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COMPOSITION OPERATORS ON THE SPACE OF ENTIRE FUNCTIONS

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Abstract

The composition operators on the space of entire functions Γ have been characterized. The invertibility of a composition operator C_ϕ interms of the invertibility of inducing map ϕ is obtained.

Preliminaries.

Let X be a non-empty set and let $V(X)$ be a vector space of complex valued functions on X. If $\phi: X\to X$ is a mapping such that $f\circ\phi\in V(X)$ whenever $f\in V(X)$, then a composition transformation C_ϕ is defined by the equation

$$C_\phi f=f\circ\phi \qquad \text{for every } f\in V(X).$$

In case $V(X)$ is a topological vector space and C_ϕ is continuous, then we call it a composition operator induced by ϕ. If $u: X\to C\setminus\{0\}$ is a mapping such that $(uC_\phi)f=u.f\circ\phi\in V(X)$ whenever $f\in V(X)$, then a weighted composition operator is a continuous linear transformation $uC_\phi: V(X)\to V(X)$ defined by

$$(uC_\phi)f=u.f\circ\phi \qquad \text{for every } f\in V(X).$$

A complex valued function $f: C\to C$ of a complex variable is called an entire function if it is analytic in the whole complex plane. If f is an entire function then there exists a sequence $\{\hat{f}_n\}$ of complex numbers such that

$$|\hat{f}_n|^{1/n} \longrightarrow 0 \qquad \text{as } n\to\infty \text{ and } f=f(z)=\sum_{n=0}^{\infty}\hat{f}_n z^n \tag{1}$$

The power series in (1) is a uniformly convergent power series. Conversely every sequence $\{\hat{f}_n\}$ of complex numbers with $|\hat{f}_n|^{1/n}\to 0$ as $n\to\infty$ defines an

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entire function f represented by (1). We can define a metric d in the class of entire functions as $d(f, g)=\sup\{|\hat{f}_0-\hat{g}_0|, |\hat{f}_n-\hat{g}_n|^{1/n}, n\geqq 1\}$. The class of entire functions topologized by this metric is denoted by Γ. It is shown in Iyer [8] that Γ is a non-normable complete metrizable locally convex topological vector space. The convergence of a sequence of entire functions in the metric topology of Γ is equivalent to the uniform convergence of entire functions in any circle of finite radius. Such a convergence in Γ will be called strong convergence in Γ.

Every continuous linear functional f on Γ is given by $f(\alpha)=\sum_{n=0}^{\infty} f_n a_n$, where $\alpha=\alpha(z)=\sum_{n=0}^{\infty} a_n z^n$ and $\{f_n\}$ is a sequence of complex numbers such that $\{|f_n|^{1/n}\}$ is a bounded sequence. The set of all bounded linear functionals on Γ is denoted by Γ^*. A sequence $\{\alpha_n\}$ in Γ is said to converge weakly to $\alpha\in\Gamma$ if and only if $f(\alpha_n)\to f(\alpha)$ for every $f\in\Gamma^*$. If for each $n\in Z_+$, we define $e_n: C\to C$ as $e_n(z)=z^n$, then the sequence $\{e_n: n\in Z_+\}$ is a basis for Γ. A sequence $\{\alpha_n\}$ in Γ is called basis for Γ if for each $\alpha\in\Gamma$ there exists a unique sequence $\{t_n(\alpha)\}$ of complex number such that $\alpha=\sum_{n=0}^{\infty} t_n(\alpha)\alpha_n$. The space Γ of entire functions has been studied extensively by Iyer ([9], [10] and [11]).

In this note we plan to study composition operators on Γ. Most of the work on composition operators is done on Hardy spaces and L^p-spaces. Nordgren [13] has summarized some known information about composition operators on L^2 and H^2 spaces. For further details about these operators we refer to Schwartz [8], Swantan [9], Cowen [6], Boyd [2], Iwanik Mayer [12], Singh [16] and Singh and Komal [17]. The weighted composition operators have been studied by Carlson [3].

We have characterized composition operators on Γ. The invertibility of C_ϕ in terms of the invertibility of ϕ is reported. Weighted composition operators on Γ have also been characterized. For $R>0$, we denote by D_R the open disc $\{z\in C: |z|<R\}$. If $f\in\Gamma$, then $M(R, f)=\sup\{|f(z)|: z\in\bar{D}_R\}$. For $z\in C$, the evaluation functional is a map $E_z: \Gamma\to C$ defined by $E_z(f)=f(z)$ for every $f\in\Gamma$. The symbol $C(\Gamma)$ denotes the set of continuous linear operators on Γ into itself.

2. Characterizations of composition operators.

In this section we obtain some characterizations of composition operators. We first prove the following lemma:

LEMMA 2.1. *Let $R>0$. Then for each $z\in D_R$ and $f\in\Gamma$,*

$$|f(z)|\leqq\frac{RM(R, f)}{R-|z|}.$$

Proof. By Cauchy integral formula

$$f(z)=\frac{1}{2\pi i}\int_{C_R}\frac{f(w)dw}{w-z},$$

where C_R is the circle $|w|=R$. Hence

$$\begin{aligned}|f(z)|&\leqq\frac{1}{2\pi}\int_{C_R}\frac{|f(w)||dw|}{|w|-|z|}\\&\leqq\frac{M(R,f)}{2\pi}\int_{C_R}\frac{|dw|}{R-|z|}\\&=\frac{RM(R,f)}{R-|z|}.\end{aligned}$$

THEOREM 2.2. *Let $\phi: C\to C$ be a mapping. Then $C_\phi\in C(\Gamma)$ if and only if ϕ is an entire function.*

Proof. Suppose ϕ is an entire function. Since composition of two entire functions is an entire function, so $f\circ\phi$ is an entire function for each $f\in\Gamma$. We prove that C_ϕ is continuous. It is enough to prove that C_ϕ is continuous at origin. Let $R>0$ be given. Then $\bar{D}_R$ is a compact subset of C. But ϕ is continuous. Therefore $\phi(\bar{D}_R)$ is also compact subset of C. Hence we can find $K\geqq M(R,\phi)$ such that $\phi(\bar{D}_R)\subset\bar{D}_K$. Now convergence in Γ is equivalent to uniform convergence in any circle of finite radius. Suppose $f_n\to 0$ strongly. Then for each $\varepsilon>0$ we can find some $n_0>0$ such that $M(K,f_n)<\varepsilon K_0/K$ where $K_0=K-M(R,\phi)$, for all $n\geqq n_0$. From Lemma 2.1, we have

$$\begin{aligned}|f_n(\phi(z))|&\leqq\frac{KM(K,f_n)}{K-|\phi(z)|}\\&\leqq\frac{KM(K,f_n)}{K-M(R,\phi)}<\varepsilon,\qquad\text{for every } z\in D_R \text{ and for all } n\geqq n_0.\end{aligned}$$

Hence $C_\phi f_n=f_n\circ\phi\to 0$ as $n\to\infty$.

Conversely, suppose $C_\phi:\Gamma\to\Gamma$ is continuous. Then $C_\phi f=f\circ\phi$ is an entire function for every $f\in\Gamma$. In particular, take $f=I$. Then $\phi=I\circ\phi=f\circ\phi$. Hence ϕ is an entire function.

THEOREM 2.3. *Let $A\in C(\Gamma)$. Then A is a composition operator if and only if $Ae_n=(Ae_1)^n$ for every $n\in Z_+$.*

Proof. Suppose A is a composition operator. Then $A=C_\phi$ for some entire function $\phi: C\to C$. Therefore

$$\begin{aligned}Ae_n&=C_\phi e_n=e_n\circ\phi=\phi^n=(e_1\circ\phi)^n\\&=(C_\phi e_1)^n=(Ae_1)^n\qquad\text{for every } n\in Z_+.\end{aligned}$$

Conversely, if the condition of the theorem is satisfied, then set $\phi=Ae_1$. Clearly ϕ is an entire function. Hence C_ϕ is a composition operator. Now

$$\begin{aligned} Af &= A\Big(\sum_{n=0}^{\infty}\hat{f}_n e_n\Big)=\sum_{n=0}^{\infty}\hat{f}_n Ae_n \\ &= \sum_{n=0}^{\infty}\hat{f}_n(Ae_1)^n=\sum_{n=0}^{\infty}\hat{f}_n\phi^n \\ &= \sum_{n=0}^{\infty}\hat{f}_n e_n\circ\phi=\sum_{n=0}^{\infty}\hat{f}_n C_\phi e_n \\ &= C_\phi\Big(\sum_{n=0}^{\infty}\hat{f}_n e_n\Big)=C_\phi f, \qquad \text{for every } f\in\Gamma. \text{ Therefore,} \end{aligned}$$

$$A=C_\phi .$$

THEOREM 2.4. *Let* $A\in C(\Gamma)$. *Then* A *is a composition operator if and only if* $A^*E\subset E$, *where* $E=\{E_z : z\in C\}$.

Proof. For each $z\in C$, the evaluation functional $E_z\in\Gamma^*$ in view of Lemma 2.1. Since

$$(C_\phi^* E_z)f=E_z(C_\phi f)=(f\circ\phi)(z)=f(\phi(z))=E_{\phi(z)}(f)$$

for every $f\in\Gamma$, so $C_\phi^*(E)\subset E$. Hence if $A=C_\phi$, then $A^*(E)\subset E$.

Conversely, if $A^*E_z=E_w$ for some $w\in C$, then define $\phi(z)=w$. Now

$$\begin{aligned} (Af)(z) &= E_z(Af)=A^*(E_z)f \\ &= E_w(f)=E_{\phi(z)}f \\ &= f(\phi(z))=(C_\phi f)(z) \end{aligned}$$

for every $z\in C$ and $f\in\Gamma$. Hence $A=C_\phi$.

THEOREM 2.5. *Let* $C_\phi\in C(\Gamma)$. *Then* $C_\phi^* : \Gamma^*\to\Gamma^*$ *is a composition operator if* $\phi(z)=\alpha z$.

Proof. Suppose $\phi(z)=\alpha z$. Define $\psi : C\to C$ by $\psi(z)=\alpha z$. We prove that $C_\phi^*=C_\psi$. Let $f\in\Gamma^*$ and $x\in\Gamma$. Then $f(z)=\sum_{n=0}^{\infty} f_n z^n$ and $x(z)=\sum_{n=0}^{\infty}\hat{x}_n z^n$. Therefore, $x(\phi(z))=\sum_{n=0}^{\infty}(\hat{x}\circ\phi)(n)z^n$. But $x(\phi(z))=\sum_{n=0}^{\infty}\hat{x}_n(\phi(z))^n=\sum_{n=0}^{\infty}\hat{x}_n(\alpha z)^n=\sum_{n=0}^{\infty}\alpha^n\hat{x}_n z^n$. Hence by unique expansion of $x(\phi(z))$, we have $(\hat{x}\circ\phi)(n)=\alpha^n\hat{x}_n$. Similarly $f(\psi(z))=f(\alpha z)=\sum_{n=0}^{\infty}\alpha^n f_n z^n$. Now $(C_\phi^* f)(x)=f(C_\phi x)=\sum_{n=0}^{\infty} f_n(\hat{x}\circ\phi)(n)=\sum_{n=0}^{\infty}\alpha^n f_n\hat{x}(n)=(f\circ\psi)(x)$ $=(C_\psi f)(x)$ for every $x\in\Gamma$ and $f\in\Gamma^*$. Therefore $C_\phi^*=C_\psi$.

3. Invertible composition operators.

A continuous linear transformation $A: \Gamma \to \Gamma$ is called invertible if there exists a continuous linear transformation $B: \Gamma \to \Gamma$ such that $A \circ B = B \circ A = I$, the identity operator on Γ. Similarly a mapping $\phi: C \to C$ is called invertible if there exists a mapping $\psi: C \to C$ such that $\phi \circ \psi = \psi \circ \phi = I$, the identity mapping on C. Let $A \in C(\Gamma)$. Then A is called an isometry if $d(Af, Ag) = d(f, g)$ for every $f, g \in \Gamma$. In this section invertible and isometric composition operators have been studied.

THEOREM 3.1. *Let $C_\phi \in \Gamma$. Then C_ϕ is invertible if and only if ϕ is invertible with $\phi^{-1} \in \Gamma$.*

Proof. Suppose C_ϕ is invertible. Then thre exists $A \in C(\Gamma)$ such that $AC_\phi = C_\phi A = I$. So we have

$$\begin{aligned} Ae_n &= Ae_1^n = A((C_\phi Ae_1)^n) = A(((Ae_1)\circ\phi)^n) \\ &= A((Ae_1)^n \circ \phi) = AC_\phi((Ae_1)^n) \\ &= (Ae_1)^n \qquad \text{for } n = 0, 1, 2, \cdots. \end{aligned}$$

By theorem 2.2 $A = C_\psi$ for some entire function ψ. It follows that $\phi \circ \psi = \psi \circ \phi = I$. This atonce implies that ϕ and ψ are bijections. Hence $\phi^{-1} \in \Gamma$.

To prove the converse, let ϕ be invertible with $\phi^{-1} \in \Gamma$. Then $C_{\phi^{-1}}$ is a composition operator. Clearly $C_\phi C_{\phi^{-1}} = C_{\phi^{-1}} C_\phi = I$. Hence C_ϕ is invertible.

COROLLARY 3.2. *Let $C_\phi \in C(\Gamma)$. Then C_ϕ is invertible if and only if $\phi(z) = az + b$, where $(0 \neq) a, b \in C$.*

Proof. By Theorem 3.1 C_ϕ is invertible if and only if ϕ is bijective on C. And this is the case if and only if $\phi(z) = az + b$ with $a \neq 0$. (In fact, if ϕ is a polynomial, then it should be linear. If it is not a polynomial, it has an essential singularity at the point at infinity, so that it can not be one-to-one).

THEOREM 3.3. *Let $C_\phi \in C(\Gamma)$. Then C_ϕ is an isometry if and only if $\phi(z) = \alpha z$ where $|\alpha| = 1$.*

Proof. Let C_ϕ be an isometry. Then, $d(C_\phi(e_1), 0) = d(e_1, 0) = 1$, so that we have $|\hat{\phi}(0)| \leqq 1$ and $|\hat{\phi}(n)|^{1/n} \leqq 1$ for $n = 1, 2, \cdots$. Also, $d(C_\phi(z + c), 0) = d(z + c, 0) = \max\{1, |c|\}$. If $|c| > 2$, then $|c + \hat{\phi}(0)| \leqq |c|$. This means that $\hat{\phi}(0) = 0$. Next, suppose $\hat{\phi}(m) \neq 0$ for some $m \geqq 2$. Since $d(C_\phi(\alpha e_1), 0) = d(\alpha e_1, 0) = |a|$ implies that $|\alpha\hat{\phi}(m)|^{1/m} \leqq |\alpha|$ or $|\alpha| \leqq |\alpha|^m / |\hat{\phi}(m)|$, which yields a contradiction by letting $\alpha \to 0$. Hence, $\phi(z) = \hat{\phi}(1) z$. That $|\hat{\phi}(1)| = 1$ follows at once from the identity $d(\phi, 0) = d(C_\phi(e_1), 0) = d(e_1, 0) = 1$.

Conversely if $\phi(z) = \alpha z$ for some $\alpha \in C$ such that $|\alpha| = 1$, then clearly C_ϕ is an isometry.

4. Weighted composition operators on Γ.

A caracterization of weighted composition operators is obtained in this section.

THEOREM 4.1. *Let* $u: C\to C$ *and* $\phi: C\to C$ *be two non-trivial mappings. Then* $uC_\phi \in C(\Gamma)$ *if and only if* u *and* ϕ *are entire functions.*

Proof. First we suppose that uC_ϕ is a continuous linear operator. Then $u.f\circ\phi$ is an entire function for every entire function f. Now, if we take f to be a constant function which is equal to 1 every where, then we have $u.f\circ\phi = u$ so that u is an entire function. Further, if we take $f=I$, the identity function then $u.f\circ\phi=u$. Suppose $u\not\equiv 0$. By the assumption we see that $u(z)(\phi(z)(^n = uC_\phi(e_n)$ is entire for every $n=0, 1, 2, \cdots$. That u is entire follows from the case $n=0$. The case $n=1$ shows that $\phi(z)$ is analytic wherever $u(z) \neq 0$. Suppose that u has a zero of order $m>0$ at a point α. If $\phi(z)$ has a pole of order k at the point α, then $C_\phi(e_n)$ has a pole of order nk there. So, for n with $nk>m$ the function $uC_\phi(e_n)$ cannot be analytic at α. In case α is an essential singularity for ϕ, $u\phi$ cannot be analytic at α. This means that ϕ should be analytic at α. Hence, ϕ is an entire function.

To prove the converse, let u and ϕ be entire functions. Since product and composition of two entire functions is an entire function, it follows that $uC_\phi f = u.f\circ\phi \in \Gamma$ for every $f\in\Gamma$. Suppose $f_n\to 0$ strongly. For a given $R>0$, as in proof of Theorem 2.2 choose $K>M(R, \phi)$ such that $\phi(\bar{D}_R)\subset\bar{D}_K$. Let $\varepsilon>0$ be given. Then there exists $n_0>0$ such that

$$M(2K, f_n) < \frac{\varepsilon}{2M(R, u)} \qquad \text{for every } n>n_0.$$

From Lemma 2.1, we have

$$\begin{aligned} |u(z)f_n(\phi(z))| &\leqq M(R, u)|f_n(\phi(z))| \\ &\leqq M(R, u)\frac{M(2K, f_n)}{2K-|\phi(z)|}\times 2K \\ &\leqq 2M(R, u)M(2K, f_n) < \varepsilon \end{aligned}$$

for each $|z|\leqq R$ and $n\geqq n_0$. Hence

$$M(R, (uC_\phi)(f_n)) < \varepsilon \qquad \text{for every } n\geqq n_0$$

Thus $uC_\phi f_n\to 0$. This proves that uC_ϕ is continuous at origin. Since uC_ϕ is linear, so uC_ϕ is continuous everywhere. This completes the proof of the theorem.

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