $\varphi_{4,p}: QP^2(c/3)\to CP^{13+p}(c)$; $\varphi_{5,p}: \mathrm{Cay}\,P^2(c/3)\to CP^{25+p}(c)(p=0, 1, 2, \cdots)$.

For the definitions of $\varphi_{i,p}(i=1, \cdots, 5; p=0, 1, 2, \cdots)$, one can consult [3, p. 254].

In this paper, we'll give a simple proof of the above Gauchman's result and prove the following.

THEOREM 4. *Let M be an n-dimensional compact totally real minimal submanifold immersed in a complex projective space $CP^m(c)$. Assume that n is odd. If $\delta(u)\leqq c/4(3-2/n)$ for all $u\in UM$, then M is totally geodesic.*

Theorem 4 improves a result by H. Gauchman in [3].

For minimal submanifolds in a sphere, we have

THEOREM 5. *Let M be an n-dimensional compact minimal submanifold immersed in a unit sphere $S^{n+p}(1)$. Let A_ξ be the Weingarten endomorphism associated to a normal vector ξ. Define $T: T_p^\perp M\times T_p^\perp M\to R$ by $T(\xi, \eta)=\mathrm{trace}\, A_\xi A_\eta$. Assume that the Ricci curvature of M satisfies $\mathrm{Ric}_M\geqq n-1-((n+2)p)/(2(n+p+2))$ and $T=k\langle\, ,\rangle$. Then the immersion of M into $S^{n+p}(1)$ is one of the following standard ones (see [11] for details)* $S^n(1)\to S^n(1)$; $RP^2(1/3)\to S^4(1)$; $S^2(1/3)\to S^4(1)$; $CP^2(4/3)\to S^7(1)$; $QP^2(4/3)\to S^{13}(1)$; $\mathrm{Cay}\,P^2(4/3)\to S^{25}(1)$.

2. Preliminaries.

Let M be an n-dimensional compact Riemannian manifold. We denote by UM the unit tangent bundle over M and by UM_p its fiber over $p\in M$. If dp, dv and dv_p denote the canonical measures on M, UM and UM_p respectively, then for any continuous function $f: UM\to R$, we have:

$$\int_{UM} f\, dv=\int_M\left\{\int_{UM_p} f\, dv_p\right\}dp\,.$$

Now, we suppose that M is isometrically immersed in an $(n+p)$-dimensional

Riemannian manifold $\bar{M}$. We denote by $\langle\, ,\rangle$ the metric of $\bar{M}$ as well as that induced on M. If σ is the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated to a normal vector ξ, we define

$$L: T_pM \longrightarrow T_pM \quad \text{and} \quad T: T_p^\perp M \times T_p^\perp M \longrightarrow R$$

by the expressions

$$Lv = \sum_{i=1}^n A_{\sigma(v, e_i)} e_i \quad \text{and} \quad T(\xi, \eta) = \text{trace}\, A_\xi A_\eta ,$$

where $T_p^\perp M$ is the normal space to M at p and $e_1, \cdots, e_n$ is an orthonormal basis of T_pM. M is called a curvature-invariant submanifold of $\bar{M}$, if $\bar{R}(X, Y)Z \in T_pM$ for all $X, Y, Z \in T_pM$, being $\bar{R}$ the curvature operator of $\bar{M}$. Then, if $\nabla\sigma$ and $\nabla^2\sigma$ denote the first and second covariant derivatives of σ respectively, one has that $\nabla\sigma$ is symmetric and $\nabla^2\sigma$ satisfies the following relation

$$(2.1) \qquad \begin{aligned}(\nabla^2\sigma)(X, Y, Z, W) = (\nabla^2\sigma)(Y, X, Z, W) + R^\perp(X, Y)\sigma(Z, W)\\ -\sigma(R(X, Y)Z, W) - \sigma(Z, R(X, Y)W)\end{aligned}$$

where $R^\perp$ and R are the curvature operators of the normal and tangent bundles over M respectively.

If Ric is the Ricci tensor of M and M is minimanlly immersed in $\bar{M}$, we have from the Gauss equation

$$(2.2) \qquad \text{Ric}(v, w) = \sum_{i=1}^n \bar{R}(v, e_i, e_i, w) - \langle Lv, w\rangle .$$

LEMMA 1 ([4]) *Let M be an n-dimensional compact minimal curvature-invariant submanifold isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\bar{M}$. Then*

$$(2.3) \qquad \begin{aligned} 0 = {}& \frac{n+4}{3}\int_{UM} |(\nabla\sigma)(v, v, v)|^2 dv + (n+4)\int_{UM} |A_{\sigma(v,v)}v|^2 dv \\ & -4\int_{UM} \langle Lv, A_{\sigma(v,v)}v\rangle dv - 2\int_{UM} T(\sigma(v, v), \sigma(v, v)) dv \\ & + \int_{UM} \sum_{i=1}^n \{\bar{R}(e_i, v, \sigma(v, e_i), \sigma(v, v)) + 2\bar{R}(e_i, v, v, A_{\sigma(v, e_i)}v)\}\, dv . \end{aligned}$$

LEMMA 2 ([4]) *Let M be an n-dimensional compact minimal submanifold isometrically immersed in a Riemannian manifold $\bar{M}$. Then, for any $p \in M$, we have*

$$(2.4) \qquad \int_{UM_p} \langle Lv, A_{\sigma(v,v)}v\rangle dv_p = \frac{2}{n+2}\int_{UM_p} |Lv|^2 dv_p$$

$$(2.5) \qquad \int_{UM_p} |\sigma(v, v)|^2 dv_p = \frac{2}{n+2}\int_{UM_p} \langle Lv, v\rangle dv_p = \frac{2}{n(n+2)}\int_{UM_p} |\sigma|^2 dv_p .$$

3. Maximal directions.

Let M be an n-dimensional compact curvature-invariant submanifold minimally immersed in $\bar{M}$. Define $S=\{(u, v) \mid u, v \in UM_p, p \in M\}$ and a function f on S by

$$f(u, v)=|\sigma(u, u)-\sigma(v, v)|^2. \tag{3.1}$$

For any $p \in M$, we can take $(\bar{u}, \bar{v}) \in UM_p \times UM_p$ with $\langle \bar{u}, \bar{v} \rangle=0$, such that $f(\bar{u}, \bar{v})=\max_{(u,v)\in UM_p\times UM_p} f(u, v)$. We shall call such a pair $(\bar{u}, \bar{v})$ a maximal direction at p. To see this, we assume that $\max_{(u,v)\in UM_p\times UM_p} f(u, v)\neq 0$, since otherwise it would be obvious. Let $(u_1, u_2) \in UM_p \times UM_p$ be such that $f(u_1, u_2)=\max_{(u,v)\in UM_p\times UM_p} f(u, v)$. Set $\xi=\dfrac{\sigma(u_1, u_1)-\sigma(u_2, u_2)}{|\sigma(u_1, u_1)-\sigma(u_2, u_2)|}$ and take an orthonormal basis $e_1, \cdots, e_n$ of T_pM which diagonalizes A_ξ. Let $\langle A_\xi e_i, e_i \rangle=\lambda_i$, $i=1, \cdots, n$ and assume further that $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$. Then, we have

$$u_1=\sum_{i=1}^n x_i e_i, \quad u_2=\sum_{i=1}^n y_i e_i, \quad \sum_{i=1}^n x_i^2=1, \quad \sum_{i=1}^n y_i^2=1,$$

and

$$\begin{aligned} |\sigma(u_1, u_1)-\sigma(u_2, u_2)| &=\langle \sigma(u_1, u_1)-\sigma(u_2, u_2), \xi \rangle \\ &=\sum_{i,j=1}^n (x_i x_j-y_j y_i)\langle \sigma(e_i, e_j), \xi \rangle \\ &=\sum_{i=1}^n (x_i^2-y_i^2)\lambda_i \leqq \lambda_1-\lambda_n \\ &=\langle \sigma(e_1, e_1)-\sigma(e_n, e_n), \xi \rangle \\ &\leqq |\sigma(e_1, e_1)-\sigma(e_n, e_n)| \\ &\leqq |\sigma(u_1, u_1)-\sigma(u_2, u_2)|. \end{aligned} \tag{3.2}$$

Thus, (e_1, e_n) is a maximal direction at p. Also, we have $\sigma(e_1, e_1)-\sigma(e_n, e_n)=|\sigma(e_1, e_1)-\sigma(e_n, e_n)|\xi$.

LEMMA 3. *Let $p \in M$ and assume that $\max_{(u,v)\in UM_p\times UM_p} f(u, v)\neq 0$. Take an orthonormal basis $e_1, \cdots, e_n$ of T_pM such that (e_1, e_n) is a maximal direction at p, $e_1, \cdots, e_n$ diagonalizes A_ξ, $\xi=\dfrac{\sigma(e_1, e_1)-\sigma(e_n, e_n)}{|\sigma(e_1, e_1)-\sigma(e_n, e_n)|}$ and that $\lambda_1=:\langle \sigma(e_1, e_1), \xi \rangle \geqq \lambda_2=:\langle \sigma(e_2, e_2), \xi \rangle \geqq \cdots \geqq \lambda_n=:\langle \sigma(e_n, e_n), \xi \rangle$. Then, at the point p, it holds*

$$\sum_{i=1}^n \langle \sigma(e_1, e_1)-\sigma(e_n, e_n), \nabla^2\sigma(e_i, e_i, e_1, e_1)-\nabla^2\sigma(e_i, e_i, e_n, e_n) \rangle \tag{3.3}$$

$$\geqq |\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi)-\bar{R}(e_i, e_n, \sigma(e_i, e_n), \xi)$$

$$+(\lambda_1-\lambda_i)\bar{R}(e_i, e_1, e_1, e_i)-(\lambda_n-\lambda_i)\bar{R}(e_i, e_n, e_n, e_i)\}$$

$$-\frac{3}{2}|\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2\cdot|\sigma|^2.$$

Proof. From (2.1), the minimality of M, and the Gauss and Ricci equations, it follows

$$\sum_{i=1}^{n}\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), \nabla^2\sigma(e_i, e_i, e_1, e_1)\rangle \tag{3.4}$$

$$=|\sigma(e_1, e_1)|-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\langle\xi, R^{\perp}(e_i, e_1)\sigma(e_1, e_i)$$

$$-\sigma(R(e_i, e_1)e_1, e_i)-\sigma(e_1, R(e_i, e_1)e_i)\rangle\}$$

$$=|\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi)+\langle A_{\sigma(e_1, e_i)}A_\xi e_i, e_1\rangle$$

$$-\langle A_\xi A_{\sigma(e_1, e_i)}e_i, e_1\rangle-\langle A_\xi e_i, R(e_i, e_1)e_1\rangle-\langle A_\xi e_1, R(e_i, e_1)e_i\rangle\}$$

$$=|\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi+(\lambda_i-\lambda_1)|\sigma(e_1, e_i)|^2$$

$$+(\lambda_1-\lambda_i)(\bar{R}(e_i, e_1, e_1, e_i)+\langle\sigma(e_i, e_i), \sigma(e_1, e_1)\rangle-|\sigma(e_1, e_i)|^2)\}$$

$$=|\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi)+(\lambda_1-\lambda_i)\bar{R}(e_i, e_1, e_1, e_i)$$

$$+2(\lambda_i-\lambda_1)|\sigma(e_1, e_i)|^2-\lambda_i\langle\sigma(e_1, e_1), \sigma(e_i, e_i)\rangle\}.$$

Similarly, we have

$$\sum_{i=1}^{n}\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), \nabla^2\sigma(e_i, e_i, e_n, e_n)\rangle \tag{3.5}$$

$$=|\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_n, \sigma(e_n, e_i), \xi)+(\lambda_n-\lambda_i)\bar{R}(e_i, e_n, e_n, e_i)$$

$$+2(\lambda_i-\lambda_n)|\sigma(e_n, e_i)|^2-\lambda_i\langle\sigma(e_n, e_n), \sigma(e_i, e_i)\rangle\}.$$

Combining (3.4) and (3.5) and noticing

$$|\sigma(e_1, e_1)-\sigma(e_n, e_n)|=\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), \xi\rangle=\lambda_1-\lambda_n,\ \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n,$$

we have

$$\sum_{i=1}^{n}\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), \nabla^2\sigma(e_i, e_i, e_1, e_1)-\nabla^2\sigma(e_i, e_i, e_n, e_n)\rangle \tag{3,6}$$

$$\geqq |\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi)-\bar{R}(e_i, e_n, \sigma(e_i, e_n), \xi)$$

$$+(\lambda_1-\lambda_i)\bar{R}(e_i, e_1, e_1, e_i)-(\lambda_n-\lambda_i)\bar{R}(e_i, e_n, e_n, e_i)\}$$

$$-|\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2\Big\{\sum_{i=1}^{n}\lambda_i^2+2\Big(\sum_{i=2}^{n}|\sigma(e_1, e_i)|^2+\sum_{i=1}^{n-1}|\sigma(e_n, e_i)|^2\Big)\Big\}.$$

On the other hand, one can easily deduce from $\left|\sigma\left(\frac{e_1+e_n}{\sqrt{2}}, \frac{e_1+e_n}{\sqrt{2}}\right)-\sigma\left(\frac{e_1-e_n}{\sqrt{2}}, \frac{e_1-e_n}{\sqrt{2}}\right)\right|^2 \leqq |\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2$ that

$$|\sigma(e_1, e_n)|^2 \leqq \frac{|\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2}{4} \leqq \frac{|\sigma(e_1, e_1)|^2+|\sigma(e_n, e_n)|^2}{2}. \tag{3.7}$$

Since

$$|\sigma|^2=\sum_{i,j=1}^{n}|\sigma(e_i, e_j)|^2 \tag{3.8}$$

$$\geqq \sum_{i=1}^{n}|\sigma(e_i, e_i)|^2+2\Big(\sum_{i=2}^{n}|\sigma(e_1, e_i)|^2+\sum_{i=2}^{n-1}|\sigma(e_n, e_i)|^2\Big),$$

we have

$$\sum_{i=1}^{n}\lambda_i^2+2\Big(\sum_{i=2}^{n}|\sigma(e_1, e_2)|^2+\sum_{i=1}^{n-1}|\sigma(e_n, e_i)|^2\Big) \tag{3.9}$$

$$\leqq \sum_{i=1}^{n}|\sigma(e_i, e_i)|^2+2\sum_{i=2}^{n}|\sigma(e_1, e_i)|^2+2\sum_{i=2}^{n-1}|\sigma(e_n, e_i)|^2$$

$$+\frac{1}{2}(|\sigma(e_1, e_1)|^2+|\sigma(e_n, e_n)|^2+2|\sigma(e_1, e_n)|^2)$$

$$\leqq |\sigma|^2+\frac{1}{2}|\sigma|^2=\frac{3}{2}|\sigma|^2.$$

Substituting (3.9) into (3.6), we get (3.3). Q.E.D.

4. Proof of Theorem 1 and 2.

Proof of Theorem 1. Let L be a function on M defined by $L(x)=\max_{(u,v)\in UM_x\times UM_x} f(u, v)$. Following an idea in [8] we prove that L is a constant function on M by using the maximum principle. It suffices to show that L is subharmonic in the generalized sense. Fix $p\in M$, let (e_1, e_n) be a maximal direction at p and $e_1, \cdots, e_n$ be an orthonormal basis of T_pM as stated in Lemma 3. From the expression of the curvature tensor of $CP^n(c)$, we have

$$|\sigma(e_1, e_1)-\sigma(e_n, e_n)| \sum_{i=1}^{n} \{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi)-\bar{R}(e_i, e_n, \sigma(e_i, e_n), \xi) \tag{4.1}$$

$$+(\lambda_1-\lambda_i)\bar{R}(e_i, e_1, e_1, e_i)-(\lambda_n-\lambda_i)\bar{R}(e_i, e_n, e_n, e_i)\}$$

$$=\frac{c}{4}\sum_{i=1}^{n}\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), Je_i\rangle^2+\frac{nc}{4}(\lambda_1-\lambda_n)|\sigma(e_1, e_1)-\sigma(e_n, e_n)|$$

$$=\frac{(n+1)c}{4}|\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2.$$

In an open neighborhood U_p of p within the cut-locus of p we shall denote by $E_1(x)$(resp. $E_n(x)$) the tangent vectors to M obtained by parallel transport of $e_1=E_1(p)$(resp. $e_n=E_n(p)$) along the unique geodesic joining x to p within the cut-locus of p. Define $g_p(x)=|\sigma(E_1(x), E_1(x))-\sigma(E_n(x), E_n(x))|^2$. Then,

$$\frac{1}{2}\Delta g_p(p)=\sum_{i=1}^{n}\{|(\nabla\sigma)(e_i, e_1, e_1)-(\nabla\sigma)(e_i, e_n, e_n)|^2 \tag{4.2}$$
$$+\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), (\nabla^2\sigma)(e_i, e_i, e_1, e_1)-(\nabla^2\sigma)(e_i, e_i, e_n, e_n)\rangle\}.$$

If $|\sigma(e_1, e_1)-\sigma(e_n, e_n)|=0$, then $\Delta g_p(p)\geqq 0$ by (4.2). If $|\sigma(e_1, e_1)-\sigma(e_n, e_n)|\neq 0$, then by (4.1), (4.2), Lemma 3 and the hypothesis on $|\sigma|^2$, we have

$$\frac{1}{2}\Delta g_p(p)\geqq|\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2\left(\frac{(n+1)c}{4}-\frac{3}{2}|\sigma|^2\right)\geqq 0.$$

For the Laplacian of continuous functions, we have the generalized definition

$$\Delta L=a\lim_{r\to 0}\frac{1}{r^2}\left(\left(\int_{B(p,r)}L\Big/\int_{B(p,r)}1\right)-L(p)\right),$$

where a is a positive constant and $B(p, r)$ denotes the geodesic ball of radius r with center p. With this definition L is subharmonic on M if and only if $\Delta L(p)\geqq 0$ at each point $p\in M$. Since $g_p(p)=L(p)$ and $g_p\leqq L$ on U_p, $\Delta L(p)\geqq\Delta g_p(p)\geqq 0$. Thus, L is subharmonic and hence $L=b=$constant on M. When $b=0$, M is totally geodesic. When $b\neq 0$, it is easy to see that $|\sigma|^2\equiv(n+1)c/6$ on M and that for any $p\in M$, by the fact that the inequalities (3.6)-(3.9) now take equality sigh, the orthonormal basis $e_1, \cdots, e_n$ of T_pM further satisfies

$$\sigma(e_1, e_i)=\sigma(e_n, e_i)=\sigma(e_i, e_j)=0, \qquad 2\leqq i, j\leqq n-1, \tag{4.3}$$
$$|\sigma(e_1, e_1)|^2=|\sigma(e_n, e_n)|^2=|\sigma(e_1, e_n)|^2=\frac{(n+1)c}{24},$$
$$\sigma(e_1, e_1)=-\sigma(e_n, e_n).$$

Substituting $\lambda_1=-\lambda_n=|\sigma(e_1, e_1)|$, $\lambda_2=\cdots=\lambda_{n-1}=0$, (4.3) and the expression of the curvature tensor of $CP^n(c)$ into (3.4), we have

$$\sum_{i=1}^{n}\langle\sigma(e_1, e_1), \nabla^2\sigma(e_i, e_i, e_1, e_1)\rangle=\sum_{i=1}^{n}\bar{R}(e_i, e_1, \sigma(e_1, e_i), \sigma(e_1, e_1)) \tag{4.4}$$
$$+\lambda_1\sum_{i=1}^{n}\{(\lambda_1-\lambda_i)\bar{R}(e_i, e_1, e_1, e_i)+2(\lambda_i-\lambda_1)|\sigma(e_1, e_i)|^2-\lambda_i\langle\sigma(e_1, e_1), \sigma(e_i, e_i)\rangle\}$$

$$=\frac{c}{4}|\sigma(e_1, e_1)|^2+\lambda_1\Big(\frac{nc\lambda_1}{4}+2(\lambda_n-\lambda_1)|\sigma(e_1, e_n)|^2-2\lambda_1|\sigma(e_1, e_1)|^2\Big)$$

$$=|\sigma(e_1, e_1)|^2\Big(\frac{(n+1)c}{4}-6|\sigma(e_1, e_1)|^2\Big)=0.$$

Similarly, we have

$$\sum_{i=1}^{n}\langle\sigma(e_n, e_n), \nabla^2\sigma(e_i, e_i, e_n, e_n)\rangle=\sum_{i=1}^{n}\langle\sigma(e_1, e_n), \nabla^2\sigma(e_i, e_i, e_1, e_n)\rangle=0.$$

Thus, we have

$$0=\frac{1}{2}\Delta|\sigma|^2=\sum_{i,j,k=1}^{n}|(\nabla\sigma)(e_i, e_j, e_k)|^2+\sum_{i=1}^{n}\{\langle\sigma(e_1, e_1), (\nabla^2\sigma)(e_i, e_i, e_1, e_1)\rangle$$

$$+2\langle\sigma(e_1, e_n), (\nabla^2\sigma)(e_i, e_i, e_1, e_n)\rangle+\langle\sigma(e_n, e_n), (\nabla^2\sigma)(e_i, e_i, e_n, e_n)\rangle\}$$

$$=\sum_{i,j,k=1}^{n}|(\nabla\sigma)(e_i, e_j, e_k)|^2.$$

Hence, M has parallel second fundamental form. Theorem 1 now follows from the classification of n-dimensional totally real minimal submanifolds in $CP^n(c)$ with parallel second fundamental form by Naitoh and Takeuchi in [7].

Proof of Theorem 2. As in the proof of Theorem 1, we show that the function $L(p)=\max_{(u,v)\in UM_p\times UM_p} f(u, v)$ is subharmonic in the generalized sense. For any $p\in M$, let $e_1, \cdots, e_n$ be an orthonormal basis of T_pM as in Lemma 3 such that (e_1, e_n) is a maximal direction at p. Then,

$$|\sigma(e_1, e_1)-\sigma(e_n, e_n)|\sum_{i=1}^{n}\{\bar{R}(e_i, e_1, \sigma(e_1, e_i), \xi)-\bar{R}(e_i, e_n, \sigma(e_i, e_n), \xi)\tag{4.5}$$

$$+(\lambda_1-\lambda_i)\bar{R}(e_i, e_1, e_1, e_i)-(\lambda_n-\lambda_i)\bar{R}(e_i, e_n, e_n, e_i)\}$$

$$=\frac{c}{4}\sum_{i=1}^{n}\langle\sigma(e_1, e_1)-\sigma(e_n, e_n), Je_i\rangle^2+\frac{nc}{4}(\lambda_1-\lambda_n)|\sigma(e_1, e_1)-\sigma(e_n, e_n)|$$

$$\geqq\frac{nc}{4}|\sigma(e_1, e_1)-\sigma(e_n, e_n)|^2.$$

Let g_p be the function defined as in the proof of Theorem 1. Then from (4.5), Lemma 3 and $|\sigma|^2\leqq nc/6$, we have $\Delta g_p(p)\geqq 0$. By the same arguments as in the proof of Theorem 1, we know that L is subharmonic (and so L=cont. on M) and thet either $|\sigma|\equiv 0$ or $|\sigma|^2\equiv nc/6$. When $|\sigma|^2\equiv nc/6$, the orthonormal basis $e_1, \cdots, e_n$ of T_pM satisfies

$$\sigma(e_1, e_i)=\sigma(e_n, e_i)=\sigma(e_i, e_j)=0, \qquad 2\leqq i, j\leqq n-1,\tag{4.6}$$

$$|\sigma(e_1, e_1)|^2=|\sigma(e_n, e_n)|^2=|\sigma(e_1, e_n)|^2=\frac{nc}{24},$$

$$\sigma(e_1, e_1)=-\sigma(e_n, e_n).$$

Using a similar calculations as in the proof of Theorem 1, we have

$$0=\frac{1}{2}\Delta|\sigma|^2=\sum_{i,j,k=1}^{n}|\nabla\sigma(e_i, e_j, e_k)|^2+\sum_{i=1}^{n}\{\langle\sigma(e_1, e_1), Je_i\rangle^2$$
$$+2\langle\sigma(e_1, e_n), Je_i\rangle^2+\langle\sigma(e_n, e_n), Je_i\rangle^2\}$$

Thus, M is $P(R)$-totally real (i.e., $\forall p\in M$, we have $\langle\sigma(X, Y), JZ\rangle=0$, for any $X, Y, Z\in T_pM$ (Ref. [5])). Furthermore, for any $p\in M$, we can obtain a locally orthonormal frame $E_1, \cdots, E_n$ in a neighborhood V_p of p by translating the orthonomal basis $e_1, \cdots, e_n$ at p as stated in (4.6) along the geodesics from p. For any $q\in V_p$, since M has parallel second fundamental form, $\{E_1(q), \cdots, E_n(q)\}$ has the same properties as $\{E_1(p)=e_1, \cdots, E_n(p)=e_n\}$ has.

Now, one can deduce by using a similar arguments as in [2, p.70] that $n=2$. Since $n=2$, it is easy to see from (4.6) that M is $\sqrt{c/12}$-isotropic. Theorem 2 now follows from the classification of $P(R)$-totally real isotropic minimal surface with parallel second fundamental form in $CP^m(c)$ by Naitoh in [5].

Remark. If M^n is a compact minimal submanifold in $S^{n+p}(1)$ with $|\sigma|^2\leqq 2n/3$, then one can deduce by the same function f defined in (3.1) that M is either totally geodesic or a Veronese surface in $S^4(1)$. This result has been proved by Xu and Chen in [12].

5. Proof of Theorem 3 and 4

Proof of Theorem 3. Let $p\in M$ and $e_1, \cdots, e_n$ be an orthonormal basis of T_pM, from the expression of the curvature tensor of $CP^m(c)$, we have

$$\sum_{i=1}^{n}\{\bar{R}(e_i, v, \sigma(v, e_i), \sigma(v, v))+2\bar{R}(e_i, v, v, A_{\sigma(v,e_i)}v)\}$$
$$=\frac{1}{2}c\langle Lv, v\rangle-\frac{1}{2}c|\sigma(v, v)|^2+\frac{1}{4}c\sum_{i=1}^{n}\langle\sigma(v, v), Je_i\rangle^2. \tag{5.1}$$

From (2.4) and Holder's inequality,

$$\frac{2}{n+2}\int_{UM_p}|Lv|^2dv_p\leqq\left\{\int_{UM_p}|Lv|^2\right\}^{1/2}\cdot\left\{\int_{UM_p}|A_{\sigma(v,v)}v|^2\right\}^{1/2}, \tag{5.2}$$

or

$$\int_{UM_p}|A_{\sigma(v,v)}v|^2dv_p\geqq\frac{2}{n+2}\int_{UM_p}\langle Lv, A_{\sigma(v,v)}v\rangle dv_p. \tag{5.3}$$

Substituting (5.1) and (5.3) into (2.3), we obtain

$$(5.4)\qquad 0=\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v, v, v)|^2dv+(n+4)\int_{UM}|A_{\sigma(v,v)}v|^2dv$$

$$-4\int_{UM}\langle Lv, A_{\sigma(v,v)}v\rangle dv-2\int_{UM}T(\sigma(v, v), \sigma(v, v))dv$$

$$+\int_{UM}\left\{\frac{c}{2}\langle Lv, v\rangle-\frac{c}{2}|\sigma(v, v)|^2+\frac{c}{4}\sum_{i=1}^{n}\langle\sigma(v, v), Je_i\rangle^2\right\}dv$$

$$\geqq\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v, v, v)|^2dv+\frac{nc}{4}\int_{UM}|\sigma(v, v)|^2dv-n\int_{UM}|A_{\sigma(v,v)}v|^2dv$$

$$-2\int_{UM}T(\sigma(v, v), \sigma(v, v))dv.$$

For any v in UM, we can put $\sigma(v, v)=|\sigma(v, v)|\xi$ for some unit vector ξ normal to M. Since $|\sigma(v, v)|^2\leqq c/12$ for any $v\in UM$, we have by Schwartz's inequality,

$$(5.5)\qquad |A_\xi u|^2\leqq(\text{maximum eigenvalue of } A_\xi)^2\leqq c/12 \quad \text{for any } u\in M.$$

Hence

$$(5.6)\qquad \frac{nc}{4}|\sigma(v, v)|^2-n|A_{\sigma(v,v)}v|^2-2T(\sigma(v, v), \sigma(v, v))$$

$$=|\sigma(v, v)|^2\left(\frac{nc}{4}-n|A_\xi v|^2-2\sum_{i=1}^{n}\langle A_\xi e_i, A_\xi e_i\rangle\right)$$

$$\geqq|\sigma(v, v)|^2\left(\frac{nc}{4}-n\cdot\frac{c}{12}-2\cdot n\cdot\frac{c}{12}\right)=0,$$

where $e_1, \cdots, e_n$ is a locally orthonormal basis of TM. It follows from (5.4) and (5.6) that M has parallel second fundamental form,

$$(5.7)\qquad \langle\sigma(X, Y), JZ\rangle=0 \quad \text{for any vectors } X, Y, Z\in T_pM.\ p\in M,$$

and that the inequalities (5.3) and (5.6) take equality sign. Hence, we have

$$(5.8)\qquad |A_{\sigma(v,v)}v|^2=\frac{c}{12}|\sigma(v, v)|^2,$$

$$(5.9)\qquad Lv=\frac{n+2}{2}A_{\sigma(v,v)}v.$$

From (5.7), we know that M is $P(R)$-totally real (see [5]). Now, given $p\in M$, let ω be the 1-form on UM_p defined by

$$\omega_v(e)=\langle\sigma(v, v), \sigma(v, e)\rangle|\sigma(v, v)|^2$$

for all $v\in UM_p$, $e\in T_vUM_p$. Integrating on UM_p the codifferential of ω, we have

(5.10) $(n+6)\int_{UM_p} |\sigma(v, v)|^4 dv_p = 4\int_{UM_p} |A_{\sigma(v,v)}v|^2 dv_p + 2\int_{UM_p} \langle Lv, v\rangle |\sigma(v, v)|^2 dv_p .$

Substituting (5.8) and (5.9) into (5.10), we find

$$\int_{UM} |\sigma(v, v)|^2 \Big(\frac{c}{12} - |\sigma(v, v)|^2\Big) dv = 0 . \tag{5.11}$$

Since $|\sigma(v, v)|^2 \leqq c/12$ for any $v \in UM$, we derive from (5.11) that either $|\sigma(v, v)| \equiv 0$ (i.e., M is totally geodesic) or $|\sigma(v, v)|^2 \equiv c/12$. When $|\sigma(v, v)|^2 \equiv c/12$, we conclude from the classifications of isotropic $P(R)$-toally real minimal submanifolds with parallel second fundamental form of a complex projective space (see [4] and [11]) that the immersion of M into $CP^m(c)$ is one of the following immersions: $\varphi_{1,p}: RP^2(c/12) \to CP^{4+p}(c)$; $\varphi_{2,p}: S^2(c/12 \to CP^{4+p}(c)$; $\varphi_{3,p}: CP^2(c/3) \to CP^{7+p}(c)$; $\varphi_{4,p}: QP^2(c/3) \to CP^{13+p}(c)$; $\varphi_{5,p}: \mathrm{Cay}\, P^2(c/3) \to CP^{25+p}(c)$ $(p=0, 1, 2, \cdots)$. This completes the proof of Theorem 3.

Proof of Theorem 4. Let $v \in UM_p$, and $\sigma(v, v) = |\sigma(v, v)|\xi$. Take an orthonormal basis $e_1, \cdots, e_n$ of T_pM such that $A_\xi e_i = \lambda_i e_i$, $i=1, \cdots, n$. Then,

$$\sum_{i=1}^{n} \lambda_i = 0 . \tag{5.12}$$

Denote by $\Lambda = \max_i \lambda_i^2$. Since n is odd, it follows from [3, p. 256] that

$$\sum_{i=1}^{n} \langle A_\xi e_i, A_\xi e_i\rangle = \sum_{i=1}^{n} \lambda_i^2 \leqq (n-1)\Lambda \leqq \frac{(n-1)c}{4(3-2/n)} . \tag{5.13}$$

Using the same arguments as in the proof of Theorem 3 and the hypothesis: $|\sigma(v, v)|^2 \leqq c/4(3-2/n)$, we conclude that M is $P(R)$-totally real with parallel second fundamental form and either $|\sigma(v, v)|^2 \equiv 0$ or $|\sigma(v, v)|^2 \equiv c/4(3-2/n)$ on UM. Using the classifications of the isotropic $P(R)$-totally real minimal submanifolds with parallel second fundamental form in a complex projective space by Naitoh ([5]), we know that the case $|\sigma(v, v)|^2 \equiv c/4(3-2/n)$ cannot occur. Thus, M is totally geodesic. This completes the proof of Theorem 4.

6. Proof of Theorem 5.

Denote by $\bar{R}$ the curvature tensor of $S^{n+p}(1)$. Let $e_1, \cdots, e_n$ be an orthonormal basis of T_pM, $p \in M$. Then,

$$\sum_{i=1}^{n} \{\bar{R}(e_i, v, \sigma(v, e_i), \sigma(v, v)) + 2R(e_i, v, v, A_{\sigma(v, e_i)}v)\} = 2\langle Lv, v\rangle - 2|\sigma(v, v)|^2 \tag{6.1}$$

Since $T = k\langle\ ,\ \rangle$, taking the trace, we have $k = |\sigma|^2/p$. Thus, it follows from Lemma 1 and Lemma 2 that

(6.2) $$0=\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v, v, v)|^2dv+\frac{2}{n+2}\int_{UM}|\sigma|^2dv-\frac{4}{pn(n+2)}\int_{UM}|\sigma|^4dv$$
$$+(n+4)\int_{UM}|A_{\sigma(v,v)}v|^2dv-4\int_{UM}\langle Lv, A_{\sigma(v,v)}v\rangle dv.$$

Suppose that $\mathrm{Ric}_M\geqq(n-1)-(p(n+2)/2(n+p+2))$. Then, from Gauss' equation, one has that $0\leqq\langle Lv, v\rangle\leqq p(n+2)/2(n+p+2)$ for all $v\in UM$. So, we have

(6.3) $$|\sigma|^2=\sum_{i=1}^{n}\langle Le_i, e_i\rangle\leqq\frac{np(n+2)}{2(n+p+2)},$$

(6.4) $$|Lv|^2\leqq\frac{p(n+2)}{2(n+p+2)}\langle Lv, v\rangle,$$

where $e_1, \cdots, e_n$ is an orthonormal basis of T_pM, $p\in M$.

By the Schwarz inequality, we have: $|\sigma(v, v)|^4\leqq|A_{\sigma(v,v)}v|^2$. So, (5.10) gives

(6.5) $$\int_{UM_p}|A_{\sigma(v,v)}v|^2dv_p\geqq\frac{2}{n+2}\int_{UM_p}\langle Lv, v\rangle|\sigma(v, v)|^2dv_p.$$

The equality in (6.5) holds if and only if M is isotropic at p. Combining (2.4), (5.3) and (6.4), we get

(6.6) $$(n+4)\int_{UM_p}|A_{\sigma(v,v)}v|^2dv_p-4\int_{UM_p}\langle Lv, A_{\sigma(v,v)}v\rangle dv_p$$
$$\geqq-\frac{2n}{n+2}\int_{UM_p}\langle Lv, A_{\sigma(v,v)}v\rangle dv_p=\frac{-4n}{(n+2)^2}\int_{UM_p}|Lv|^2dv_p$$
$$\geqq-\frac{4n}{(n+2)^2}\cdot\frac{p(n+2)}{2(n+p+2)}\int_{UM_p}\langle Lv, v\rangle dv_p$$
$$=-\frac{2np}{(n+2)(n+p+2)}\cdot\frac{1}{n}\int_{UM_p}|\sigma|^2dv_p.$$

Substituting (6.3) and (6.6) into (6.2), we find

(6.7) $$0\geqq\int_{UM}\frac{2|\sigma|^2}{(n+p+2)}\left\{1-\frac{2(n+p+2)}{np(n+2)}|\sigma|^2\right\}dv\geqq0.$$

Thus, M is isotropic with parallel second fundamental form. Using [11], we know that M is a compact rank one symmetric space, and the immersion of M into $S^{n+p}(1)$ is one of the following standard ones: $S^n(1)\to S^n(1)$; $RP^2(1/3)\to S^4(1)$; $S^2(1/3)\to S^4(1)$; $CP^2(4/3)\to S^7(1)$; $QP^2(4/3)\to S^{13}(1)$; Cay $P^2(4/3)\to S^{25}(1)$.

Q.E.D.

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