A. INOUE AND Y. MAEDA
KODAI MATH. J.
14 (1991), 72—112

FOUNDATIONS OF CALCULUS ON SUPER EUCLIDEAN SPACE $\mathfrak{R}^{m|n}$ BASED ON A FRÉCHET-GRASSMANN ALGEBRA

Dedicated to Professor T. Kimura on the occasion of his 60th birthday

BY ATSUSHI INOUE AND YOSHIAKI MAEDA

Abstract.

We define a Fréchet-Grassmann algebra with infinitely many generators as the supernumber algebra. Using this, we define a so-called super Euclidean space and may develop elementary analysis on it. In doing this, we clarify the relation between Grassmann generators and odd variables. Moreover, we construct a certain Hamilton flow on the super Euclidean space, corresponding to the 'classical' orbit of the Pauli equation, for which we define the action integral, van Vleck determinant etc. as similar as we do on the Euclidean space.

Introduction

After the pioneering works of Martin [20, 21] in 1959, who considered a generalization of the classical mechanics on a ring with arbitrary generators, Berezin started independently his endeavor of a generalization of analysis in which the Grassmann variables would play a part on equal footing with real variables. (One may find more general idea in Manin [19] where he claimed that there should be at least 'three dimensions = ordinary, odd and arithmetic dimensions' in geometry.) There are many works by Berezin, but seemingly he did not distinguish the Grassmann generators and the (odd) variables because he considered his supermanifold rather sheaf theoretically. Roughly speaking, for an (ordinary) C^∞-manifold X of $\dim X=m$, he considered a ringed space $(X, \mathcal{A}(X))$ as his supermanifold of dimension $\mathcal{A}(X)=C^\infty(X)\otimes\Lambda(R^n)$. See, his book edited by Kirillov [2] and Leites [17].

Supersymmetric theory is now widely used by physicists, and the need of an infinite number of generators is recognized by some of them especially when they want to 'quantize classical systems'. Therefore, there are many trials to define the 'supernumber' based on the Grassmann algebra with infinitely many generators. For example, Rogers [23] introduced a Banach-Grassmann algebra

Received May 10, 1990.

modelled on the real sequence space l^1 and using the standard theory of differential calculus on Banach spaces, she defined her 'G^∞ functions'. On the other hand, De Witt, in p. 3 of his book [6], asserted that he could develop the analysis even if there exists a very weak topology in his ground ring: "In the formal limit $L\rightarrow\infty$ they many continue to be regarded as vector spaces, but we shall not give them a norm or even a topology" (L is the number of Grassmann generators, 'they' stands for Λ_L, $\Lambda_{L,ev}$ and $\Lambda_{L,od}$ where $\Lambda_L=a$ Grassmann algebra with L generators). More precisely, he introduced a non-Hausdorff topology in his superspace based on his Grassmann algebra. Thus, Rogers [25] was offended by saying: "To those physicists who use supermanifolds, but do not often lie awake at night worrying about the finer points of analysis, the message of this paper is simple—if you need more generators for your Grassmann algebra, help yourself !".

In this paper, we introduce a Fréchet algebra with degree, called a Fréchet-Grassmann algebra over $\boldsymbol{C}$, modelled on the sequence space $\boldsymbol{\omega}$. Briefly speaking, it is the set of formal power series of infinitely many indeterminate letters which satisfies the Grassmann relations. If it is considered as the ground ring, we call it the supernumber algebra. Moreover, our (real) supernumber algebra is assumed to be real in the body direction and complex in the soul direction, whose reason will be given in § 4. After introducing the (real) supernumber algebra, we define the super Euclidean spaces in § 1. Supersmooth functions are defined on 'saturated domains' in our super Euclidean space and the differential calculus containing Taylor's formula, composition of functions, implicit function theorem, etc. are proved in § 2. It seems meaningfull to remark here that our definition of supersmooth functions is considerably different from others in the sense we define it from scratch by the so-called z-expansion not introducing the Fréchet or Gâteaux type differentiability. In other word, we may consider 'H^∞-functions' whose coefficients in the z-expansion are generated by supernumber algebra-valued C^∞ functions. This answers partly the 'interesting' problem posed at the last line of Bryant [4]. In § 3, we give the definition of integrations also with the change of variables under integral sign. Lastly, in § 4, as an application of § 2 and § 3, we solve a Hamilton equation on the super Euclidean space. These equations themselves are given in Berezin & Marinov [3], Casalbuoni [5] and Manẽs & Zumino [18] without considering the existence proof of solutions nor paying attention to the number of Grassmann generators.

The main difference between our treatment and others is that we never reduce the problem to the case of the finite number of Grassmann generators. Therefore, we present the fundamentals of the so-called superanalysis from our point of view, though this paper is a refined version of the portion of our (unpublished) treatise in [11]. As an application, we constructed a fundamental solution of Pauli equatious in Inoue & Maeda [10, 11], where we used the Feynman's heuristic derivation of his path integral. Concerning our references, we never want to claim those completeness because there are too many articles prefixed 'super'. In writing this paper, we have been stimulated mainly by [18] and Vladimirov & Volovich [26, 27].

§ 1. The supernumber algebra and the super Euclidean space

Let us prepare a set of countably infinite distinct symbols $\{\sigma_j\}_{j\in N}$ satisfying the relations

$$\sigma_i\sigma_j+\sigma_j\sigma_i=0 \qquad \text{for any } i, j=1, 2, \cdots. \tag{1.1}$$

Remark. A concrete realization of this set $\{\sigma_j\}_{j\in N}$ in l^1 is given in [23]. Berezin [1] gave another realization of it as operators in the Fock space. See, for more algebraic treatment, Kostant & Sternberg [15].

We define a set by

$$\Lambda^C=\Big\{x=\sum_{\text{finite sum}} x_I\sigma^I\,;\ x_I\in \boldsymbol{C}\Big\} \tag{1.2}$$

where

$$\mathfrak{I}=\{I=(i_1, i_2, \cdots, i_k, \cdots)\in\{0, 1\}^N;\ |I|<\infty\} \qquad \text{with } |I|=\sum_k i_k$$

and

$$\sigma^I=\sigma_1^{i_1}\sigma_2^{i_2}\cdots \qquad \text{with } \sigma^{\tilde{0}}=1,\quad \tilde{0}=(0, 0, \cdots).$$

It forms an algebra by introducing sum and product as follows:

$$x+y=\sum_I(x_I+y_I)\sigma^I \quad \text{and} \quad xy=\sum_I(xy)_I\sigma^I \tag{1.3}$$

$$\text{with } (xy)_I=\sum_{I=J+K}(-1)^{\tau(I;J,K)}x_Jy_K.$$

Here, the indeces $\tau(I; J, K)$, or more generally $\tau(I; J_1, \cdots, J_k)$ are defined by

$$(-1)^{\tau(I;J_1,\cdots,J_k)}\sigma^{J_1}\cdots\sigma^{J_k}=\sigma^I \tag{1.4}$$

when I is decomposed by $I=J_1+\cdots+J_k$. But for notational simplicity, we will use $(-1)^{\tau(*)}$ without specifying the decomposition if there occurs no confusion.

We call this a Grassmann algebra over $\boldsymbol{C}$ with infinite generators $\{\sigma_j\}_{j\in N}$. Moreover, we may introduce the topology of Λ^C as follows: Elements $x^{(n)}$ converges to x in Λ^C if and only if for any $\varepsilon>0$, there exist integers L and n_0 such that (i) $x^{(n)}$ and x belong to Λ_L^C when $n>n_0$ and (ii) $|x_I^{(n)}-x_I|<\varepsilon$ when $n>n_0$. Here, we put

$$(1.5)\quad \begin{cases} \Lambda_L^C=\{x=\sum x_I\sigma^I \text{ (summation is taken for } I \text{ satisfying } i_k=0 \\ \qquad\qquad\qquad\qquad \text{for } k>L);\ x_I\in \boldsymbol{C}\} \\ \cong \Lambda^C(\boldsymbol{R}^L)=\text{the exterior algebra of forms on } R^L \\ \qquad\qquad\qquad\qquad \text{with coefficients in } \boldsymbol{C}\cong \boldsymbol{C}^{2^L}. \end{cases}$$

Instead of this, we consider following sets rather formally (but later 'proved as rigorous'):

$$(1.6)\qquad \mathfrak{C}=\left\{x=\sum_{I\in\mathcal{I}} x_I\sigma^I\,;\ x_I\in\boldsymbol{C}\right\},$$

$$(1.7)\qquad \begin{cases} \mathfrak{C}_{(0)}=\mathfrak{C}_{[0]}=\boldsymbol{C}\,, \\ \mathfrak{C}_{(j)}=\{x=\sum_{|I|\le j} x_I\sigma^I\} \quad \text{and} \\ \mathfrak{C}_{[j]}=\{x=\sum_{|I|=j} x_I\sigma^I\}=\mathfrak{C}_{(j)}/\mathfrak{C}_{(j-1)}, \end{cases}$$

To give the concrete meaning of the above summation expressions in (1.6) rnd (1.7), we recall the sequence spaces $\mathfrak{C}$ and ϕ in the terminology of Köthe [16]. That is, we define

$$(1.8)\qquad \begin{cases} \phi=\{\mathfrak{x}=(x_k)=(x_1, x_2, \cdots, x_k, \cdots);\ x_k\in\boldsymbol{C} \\ \qquad\qquad \text{and } x_k=0 \text{ except for finitely many } k\}, \\ \omega=\{u=(u_k)=(u_1, u_2, \cdots, u_k, \cdots);\ u_k\in\boldsymbol{C}\}. \end{cases}$$

For $X\supset\phi$, we define also the space $X^\times$ by

$$X^\times=\left\{\mathfrak{u}=(u_k)\,;\ \sum_k |u_k|\,|x_k|<\infty \text{ for any } \mathfrak{x}=(x_k)\in X\right\}$$

then, we get

$$\phi^\times=\omega \quad\text{and}\quad \omega^\times=\phi\,.$$

We introduce the (normal) topology in X and $X^\times$ by defining the seminorms

$$(1.9)\qquad p_u(\mathfrak{r})=\sum_K |u_k|\,|x_k|=p_{\mathfrak{x}}(\mathfrak{u}) \qquad \text{for } \mathfrak{x}\in X \text{ and } \mathfrak{u}\in X^\times.$$

Especially $\mathfrak{x}^{(n)}$ converges to $\mathfrak{x}$ in ϕ if and only if for any $\varepsilon>0$, there exist L and n_0 such that

$$(1.10)\qquad \begin{cases} \text{(i)}\ \ x_k^{(n)}=x_k=0 \qquad \text{for } k>L \text{ when } n\geqq n_0, \text{ and} \\ \text{(ii)}\ \ |x_k^{(n)}-x_k|<\varepsilon \qquad \text{for } k\leqq L \text{ when } n\geqq n_0\,. \end{cases}$$

Analogously, $\mathfrak{u}^{(n)}$ converges to $\mathfrak{u}$ in ω if and only if any $\varepsilon>0$ and each k, there exists $n_0=n_0(\varepsilon, k)$ such that

$$(1.11)\qquad |u_k^{(n)}-u_k|<\varepsilon \qquad \text{when } n\geqq n_0\,.$$

Clearly, ω forms a Fréchet space because the topology above in ω is equivalent to the one defined by countable seminorms $\{\mathfrak{p}_I(\mathfrak{u})\}_{I\in\mathfrak{I}}$ where $\mathfrak{p}_I(\mathfrak{u})=|u_{r(I)}|$ for $I=(i_k)\in\mathfrak{I}$ and $\mathfrak{u}\in\omega$. Here we used the isomorphism between N and $\mathfrak{I}$ defined by

(1.12) $$I\to r(I)=1+\frac{1}{2}\sum_k 2^k i_k \qquad \text{for } I=(i_k)\in\mathfrak{I}.$$

For each $p\in N$, we define an element $\mathfrak{e}_p=(\overbrace{0,\cdots,0,1}^{p},0,\cdots)\in\omega$. Using $r(I)$ in (1.12), we define a map

$$T:\sigma^I \longrightarrow \mathfrak{e}_{r(I)} \qquad \text{for } I=(i_k).$$

Extending this linearly, we put

(1.13) $$T(x)=\sum_{|I|\leq j} x_I\mathfrak{e}_{r(I)}\in\omega \qquad \text{for } x=\sum_{|I|\leq j} x_I\sigma^t\in\mathfrak{C}_{(j)}.$$

Then, we have

(1.14) $$\bigcup_{j=0}^{\infty} T(\mathfrak{C}_{(j)})=\sum_{j=0}^{\infty} T(\mathfrak{C}_{[j]})=\omega$$

because $T(\mathfrak{C}_{[j]})$ and $T(\mathfrak{C}_{[k]})$ are disjoint sets in ω if $j\neq k$ and r is an isomorphism from $\mathfrak{I}$ onto N. Therefore, it is reasonable to write as in (1.6) and more precisely.

(1.15) $$\mathfrak{C}=\sum_{j=0}^{\infty}\mathfrak{C}_{[j]}; \quad \text{that is } x=\sum_{j=0}^{\infty}x_{[j]} \quad \text{with } x_{[j]}=\sum_{|I|=j}x_I\sigma^I.$$

Here, $x_{[j]}$ is called the j-th degree component of $x\in\mathfrak{C}$. We have just gave the meaning of the summations in (1.6) and (1.7) by using the summation in ω. (See, (2) of Remarks after Theorem 1.2 below.)

Topology. We introduce the weakest topology in $\mathfrak{C}$ which makes the map T continuous from $\mathfrak{C}$ to ω, that is, $x=\sum_{I\in\mathfrak{I}}x_I\sigma^I\to 0$ in $\mathfrak{C}$ if and only if $\mathrm{proj}_I(x)\to 0$ for each $I\in\mathfrak{I}$ with $\mathrm{proj}_I(x)=x_I$; it is equivalent to the metric $\mathrm{dist}(x,y)=\mathrm{dist}(x-y)$ defined by

(1.16) $$\mathrm{dist}(x)=\sum_{I\in\mathfrak{I}}\frac{1}{2^{r(I)}}\frac{|\mathrm{proj}_I(x)|}{1+|\mathrm{proj}_I(x)|} \qquad \text{for any } x\in\mathfrak{C}.$$

Algebraic operations. For any x, $y\in\mathfrak{C}$, we define

(1.17) $$x+y=\sum_{j=0}^{\infty}(x+y)_{[j]} \quad \text{with } (x+y)_{[j]}=x_{[j]}+y_{[j]} \quad \text{for } j\geqq 0$$

and

(1.18) $$xy=\sum_{j=0}^{\infty}(xy)_{[j]} \quad \text{where } (xy)_{[j]}=\sum_{k=0}^{j}x_{[j-k]}y_{[k]}=\sum_{|I|=j}(xy)_I\sigma^I.$$

Here, $(xy)_I=\sum_{I=J+K}(-1)^{\tau(I;J,K)}x_J y_K\in\boldsymbol{C}$ is well-defined because for any set $I\in\mathfrak{T}$, there exist only finitely many decompositions by sets J, K satisfying $I=J+K$. By definition, we get

$$(1.19)\qquad \begin{cases} \mathfrak{C}_{(j)}\subset\mathfrak{C}_{(k)} & \text{for } j\leqq k\,, \\ \mathfrak{C}=\bigcup_{j=0}^{\infty}\mathfrak{C}_{(j)} & \text{with } \bigcap_{j=0}^{\infty}\mathfrak{C}_{(j)}=0\,, \end{cases}$$

$$(1.20)\qquad \mathfrak{C}_{[j]}\cdot\mathfrak{C}_{[k]}\subset\mathfrak{C}_{[j+k]} \quad\text{and}\quad \mathfrak{C}_{(j)}\cdot\mathfrak{C}_{(k)}\subset\mathfrak{C}_{(j+k)}\,.$$

Remarks. (1) The second relation in (1.20) also holds for Clifford algebras but the first one is specific to the Grassmann relation (1.1). (2) As $\{\mathfrak{C}_{(j)}\}$ forms a filter by (1.19) and (1.20), it gives a 0-neighbourhood base of the linear topology of $\mathfrak{C}$ which is equivalent to the above one defined by (1.16). (See [16] for the linear topology of vector spaces.)

Moreover, we get

LEMMA 1.1. *The product defined by* (1.18) *is continuous from* $\mathfrak{C}\times\mathfrak{C}\to\mathfrak{C}$.

Proof. It is simple by remarking that there exist $2^{|I|}$ elements $J\in\mathfrak{T}$ satisfying $J\subset I$ and that

$$|(xy)_I|\leqq\sum_{I=J+K}|x_J|\,|y_K| \qquad \text{for any } x,\,y\in\mathfrak{C}\,. \qquad \blacksquare$$

To summatize, we get

THEOREM 1.2. $\mathfrak{C}$ *forms a Fréchet-Grassmann algebra over* $\boldsymbol{C}$, *that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

Proof. Clearly, we get

$$\begin{cases} x(yz)=(xy)z & (\textit{associativity}), \\ x(y+z)=xy+xz & (\textit{distributivity}). \end{cases}$$

Other properties have been proved. $\blacksquare$

Remarks. (1) Introducing the topology corresponding to (1.10), Λ^C defined in (1.2) is made to be algebraically and topologically isomorphic to ϕ. (2) We may consider that an element of $x\in\mathfrak{C}$ stands for the 'state' such that the position labeled by σ^I is occupied by $x_I\in\boldsymbol{C}$. In other word, considering $\{\sigma_i\}$ as the countable indeterminate letters, it seems reasonable to regard $\mathfrak{C}$ as the set of certain formal power series (same letter appears only once in each monomials) with simple topology. Therefore, it is permitted to reorder the terms freely under 'summation sign'. That is, the summation $\sum_{I\in\mathfrak{T}}x_I e_{\tau(I)}$ is 'unconditionally (though not absolutely) convergent' (diverting the terminology of basis problem in Banach spaces) and so is $\sum_{I\in\mathfrak{I}}x_I\sigma^I$. In this respect, the real Banach-Grassmann algebra introduced by Rogers consists of the absolutely convergent sequence

$$\|x\|=\sum_{I\in\mathfrak{I}}|x_I|<\infty \text{ for } x=\sum_{I\in\mathfrak{I}}x_I\sigma^I \text{ with } x_I\in R, \text{ and it satisfies } \|xy\|\leqq\|x\|\|y\|.$$

Using (1.15), we decompose

$$x=x_B+x_S \quad \text{where } x_S=\sum_{1\leq j<\infty}x_{[j]} \text{ and } x_B=x_{\tilde{0}}=x_{[0]} \tag{1.21}$$

and the number x_B is called *the body* (*part*) of x and the remainder x_S is called *the soul* (*part*) of x, respectively. We define the map π_B from $\mathfrak{C}$ to $\boldsymbol{C}$ by $\pi_B(x)=x_B$, called the body projection (or called the augmentation map in [23]). Aside the decomposition (1.15), we have the following as a vector space.

$$\mathfrak{C}=\mathfrak{C}_{ev}\oplus\mathfrak{C}_{od}\,. \tag{1.22}$$

Here, we put

$$\mathfrak{C}_{ev}=\{x\in\mathfrak{C};\, x=\sum_{|I|=even}x_I\sigma^I\} \quad\text{and}\quad \mathfrak{C}_{od}=\{x\in\mathfrak{C};\, x=\sum_{|I|=odd}x_I\sigma^I\}. \tag{1.23}$$

Important Remark. $\mathfrak{C}$ does not form a field because $x^2=0$ for any $x\in\mathfrak{C}_{od}$. But, if $x, y\in\mathfrak{C}$ satisfy $xy=0$ for any $y\in\mathfrak{C}_{od}$, then $x=0$. The decomposition of x with respect to degree in (1.15) is unique. These properties are shared only if the number of Grassmann generators is infinite.

$\mathfrak{C}$ is called the (*complex*) *supernumber algebra* over $\boldsymbol{C}$ and any element x of $\mathfrak{C}$ is called (*complex*) *supernumber*. Moreover, it splits into its even and odd parts, called (*complex*) *even number* and (*complex*) *odd number*, respectively;

$$x=x_{ev}+x_{od}=\sum_{|a|=even}x_a\sigma^a+\sum_{|a|=odd}x_a\sigma^a=\sum_{j=even}x_{[j]}+\sum_{j=odd}x_{[j]}\,. \tag{1.24}$$

We define the *parity* p as $p(x)=0$ for $x\in\mathfrak{C}_{ev}$ and $p(x)=1$ for $x\in\mathfrak{C}_{od}$ and we call the element x in $\mathfrak{C}$ is *homogeneous* if $p(x)=0$ or 1.

Now, we define our *supernumber algebra* over $\boldsymbol{R}$ (but not over $\boldsymbol{C}$) by

$$\mathfrak{R}=\pi_B^{-1}(\boldsymbol{R})\cap\mathfrak{C}=\Big\{x=\sum_{I\in\mathfrak{I}}x_I\sigma^I;\, x_B\in\boldsymbol{R} \text{ and } x_I\in\boldsymbol{C} \text{ for } |I|\neq0\Big\}. \tag{1.25}$$

Defining as same as before, we have

$$\mathfrak{R}=\mathfrak{R}_{ev}\oplus\mathfrak{R}_{od}\,, \qquad \mathfrak{R}=\sum_{j=0}^{\infty}\mathfrak{R}_{[j]}\,. \tag{1.26}$$

$\mathfrak{R}_{(j)}$ and other terminologies are analogously introduced.

DEFINITION 1.3. The super Euclidean space of dimension $m|n$ is defined by

$$\mathfrak{R}^{m|n}=\mathfrak{R}_{ev}^m\times\mathfrak{R}_{od}^n \tag{1.27}$$

whose element is denoted by $X=(X_\kappa)=(x, \theta)\in\mathfrak{R}^{m|n}$ with $x=(x_1, x_2, \cdots, x_m)\in\mathfrak{R}_{ev}^m$ and $\theta=(\theta_1, \theta_2, \cdots, \theta_n)\in\mathfrak{R}_{od}^n$. The topology of $\mathfrak{R}^{m|n}$ is induced from the metric defined by $\mathrm{dist}_{m|n}(X, Y)=\mathrm{dist}_{m|n}(X-Y)$ for $X, Y\in\mathfrak{R}^{m|n}$, where we put

$$\mathrm{dist}_{m|n}(X)=\sum_{j=1}^{m}\Big(\sum_{I\in\mathfrak{T}}\frac{1}{2^{r(I)}}\frac{|\mathrm{proj}_I(x_j)|}{1+|\mathrm{proj}_I(x_j)|}\Big)+\sum_{s=1}^{n}\Big(\sum_{I\in\mathfrak{T}}\frac{1}{2^{r(I)}}\frac{|\mathrm{proj}_I(\theta_s)|}{1+|\mathrm{proj}_I(\theta_s)|}\Big). \tag{1.28}$$

Clearly, $\mathrm{dist}_{1|1}(X)=\mathrm{dist}(X)$ for $X\in\mathfrak{R}^{1|1}\cong\mathfrak{R}\subset\mathfrak{C}$. Analogously, the complex superspace of dimension $m|n$ is defined by

$$\mathfrak{C}^{m|n}=\mathfrak{C}_{ev}^m\times\mathfrak{C}_{od}^n\,. \tag{1.29}$$

We generalize the body map π_B as that from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to $\boldsymbol{R}^m$ by $\pi_B X=\pi_B x=(\pi_B x_1, \cdots, \pi_B x_m)\in\boldsymbol{R}^m$ for $X=(x, \theta)\in\mathfrak{R}^{m|n}$.

Remarks. (1) Defining $\mathfrak{R}$ in (1.25), we used both $\boldsymbol{R}$ and $\boldsymbol{C}$. The reason of this definition is explained in § 4 where we solve a certain Hamiltonian equation stemming from the Pauli equation. (2) de Witt [6] introduces his space $R_{dW}^{m|n}=(\Lambda_{ev}^R)^m\times(\Lambda_{od}^R)^n$. Here, $\Lambda_{ev}^R=\lim_{L\to\infty}\Lambda_{ev}^R(\boldsymbol{R}^L)$ and $\Lambda_{ev}^R(\boldsymbol{R}^L)$ is isomorphic to the exterior algebra of even forms on $\boldsymbol{R}^L$ with real coeffieicnts. Λ_{od}^R and $\Lambda^R=\Lambda_{ev}^R+\Lambda_{od}^R$ are 'defined' analogously. In the above, the meaning of '$\lim_{L\to\infty}$' is not so clear. And his topology in $R_{dW}^{m|n}$ is the weakest topology which makes continuous the projection π_B from $R_{dW}^{m|n}$ to $\boldsymbol{R}^m$. This does not give the Hausdorff topology in $R_{dW}^{m|n}$ but he claims that it is not serious in his analysis. (3) Rogers [23] defines her space $R_R^{m|n}$ based on the real Banach-Grassmann algebra l^1 in order to develop her theory of superanalysis, using the known differential calculus for functions on Banach spaces. But we are not sure whether such a strong topology is really necessary. Or rather, we claim in the following that though generally speaking, the differential calculus on locally convex spaces are rather troublesome, see for example, Keller [13], Yamamuro [29], but we may carry out almost the same procedures as she done in [23] using the ring structure directly in our Fréchet-Grassmann algebra, (4) Matsumoto & Kakazu [22], Yagi [28] and Bryant [4], in order to refine the idea of DeWitt, defined a Fréchet space which is the projective limit of the Banach space modelled on the exterior algebra of forms on $\boldsymbol{R}^L$ with real coefficients, though the grading and the ring structure of it is obscured by their construction. (5) See also the papers [26], Jadczyk & Pilch [12] and Hoyos et al. [8].

§ 2. Supersmooth functions and their basic properties

DEFINITION 2.1. *A set* $U_{ev}\subset\mathfrak{R}^{m|0}=\mathfrak{R}_{ev}^m$ is called a even superdomain if $\pi_B(U)\subset\boldsymbol{R}^m$ is open and connected and $\pi_B^{-1}(\pi_B(U_{ev}))=U_{ev}$. When $U\subset\mathfrak{R}^{m|n}$ is represented by $U=U_{ev}\times\mathfrak{R}_{od}^n$ with a even superdomain $U_{ev}\subset\mathfrak{R}^{m|0}$, U is called a superdomain.

Remark. This definition of superdomain corresponds to the 'saturated' domain which appeared in [12] and [8]. This saturated domain seems not suitable to construct 'supermanifolds' with non-trivial fermion sectors, which will be discussed in the separate paper.

PROPOSITION 2.2. *Let $U_{ev}\subset\mathfrak{C}^{m|0}$ be a even superdomain. Assume that f is a smooth mapping from $U_B=\pi_B(U_{ev})$ into $\mathfrak{C}$, denoted simply by $f\in C^\infty(U_B;\mathfrak{C})$. That is, we have the expression*

$$f(q)=\sum_J f_J(q)\sigma^J \qquad \text{with } f_J(q)\in C^\infty(U_B;\boldsymbol{C}). \tag{2.1}$$

Then, we may define a mapping $\tilde{f}$ of U_{ev} into $\mathfrak{C}$ called the Grassmann continuation of f by

$$\tilde{f}(x)=\sum_{|\alpha|\geq 0}\frac{1}{\alpha!}\partial_q^\alpha f(x_B)x_S^\alpha \qquad \text{where } \partial_q^\alpha f(x_B)=\sum_J \partial_q^\alpha f_J(x_B)\sigma^J. \tag{2.2}$$

Here, we put $x=(x_1,\cdots,x_m)$, $x=x_B+x_S$ with $x_B=(x_{1,B},\cdots,x_{m,B})=(q_1,\cdots,q_m)=q\in U_B$, $x_S=(x_{1,S},\cdots,x_{m,S})$ and $x^\alpha=x_1^{\alpha_1}\cdots x_m^{\alpha_n}$.

Proof. Denoting by $x_{1,S,[k_1]}$, the k_1-th degree component of $x_{1,S}$, we get

$$(x_{1S}^{\alpha_1})_{[k_1]}=\sum(x_{1,s,[r_1]})^{p_{1,1}}\cdots(x_{1,s,[r_l]})^{p_{1,l}}.$$

Here, the summation is taken for all partitions of an integer α_1 into $\alpha_1=p_{1,1}+\cdots+p_{1,l}$ satisfying $\sum_{i=1}^l r_i p_{1,i}=k_1$. Using these notations, we put

$$\tilde{f}_{[k]}(x)=\sum_{\substack{|\alpha|\leq k\\ k_0+k_1+\cdots+k_m=k}}\frac{1}{\alpha!}(\partial_q^\alpha f)_{[k_0]}(x_B)(x_{1,S}^{\alpha_1})_{[k_1]}\cdots(x_{m,S}^{\alpha_m})_{[k_m]} \tag{2.3}$$

$$\text{where } (\partial_q^\alpha f)_{[k_0]}(x_B)=\sum_{|J|=k_0}\partial_q^\alpha f_J(x_B)\sigma^J.$$

That is,

$$\tilde{f}_{[0]}(x)=f_{[0]}(x_B),$$

$$\tilde{f}_{[1]}(x)=f_{[1]}(x_B),$$

$$\tilde{f}_{[2]}(x)=f_{[2]}(x_B)+\sum_{j=1}^m(\partial_{q_j}f)_{[0]}(x_B)(x_{j,s})_{[2]},$$

$$\tilde{f}_{[3]}(x)=f_{[3]}(x_B)+\sum_{j=1}^m(\partial_{q_j}f)_{[1]}(x_B)(x_{j,S})_{[2]},$$

$$\tilde{f}_{[4]}(x)=f_{[4]}(x_B)+\sum_{j=1}^m(\partial_{q_j}f)_{[2]}(x_B)(x_{j,S})_{[2]}$$
$$+\frac{1}{2}\sum_{j=1}^m(\partial_{q_j}^2 f)_{[0]}(x_B)(x_{j,S}^2)_{[4]}+\sum_{j\neq k}(\partial_{q_jq_k}^2 f)_{[0]}(x_B)(x_{j,S})_{[2]}(x_{k,S})_{[2]}, \text{ etc.}$$

Since $\tilde{f}_{[j]}(x)\neq\tilde{f}_{[k]}(x)$ $(j\neq k)$ in $\mathfrak{C}$, we may take the sum $\sum_{j=0}^{\infty}\tilde{f}_{[j]}(x)\in\mathfrak{C}$, which is denoted by $\tilde{f}(x)$. Therefore, rearranging the above 'summation', we get the 'familiar' expression as in (2.2). ■

Remarks. (1) More primitively, we may represent $\tilde{f}(x)=\sum_H\tilde{f}_H(x)\sigma^H$ where

$$\tilde{f}_H(x)=\sum_{\substack{H=J+I_1^{(1)}+\cdots+I_m^{(\alpha_m)}\\ \alpha=(\alpha_0,\cdots,\alpha_m)}}(-1)^{r(*)}\frac{1}{\alpha!}\partial_q^\alpha f_J(x_B)x_{1,I_1^{(1)}}\cdots x_{m,I_m^{(\alpha_m)}}$$

but this representation obscures the form of $\tilde{f}$ given in (2.2). (2) Defining H^∞-functions, Rogers [25] used C^∞-functions with values in $\boldsymbol{R}$ defined on an open connected set U in her topology.

COROLLARY 2.3. *If f and $\tilde{f}$ be given as above, then* (i) *$\tilde{f}$ is continuous and* (ii) *$\tilde{f}(x)=0$ in U implies $f(x_B)=0$ in U_B. Moreover, if we define the partial derivatives of $\tilde{f}$ by*

$$(2.4)\qquad \partial_{x_j}\tilde{f}(x)=\frac{d}{dt}\tilde{f}(x+te_{(j)})\Big|_{t=0}\qquad \text{where } e_{(j)}=(\overbrace{0,\cdots,0,1}^{j},0,\cdots,0)\in\mathfrak{R}^{m|0},$$

then we get

$$(2.5)\qquad \partial_{x_j}\tilde{f}(x)=\widetilde{\partial_{q_j}f}(x)\qquad \text{for } j=1,\cdots,m.$$

Proof. Let $y_j=y_{j,B}+y_{j,S}\in\mathfrak{R}_{ev}$. For $y_{(j)}=(\overbrace{0,\cdots,0,y_j}^{j},0,\cdots,0)\in\mathfrak{R}^{m|0}$, as

$$\frac{d}{dt}\tilde{f}(x+ty_{(j)})=\frac{d}{dt}\Big\{\sum_\alpha\frac{1}{\alpha!}\Big(\sum_J\partial_q^\alpha f_J(x_B+ty_{(j),B})\sigma^J\Big)(x_S+ty_{(j),S})^\alpha\Big\},$$

we get easily

$$\frac{d}{dt}\tilde{f}(x+ty_{(j)})\Big|_{t=0}=y_j\sum_\alpha\frac{1}{\alpha!}\partial_q^\alpha\partial_{q_j}f(x_B)(x_S)^\alpha=y_j\widetilde{\partial_{q_j}f}(x).$$

Putting $y_j=1$, we have (2.5). ■

Remark. By the same argument as above, we get

$$(2.6)\qquad \frac{d}{dt}\tilde{f}(x+ty)\Big|_{t=0}=\sum_{j=1}^m y_j\sum_\alpha\frac{1}{\alpha!}\partial_q^\alpha\partial_{q_j}f(x_B)(x_S)^\alpha \text{ where } y=(y_1,\cdots,y_m)\in\mathfrak{R}^{m|0}.$$

DEFINITION 2.4. (1) For a given even superdomain $U_{ev}\subset\mathfrak{R}^{m|0}$, mapping $\tilde{f}$ from U_{ev} into $\mathfrak{C}$ is called a supersmooth function if $\tilde{f}$ is the Grassmann continuation of a smooth mapping f from $U_B=\pi_B(U_{ev})$ into $\mathfrak{C}$. We denote by $C_{SS}(U_{ev};\mathfrak{C})$, the set of supersmooth function on U_{ev}. Hereafter, for the sake of notational simplicity, $\tilde{f}$ is written simply as f unless there occurs confusion.

(2) A mapping f from a superdomain $U\subset\mathfrak{R}^{m|n}$ to $\mathfrak{C}$ is called supersmooth,

denoted by $f\in\mathcal{C}_{SS}(U;\mathfrak{C})$, if it has the following form:

$$f(x,\theta)=\sum_{|a|\leq n}f_a(x)\theta^a \tag{2.7}$$

with $a=(a_1,\cdots,a_n)\in\{0,1\}^n$, $\theta^a=\theta_1^{a_1}\cdots\theta_n^{a_n}$ and $f_a(x)\in\mathcal{C}_{SS}(U_{ev};\mathfrak{C})$. In the following, supersmooth functions are assumed to be homogeneous (i.e., $f_a(x)$ is homogeneous for each a), unless otherwise mentioned and we denote the set of them by $\mathcal{C}_{SS}(U;\mathfrak{C})$.

(3) For $f\in\mathcal{C}_{SS}(U;\mathfrak{C})$, $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$, we put

$$\begin{cases} F_j(X)=\sum_{|a|\leq n}\partial_{x_j}f_a(x)\theta^a\,, \\ F_{s+m}(X)=\sum_{|a|\leq n}(-1)^{l(a)+p(f_a(x))}f_a(x)\theta_1^{a_1}\cdots\theta_s^{a_s-1}\cdots\theta_n^{a_n} \end{cases} \tag{2.8}$$

where $l(a)=\sum_{j=1}^{s-1}a_j$ and $\theta_s^{-1}=0$. $F_k(X)$ are called the partial derivatives of f with respect to X_κ at $X=(x,\theta)$ and are denoted by

$$\begin{cases} F_j(X)=\dfrac{\partial}{\partial x_j}f(x,\theta)=\partial_{x_j}f(x,\theta) & \text{for } j=1,2,\cdots,m. \\ F_{m+s}(X)=\dfrac{\vec{\partial}}{\partial\theta_s}f(x,\theta)=\vec{\partial}_{\theta_s}f(x,\theta), & \text{for } s=1,2,\cdots,n \end{cases} \tag{2.9}$$

or simply by

$$F_\kappa(X)=\partial_{X_\kappa}f(X) \qquad \text{for } \kappa=1,\cdots,m+n\,. \tag{2.10}$$

Remarks. (1) We only use the derivatives defined above which are called the *left derivatives* with respect to odd variables. Because, after bringing the variable θ_k to the left in each monomial, we replace it with 1. (Some people call these as right derivatives, cf. [5] etc.) Similarly, we define the *right derivatives* with respect to odd variables as follows: For $f\in\mathcal{C}_{SS}(U;\mathfrak{C})$, $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$ we put

$$\begin{cases} F_j^{(r)}(X)=\sum_{|a|\leq n}\partial_{x_j}f_a(x)\theta^a\,, \\ F_{s+m}^{(r)}(X)=\sum_{|a|\leq n}(-1)^{r(a)}f_a(x)\theta_1^{a_1}\cdots\theta_s^{a_s-1}\cdots\theta_n^{a_n} \end{cases}$$

where $r(a)=\sum_{j=s+1}^n a_j$. $F_\kappa^{(r)}(X)$ are called the (right) partial derivatives of f with respect to X_κ at $X=(x,\theta)$ and are denoted by

$$F_j^{(r)}(X)=\frac{\partial}{\partial x_j}f(x,\theta)=\partial_{x_j}f(x,\theta),\quad F_{m+s}^{(r)}(X)=f(x,\theta)\frac{\overleftarrow{\partial}}{\partial\theta_s}=f(x,\theta)\overleftarrow{\partial}_{\theta_s},$$

for $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$. (2) As we use the infinite dimensional Grassmann algebras, the expression (2.8) is unique. In fact, $\sum_a f_a(x)\theta^a\equiv 0$ on U implies $f_a(x)\equiv 0$ (see, p. 322 in [26]). (3) The higher derivatives are defined analogously and we use the following notations.

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} \quad\text{and}\quad \vec{\partial}_\theta^a=\vec{\partial}_{\theta_1}^{a_1}\cdots\vec{\partial}_{\theta_n}^{a_n}\,.$$

Repeating the argument in proving Corollary 2.3, we get the following formula for $f\in C_{SS}(U;\mathfrak{C})$:

$$\frac{d}{dt}f(X+tY)\Big|_{t=0}=\sum_{j=1}^{m}y_j\frac{\partial}{\partial x_j}f(X)+\sum_{s=1}^{m}\omega_s\frac{\vec{\partial}}{\partial\theta_s}f(X) \tag{2.11}$$

where $X=(x,\theta)$, $Y=(y,\omega)\in\mathfrak{R}^{m|n}$ such that $X+tY\in U$ for any $t\in[0,1]$.

To understand the meaning of supersmoothness, we consider the dependence with respect to the 'coordinate' more precisely.

PROPOSITION 2.5. *Let* $f=\sum_I f_I(X)\sigma^I\in C_{SS}(U;\mathfrak{C})$ *where* U *is a superdomain in* $\mathfrak{R}^{m|n}$. *Let* $X=(X_\kappa)$ *be represented by* $X_\kappa=\sum_I X_{\kappa,I}\sigma^I$ *where* $\kappa=1,\cdots,m+n$, $X_{\kappa,I}\in\boldsymbol{C}$ *for* $|I|\neq 0$ *and* $X_{\kappa,0}\in R$. *Then,* $f(X)$, *considered as a function of countably many variables* $\{X_{\kappa,I}\}$ *with values in* $\mathfrak{C}$, *satisfies the following (Cauchy-Riemann type) equations.*

$$\begin{cases}\dfrac{\partial}{\partial X_{\kappa,I}}f(X)=\sigma^I\dfrac{\partial}{\partial X_{\kappa,0}}f(X) & \text{for } 1\leqq\kappa\leqq m,\ |I|=even,\\[2ex] \sigma^K\dfrac{\partial}{\partial X_{\kappa,J}}f(X)+\sigma^J\dfrac{\partial}{\partial X_{\kappa,K}}f(X)=0 & \text{for } m+1\leqq\kappa\leqq m+n,\ |J|=odd=|K|.\end{cases} \tag{2.12}$$

Here, we define

$$\frac{\partial}{\partial X_{\kappa,I}}f(X)=\frac{d}{dt}f(X+tY_{(\kappa,I)})\Big|_{t=0}, \tag{2.13}$$

with $Y_{(\kappa,I)}=(\overbrace{0,\cdots,0,\sigma^I}^{\kappa},0,\cdots,0)\in\mathfrak{R}^{m|n}$.

Proof. Replacing Y with $Y_{(\kappa,J)}$ with $1\leqq\kappa\leqq m$ and $|J|=$even in (2.11), we get readily the first equation of (2.12). Here, we have used (2.5). Considering $Y_{(\kappa,J)}$ or $Y_{(\kappa,K)}$ for $m+1\leqq\kappa\leqq m+n$ and $|J|=$odd$=|K|$ in (2.12) and multiplying σ^K or σ^J from the left respectively, we have the second equality in (2.12) readily. ■

Remark. In order to obtain the converse statement of Proposition 2.5 (see [26], [28]), it seems better to modify a general theory of differential calculus on locally convex spaces developped in [13], [29] etc. For example, we may introduce 'k-times super Fréchet or Gâteaux-differentiability' as similar as proposed in [22], but this will not be pursued here.

PROPOSITION 2.6 (Taylor's formula). *Let* $X=(x,\theta)$, $Y=(y,\omega)\in U\subset\mathfrak{R}^{m|n}$ *satisfying* $Y+t(X-Y)\in U$ *for* $0\leqq t\leqq 1$. *For* $f\in C_{SS}(U;\mathfrak{C})$, *Taylor's formula holds. That is, for any positive integer* p, *we have*

$$f(x,\theta)-\sum_{\substack{|\alpha|+|a|\leq p\\ |a|\leq n}}\frac{1}{\alpha!}(x-y)^\alpha(\theta-\omega)^a\partial_x^\alpha\vec{\partial}_\theta^a f(y,\omega)=\tau_p(X,Y) \tag{2.14}$$

where

(2.15) $\tau_p(X, Y)$

$$=\sum_{\substack{|\alpha|+|\bar{a}|=p+1,\\ |\bar{a}|\leq n}}(x-y)^{\alpha}(\theta-\omega)^{\bar{a}}\int_0^1 dt\frac{1}{p!}(1-t)^p\partial_x^{\alpha}\vec{\partial}_\theta^{\bar{a}}f(y+t(x-y),\ \omega+t(\theta-\omega)).$$

Proof. Use the following equality

$$\int_0^1 dt\frac{(1-t)^p}{p!}\left(\frac{d}{dt}\right)^{p+1}f(y+t(x-y),\ \omega+t(\theta-\omega))$$

$$=\sum_{|\alpha|+|\bar{a}|=p+1}(x-y)^{\alpha}(\theta-\omega)^{\bar{a}}\int_0^1 dt\frac{1}{p!}(1-t)^p\partial_x^{\alpha}\vec{\partial}_\theta^{\bar{a}}f(y+t(x-y),\ \omega+t(\theta-\omega)).$$

Using the integration by parts in the left hand side, we get that of (2.14). ■

To state other properties of supersmooth functions, we prepare the linear algebra on super Euclidean space briefly.

DEFINITION 2.7. M, a rectangular array whose cells are indexed by pairs consisting of a row number and a column number, is called a supermatrix if it satisfies the following:

(1) A $(m+n)\times(r+s)$ matrix M is decomposed blockwisely as $M=\begin{bmatrix}A & C\\ D & B\end{bmatrix}$ where A, B, C and D are $m\times r$, $n\times s$, $m\times s$ and $n\times r$ matrices with elements in $\mathfrak{R}$, respectively.

(2) One of the following conditions is satisfied: Either

$$\begin{cases} p(M)=0, \text{ that is, } p(A_{jk})=0=p(B_{vu}) \text{ and } p(C_{uj})=1=p(D_{ju}) \text{ or}\\ p(M)=1, \text{ that is, } p(A_{jk})=1=p(B_{vu}) \text{ and } p(C_{uj})=0=p(D_{ju}), \end{cases}$$

We call M is even (resp. odd) if $p(M)=0$ (resp. $p(M)=1$). Moreover, we many decompose M as $M=M_B+M_S$ where

$$M_B=\begin{cases}\begin{bmatrix}A_B & 0\\ 0 & B_B\end{bmatrix} & \text{when } p(M)=0\\[2ex] \begin{bmatrix}0 & C_B\\ D_B & 0\end{bmatrix} & \text{when } p(M)=1\end{cases}$$

It is clear that for $(m+n)\times(r+s)$ matrix M and $(r+s)\times(p+q)$ matrix N, we define the product MN as $(MN)_{ij}=\sum_k M_{ik}N_{kj}$ and the parity of MN is given by $p(MN)=p(M)+p(N)$. Moreover, we define $\mathrm{Mat}_{m|n}(\mathfrak{R})$ as the algebra of $(m+n)\times(m+n)$ supermatrices.

DEFINITION 2.8. Let $M=\begin{bmatrix}A & C\\ D & B\end{bmatrix}\in\mathrm{Mat}_{m|n}(\mathfrak{R})$. We define the supertrace of M by

(2.16) $$str\ M=\sum_k(-1)^{(p(M)+1)p_{row}(k)}M_{kk}=tr\ A-(-1)^{p(M)}\ tr\ B\,.$$

Here,

$$p_{row}(k)=\begin{cases}\left.\begin{array}{ll}0 & \text{for } 1\leqq k\leqq m,\\ 1 & \text{for } m+1\leqq k\leqq m+n\end{array}\right\} & \text{for } p(M)=0,\\ \left.\begin{array}{ll}1 & \text{for } 1\leqq k\leqq m,\\ 0 & \text{for } m+1\leqq k\leqq m+n\end{array}\right\} & \text{for } p(M)=1,\end{cases}$$

If $M\in \mathrm{Mat}_{m|n}(\mathfrak{R})$ is even, then M acts on $\mathfrak{R}^{m|n}$ linearly. Denoting this by T_M, we call it a super linear transformation on $\mathfrak{R}^{m|n}$ and M is called the representative matrix of T_M.

PROPOSITION 2.9. *Let $M\in \mathrm{Mat}_{m|n}(\mathfrak{R})$ be even and assume* $\det M_B\neq 0$. *Then, for given $Y\in\mathfrak{R}^{m|n}$,*

$$T_M X=Y \tag{2.17}$$

has the unique solution $X\in\mathfrak{R}^{m|n}$, which is denoted by $X=M^{-1}Y$.

Proof. Since M_B has the inverse matrix M_B^{-1}, (2.17) is reduced to

$$X+N_S X=Y', \qquad Y'=M_B^{-1}Y$$

where $N_S=M_B^{-1}M_S$. Remark that $N_S X_{[j]}\in\sum_{k\geqq j+1}^{\infty}\mathfrak{R}_{[k]}$ for $j\geqq 0$. Decomposing by order, we get

$$X_{[j]}=Y'_{[j]}-(N_S X_{(j-1)})_{[j]} \qquad \text{for } j=1, 2, \cdots.$$

As $X_{(0)}=X_{[0]}=Y'_{[0]}$, we get $X_{[j]}$ from $X_{(j-1)}$ for $j\geqq 1$ by induction. ■

DEFINITION 2.10. $M\in \mathrm{Mat}_{m|n}(\mathfrak{R})$ is called invertible or non-singular if M_B is invertible, i.e. $(\det A_B)(\det B_B)\neq 0$ if $p(M)=0$ or $(\det C_B)(\det D_B)\neq 0$ if $p(M)=1$.

DEFINITION 2.11. Let M be a supermatrix. When $\det B_B\neq 0$, we put

$$\mathit{sdet}\, M=(\det(A-CB^{-1}D))(\det B)^{-1} \tag{2.18}$$

and call it superdeterminant or Berezinian of M.

Remark. Let $B=(B_{jk})$ be $(q\times q)$-matrix with elements in $\mathfrak{R}_{ev}$. As $\mathfrak{R}_{ev}$ is a commutative ring, we may define $\det B$ as usual:

$$\det B=\sum_{\rho\in\mathfrak{p}_q}\mathrm{sgn}(\rho)B_{1\,\rho(1)}\cdots B_{q\,\rho(q)}.$$

Following decomposition of a even supermatrix M will be useful.

$$\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} I_m & CB^{-1} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A-CB^{-1}D & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_m & 0 \\ B^{-1}D & I_n \end{bmatrix}$$

$$= \begin{bmatrix} I_m & 0 \\ DA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B-DA^{-1}C \end{bmatrix} \begin{bmatrix} I_m & A^{-1}C \\ 0 & I_n \end{bmatrix}.$$

PROPOSITION 2.12. *Let M, N be even super matrices in* $\mathrm{Mat}_{m|n}(\mathfrak{R})$.

(1) *If M is invertible, then we have sdet $M \neq 0$. Moreover, if A is nonsingular, then*

$$(sdet\ M)^{-1} = (\det A)^{-1}(\det(B-DA^{-1}C)).$$

(2) *sdet $(MN) = ($sdet $M)($sdet $N)$.*

(3) *str and sdet are (even) matrix invariants. That is, if N is invertible, then*

$$str\ M = str\,(NMN^{-1}), \qquad sdet\ M = sdet\,(NMN^{-1}).$$

(4) *Let $M(x,\theta) = \begin{bmatrix} A(x,\theta) & C(x,\theta) \\ D(x,\theta) & B(x,\theta) \end{bmatrix}$ be a even invertible supermatrix such that each matrix elements are supersmooth in $X=(x,\theta)$. Then, we have*

$$\partial_X(sdet\ M(X)) = (sdet\ M(X))\, str\,(M^{-1}(X)(\partial_X M(X))) \tag{2.19}$$

$$= (sdet\ M(X))\, str\,((\partial_X M(X))M^{-1}(X)).$$

Proof. See the proofs in [2], [6], [17] or [27].

Now, return to state our elementary analysis.

For $f(X) \in \mathcal{C}_{SS}(U;\mathfrak{C})$ on a superdomain $U \subset \mathfrak{R}^{m|n}$, we put

$$d_X f(x,\theta) = [\partial_{x_j} f(x,\theta), \vec{\partial}_{\theta_r} f(x,\theta)] \in \mathfrak{C}^{m+n} \tag{2.20}$$

and call it the Jacobian matrix (or differential) of f at $X=(x,\theta)$.

From Definition 2.4, we get readily

PROPOSITION 2.13. *Let U be a superdomain in $\mathfrak{R}^{m|n}$. For $f, g \in \mathcal{C}_{SS}(U;\mathfrak{C})$, the product fg belongs to $\mathcal{C}_{SS}(U;\mathfrak{C})$ and the differentials $d_X f(X)$ and $d_X g(X)$ may be regarded as continuous linear mappings from $\mathfrak{R}^{m|n}$ into $\mathfrak{C}^{m+n}$. Moreover, they satisfy the following:*

(1) *For any homogeneous elements $\lambda, \mu \in \mathfrak{C}$, we have*

$$d_X(\lambda f + \mu g)(X) = (-1)^{p(\lambda)p(X)}\lambda d_X f(X) + (-1)^{p(\mu)p(X)}\mu d_X g(X). \tag{2.21}$$

(2) *(Leibnitz formula)*

$$\partial_{X_\kappa}[f(X)g(X)] = (\partial_{X_\kappa} f(X))g(X) + (-1)^{p(X_\kappa)p(f(X))} f(X)(\partial_{X_\kappa} g(X)). \tag{2.22}$$

Proof. For the product, as we get

$$(fg)(x_B)=\Big(\sum_I f_I(x_B)\sigma^I\Big)\Big(\sum_J g_J(x_B)\sigma^J\Big)=\sum_H h_H(x_B)\sigma^H$$

where $h_H(x_B)=\sum_{H=I+J}(-1)^{r(H;I,J)}f_I(x_B)g_J(x_B)\in C^\infty(U_B;\boldsymbol{C})$, so we have the desired result. (2.21) is obvious. To get (2.22), use the formula (2.11). ∎

DEFINITION 2.14. Let $U\subset\mathfrak{R}^{m|n}$ and $U'\subset\mathfrak{R}^{m'|n'}$ be superdomains and let φ be a continuous mapping from U to U', denoted by $\varphi(X)=(\varphi_1(X),\cdots,\varphi_{m'}(X),\varphi_{m'+1}(X),\cdots,\varphi_{m'+n'}(X))\in\mathfrak{R}^{m'|n'}$. φ is called a supersmooth mapping from U to U' if each $\varphi_\kappa(X)\in\mathcal{C}_{SS}(U;\mathfrak{C})$ for $\kappa=1,\cdots,m'+n'$ and $\varphi(U)\subset U'$.

PROPOSITION 2.15 (Composition of supersmooth mappings). *Let $U\subset\mathfrak{R}^{m|n}$ and $U'\subset\mathfrak{R}^{m'|n'}$ be superdomains and let $\Phi:U\to U'$ and $\Phi':U'\to\mathfrak{R}^{m''|n''}$ be supersmooth mappings. Then, the composition $\Psi=\Phi'\circ\Phi:U\to\mathfrak{R}^{m''|n''}$ gives a supersmooth mapping and*

$$d_X\Psi(X)=[d_Y\Phi'(Y)]|_{Y=\Phi(X)}[d_X\Phi(X)]\,. \tag{2.23}$$

Proof. (1) First of all, we prove our assertion for the case m, m' are arbitrary, $n=n'=0$ and $m''=n''=1$: Let $U_{ev}\subset\mathfrak{R}^{m|0}$ and $U'_{ev}\subset\mathfrak{R}^{m'|0}$ be even superdomains and let $\varphi:U_{ev}\to U'_{ev}$ be a supersmooth mapping represented by $\varphi(x)=(\varphi_1(x),\cdots,\varphi_{m'}(x))$ with $\varphi_j(x)\in\mathcal{C}_{SS}(U_{ev};\mathfrak{C})$. For any $f\in\mathcal{C}_{SS}(U'_{ev};\mathfrak{C})$, we want to claim that $(\varphi^*f)(x)=(f\circ\varphi)(x)=f(\varphi(x))$, is well-defined and belongs to $\mathcal{C}_{SS}(U_{ev};\mathfrak{C})$. Putting

$$y=\varphi(x_B)=\varphi_B(x_B)+\varphi_S(x_B)=y_B+y_S \qquad \text{with } \varphi_S(x_B)=\sum_{|J|\ge1}\varphi_J(x_B)\sigma^J\,,$$

we define, by using the supersmoothness of f and φ,

$$f(\varphi(x_B))_{[k]}=\sum_{\substack{|\alpha|\le k\\ k_0+k_1+\cdots+k_m=k}}\frac{1}{\alpha!}(\partial_y^\alpha f)_{[k_0]}(y_B)(y_{1,S}^{\alpha_1})_{[k_1]}\cdots(y_{m,S}^{\alpha_m})_{[k_m]}\Big|_{y=\varphi(x_B)}. \tag{2.24}$$

By the same reasoning as in the proof of Proposition 2.2, $f(\varphi(x_B))_{[k]}$ is well-defined and belongs to $C^\infty(U_B;\mathfrak{C}_{[k]})$, so $f(\varphi(x_B))=\sum_{k=0}^\infty f(\varphi(x_B))_{[k]}\in C^\infty(U_B;\mathfrak{C})$. Therefore, it has the Grassmann continuation which should be denoted by $(f\circ\varphi)(x)$. On the other hand, as we get from (2.24),

$$\begin{aligned}
&\partial_{x_j,B}(f\circ\varphi)_{[k]}(x_B)\\
&=\sum_{\substack{\ell,|\alpha|\le k,\\ k_0+k_1+\cdots+k_m=k}}\frac{1}{\alpha!}(\partial_y^\alpha\partial_{y_\ell}f)_{[k_0]}(y_B)\frac{\partial\varphi_{\ell,B}(x_B)}{\partial x_{j,B}}(y_{1,S}^{\alpha_1})_{[k_1]}\cdots(y_{m,S}^{\alpha_m})_{[k_m]}\Big|_{y=\varphi(x_B)}\\
&=\sum_{\substack{\ell,|\alpha|\le k\\ k_0+k_1+\cdots+\\ k'_\ell+k''_\ell+\cdots+k_m=k}}\frac{1}{\alpha!}(\partial_y^\alpha\partial_{y_\ell}f)_{[k_0]}(y_B)\\
&\qquad\times(y_{1,S}^{\alpha_1})_{[k_1]}\cdots\alpha_\ell(y_{\ell,S}^{\alpha_\ell-1})_{[k'_\ell]}\Big(\frac{\partial\varphi_{\ell,S}(x_B)}{\partial x_{j,B}}\Big)_{[k''_\ell]}\cdots(y_{m,S}^{\alpha_m})_{[k_m]}\Big|_{y=\varphi(x_B)}
\end{aligned} \tag{2.25}$$

$$=\sum_{\ell}\sum_{k_0=0}^{k}(\partial_{y_\ell}f(\varphi(x_B)))_{[k_0]}\Big(\frac{\partial\varphi_\ell(x_B)}{\partial x_{j,B}}\Big)_{[k-k_0]}.$$

This is the desired result (2.23) in the case of (1).

(2) Now, we treat the case m, m', n, n' are arbitrary and $m''=n''=1$: Let $U\subset\mathfrak{R}^{m|n}$ and $U'\subset\mathfrak{R}^{m'|n'}$ be superdomains and let $\varphi: U\to U'$ and $f: U'\to\mathfrak{C}$ be supersmooth mappings. Put $\varphi(x,\theta)=(\varphi_\kappa(x,\theta))$, $1\leqq\kappa\leqq m'+n'$ where $\varphi_\kappa(x,\theta)=\sum_a\varphi_{\kappa,a}(x)\theta^a$ and $f(y,\omega)=\sum_b f_b(y)\omega^b$ with $b=(b_1,\cdots,b_{n'})\in\{0,1\}^{n'}$. We decompose

$$\varphi_j(x,\theta)=Y_j=Y_j^{(0)}+Y_j^{(1)}\qquad\text{for } 1\leqq j\leqq m'$$

where

$$\begin{cases} Y_j^{(0)}=\varphi_{j,\tilde{0}}(x)=Y_{j,B}^{(0)}+Y_{j,S}^{(0)} & \text{with } Y_{j,B}^{(0)}=\varphi_{j,\tilde{0},B}(x),\ Y_{j,S}^{(0)}=\varphi_{j,\tilde{0},S}(x),\\ Y_j^{(1)}=\sum_{1\leq|a|\leq n}\varphi_{j,a}(x)\theta^a. \end{cases}$$

Then, we consider formally

$$\tilde{F}(x,\theta)=\sum_b f_b(Y_1,\cdots,Y_{m'})(Y_{m'+1})^{b_1}\cdots(Y_{m'+n'})^{b_{n'}}. \tag{2.26}$$

Remarking that $Y_j^{(1)}Y_j^{(1)}=0$, we apply Taylor's formula for $f_b(Y^{(0)}+Y^{(1)})$ at $Y=Y^{(0)}$ to get

$$f_b(Y^{(0)}+Y^{(1)})=f_b(Y^{(0)})+\sum_{j=1}^{m'}\partial_{y_j}f_b(Y^{(0)})Y_j^{(1)}+\cdots+\partial_{y_1}\cdots\partial_{y_{m'}}f_b(Y^{(0)})Y_1^{(1)}\cdots Y_{m'}^{(1)}. \tag{2.27}$$

On the other hand, as

$$f_b(Y^{(0)})=\sum_\alpha\frac{1}{\alpha!}\partial_Y^\alpha f_b(Y_B^{(0)})(Y_S^{(0)})^\alpha, \tag{2.28}$$

we get easily

$$f_b(\varphi_1(x,\theta),\cdots,\varphi_{m'}(x,\theta))=\sum_c g_{b,c}(x)\theta^c \tag{2.29}$$

where $g_{b,c}(x)$ is a supersmooth function on U_{ev} composed by the products of supersmooth functions $\partial_y^\alpha f(\varphi_B(x))$ and $\varphi_{\kappa,a}(x)$. Combining these, we get

$$\begin{aligned}\tilde{F}(x,\theta)&=\sum_b(\sum_c g_{b,c}(x)\theta^c)(\sum_{\tilde{a}_1}\varphi_{m'+1,\tilde{a}_1}(x)\theta^{\tilde{a}_1})^{b_1}\cdots(\sum_{\tilde{a}_{n'}}\varphi_{m'+n',\tilde{a}_{n'}}(x)\theta^{\tilde{a}_{n'}})^{b_{n'}}\\ &=\sum_d\tilde{F}_d(x)\theta^d,\end{aligned} \tag{2.30}$$

where $d=(d_s)$, $c=(c_s)$, $\tilde{a}_s=(\tilde{a}_{s,r})$, $d_s=c_s+b_1\tilde{a}_{1,s}+\cdots+b_{n'}\tilde{a}_{n',s}$ with $1\leqq s\leqq n$ and $1\leqq r\leqq n'$. Therefore, we get $\tilde{F}_d(x)\in\mathcal{C}_{SS}(U_{ev};\mathfrak{C})$, that is, $\tilde{F}(x,\theta)=f(\varphi(x,\theta))\in\mathcal{C}_{SS}(U;\mathfrak{C})$. To get (2.23), we differentiate (2.26) with respect to x_k,

$$\partial_{x_k}\tilde{F}(x,\theta)=\sum_{j=1}^{m'}\sum_b \partial_{y_j}f_b(\varphi_{ev}(x,\theta))\frac{\partial\varphi_j(x,\theta)}{\partial x_k}(\varphi_{od}(x,\theta))^b$$
$$+\sum f_b(\varphi_{ev}(x,\theta))\sum_{s=m'+1}^{m'+n'}(-1)^{b_1+\cdots+b_{s-1}}b_s\frac{\partial\varphi_s(x,\theta)}{\partial x_k}\prod_{l=1}^{(s,n')}\varphi_{m'+l}(x,\theta)^{b_l}.$$

Here, $\prod_{l=1}^{(s,n')}\varphi_{m'+l}(x,\theta)^{b_l}=\overbrace{\varphi_{m'+1}(x,\theta)^{b_1}\cdots 1}^{s}\cdots\varphi_{m'+n'}(x,\theta)^{b_{n'}}$, $\varphi_{ev}(x,\theta)=(\varphi_j(x,\theta))_{j=1}^{m'}$ and $\varphi_{od}(x,\theta)=(\varphi_{m'+s}(x,\theta))_{s=1}^{n'}$.

Taking derivatives with respect to θ_r, we get the similar expression as above and combining these, we have

$$[\partial_{x_k}\tilde{F}(x,\theta),\vec{\partial}_{\theta_r}\tilde{F}(x,\theta)]=\begin{bmatrix}\dfrac{\partial\varphi_j(x,\theta)}{\partial x_k},\cdots,\dfrac{\vec{\partial}\varphi_j(x,\theta)}{\partial\theta_r}\\ \dfrac{\partial\varphi_s(x,\theta)}{\partial x_k},\cdots,\dfrac{\vec{\partial}\varphi_s(x,\theta)}{\partial\theta_r}\end{bmatrix}\begin{bmatrix}\dfrac{\partial f(y,\omega)}{\partial y_j},\dfrac{\vec{\partial}f(y,\omega)}{\partial\omega_s}\end{bmatrix},$$

this is, (2.23) in the case of (2).

(3) For the general situation mentioned above, using the arguments in (2) repeatedly, we get the result after tedious but straightfoward calculations. ■

DEFINITION 2.16. Let $U\subset\mathfrak{R}^{m|n}$ and $U'\subset\mathfrak{R}^{m'|n'}$ be superdomains and let $\varphi: U\to U'$ be a supersmooth mapping represented by $\varphi(X)=(\varphi_1(X),\cdots,\varphi_{m'+n'}(X))$ with $\varphi_\kappa(X)\in\mathcal{C}_{SS}(U;\mathfrak{C})$. (1) φ is called a supersmooth diffeomorphism if (i) φ is a homeomorphism between U and U' and (ii) φ and φ^{-1} are supersmooth mappings. (2) For any $f\in\mathcal{C}_{SS}(U';\mathfrak{C})$, $(\varphi^*f)(X)=(f\circ\varphi)(X)=f(\varphi(X))$, called the pull back of f, is well-defined and belongs to $\mathcal{C}_{SS}(U;\mathfrak{C})$.

Remarks. (1) It is easy to see that if φ is a supersmooth diffeomorphism, then $\varphi_B=\pi_B\circ\varphi$ is an (ordinary) C^∞ diffeomorphism from U_B to U'_B. (2) If we introduce the topologies in $\mathcal{C}_{SS}(U';\mathfrak{C})$ and $\mathcal{C}_{SS}(U;\mathfrak{C})$ properly, φ^* gives a continuous linear mapping from $\mathcal{C}_{SS}(U;\mathfrak{C})$ to $\mathcal{C}_{SS}(U;\mathfrak{C})$. Moreover, if $\varphi: U\to U'$ is a supersmooth diffeomorphism, then φ^* defines an automorphism from $\mathcal{C}_{SS}(U';\mathfrak{C})$ to $\mathcal{C}_{SS}(U;\mathfrak{C})$.

PROPOSITION 2.17 (Inverse function theorem). *Let U be a superdomain in $\mathfrak{R}^{m|n}$ and let $G(X): U\subset\mathfrak{R}^{m|n}\to\mathfrak{R}^{m|n}$ be a supersmooth mapping. We assume the super matrix $[d_XG(X)]$ is invertible at $X=\tilde{X}_B\in\pi_B(U)$. Then, there exists a superdomain U', a neighbourhood of $\tilde{Y}=G(\tilde{X})$ and a unique supersmooth mapping F satisfying $F(G(X))=X$ and we have*

$$d_YF(Y)=(d_XG(X))^{-1}|_{X=F(Y)}\quad in\ U'. \tag{2.31}$$

Proof. (1) First of all, we treat the case $m=1$ and $n=0$, that is, U_{ev}, $U'_{ev}\subset\mathfrak{R}^{1|0}$. Let $g: U_{ev}\to U'_{ev}$ be a supersmooth function represented by

$$y=g(x_B)=g_B(x_B)+\sum_{|J|=even\geq 2} g_J(x_B)\sigma^J=y_B+y_S\,.$$

Here, $g_B(x_B)\in C^\infty(U_B;\boldsymbol{R})$ and $g_J(x_B)\in C^\infty(U_B;\boldsymbol{C})$. By assumption that $g_B'(\tilde{x}_B)\neq 0$, there exists a smooth function f_B such $f_B((g_B(x_B))=x_B$ near $x_B=\tilde{x}_B$. We want to construct a family of functions $f_I\in C^\infty(U_B';\boldsymbol{C})$ such that $f(y_B)=f_B(y_B)+f_S(y_B)$, $f_S(y_B)=\sum_{|I|=even\geq 2}f_I(y_B)\sigma^I$ satisfying $f(g(x_B))=x_B$ near $x_B=\tilde{x}_B$. As we should have

$$\begin{aligned}(2.32)\qquad x_B&=f_B(y_B+y_S)+f_S(y_B+y_S)\\&=f_B(y_B)+\sum_{k\geq 1}\frac{1}{k!}f_B^{(k)}(y_B)y_S{}^k+\sum_{\ell\geq 0}\frac{1}{\ell!}f_S^{(\ell)}(y_B)y_S{}^\ell\,,\end{aligned}$$

we get

$$(2.33)\qquad f_S(y_B)=-\sum_{k\geq 1}\frac{1}{k!}f_B^{(k)}(y_B)y_S{}^k-\sum_{k\geq 1}\frac{1}{k!}f_S^{(k)}(y_B)y_S{}^k\,.$$

We prove our statement using the induction with respect to the degree. The degree 2 part of (2.33) is given by

$$(2.34)\qquad f_S(y_B)_{[2]}=-f_B'(y_B)y_{S,[2]}\,.$$

In other word, for I such that $|I|=2$, we may define functions $f_I(y_B)$ by

$$f_I(y_B)=-f_B'(y_B)g_I(f_B(y_B))(=-f_B'(g_B(x_B))g_I(x_B))\,.$$

Assuming that f_S are defined for degrees less than $2i$, we put,

$$(2.35)\qquad f_S(y_B)_{[2i+2]}=-\sum_{k\geq 1}\frac{1}{k!}f_B^{(k)}(y_B)(y_S^k)_{[2i+2]}-\sum_{k\geq 1}\sum_{j=0}^{i}\frac{1}{k!}(f_S^{(k)}(y_B))_{[2j]}(y_S^k)_{[2i+2-2j]}.$$

So, we may define $f(y_B)=\sum_{j=0}^\infty f(y_B)_{[2j]}=f_B(y_B)+\sum_{j=1}^\infty f_S(y_B)_{[2j]}\in C^\infty(U_B';\mathfrak{C})$. Taking the Grassmann continuation of $f(y_B)$ and remarking $\partial_x f(g(x))=1$, we get the desised result.

(2) We next consider the case $m=n=1$, that is, $U, U'\subset\mathfrak{R}^{1|1}$. Let $G(x,\theta)$ $(g_{ev}(x,\theta), g_{od}(x,\theta)): U\to U'$ be a supersmooth mapping given by

$$(2.36)\qquad g_{ev}(x,\theta)=g_{ev,0}(x)+g_{ev,1}(x)\theta\,,\qquad g_{od}(x,\theta)=g_{od,1}(x)+g_{od,0}(x)\theta\,.$$

For simplicity, we put

$$g_{ev}(x_B,\theta)=y_B+y_S+\bar{y}\theta\quad\text{where}\left\{\begin{array}{l}y_B=g_{ev,0,B}(x_B),\ y_S=\sum_{|I|=even\geq 2}g_{ev,0,I}(x_B)\sigma^I,\\ \bar{y}=\sum_{|\bar{I}|=odd\geq 1}g_{ev,1,\bar{I}}(x_B)\sigma^{\bar{I}},\end{array}\right.$$

and

$$g_{od}(x_B,\theta)=\omega+\bar{\omega}\theta\quad\text{where}\left\{\begin{array}{l}\omega=\sum_{|\bar{I}|=odd\geq 1}g_{od,1,\bar{I}}(x_B)\sigma^{\bar{I}},\\ \bar{\omega}=\bar{\omega}_B+\bar{\omega}_S\\ \quad=g_{od,0,B}(x_B)+\sum_{|I|=even\geq 2}g_{od,0,I}(x_B)\sigma^I.\end{array}\right.$$

From $\tilde{Y}=G(\tilde{X})$ and the invertibility of $d_X G(X)|_{X=\tilde{X}}$, we get

$$g_{ev,0,B}(\tilde{x}_B)=\tilde{y}_B\,, \qquad g'_{ev,0,B}(\tilde{x}_B)g_{od,0,B}(\tilde{x}_B)\neq 0\,. \tag{2.37}$$

Now, we seek a function $F(Y)=F(y,\omega)=(f_{ev}(y,\omega), f_{od}(y,\omega)): U'\to U$ represented by

$$f_{ev}(y,\omega)=f_{ev,0}(y)+f_{ev,1}(y)\omega, \qquad f_{od}(y,\omega)=f_{od,1}(y)+f_{od,0}(y)\omega$$

which satifies $F(G(X))=X$ near $X=(x,\theta)=(\tilde{x},\tilde{\theta})=\tilde{X}$. Here, we put

$$\begin{cases} f_{ev,0}(y_B)=f_{ev,0,B}(y_B)+\sum_{|I|=even\geq 2} f_{ev,0,I}(y_B)\sigma^I, \\ f_{ev,1}(y_B)=\sum_{|\bar{I}|=odd\geq 1} f_{ev,1,\bar{I}}(y_B)\sigma^{\bar{I}}, \\ f_{od,1}(y_B)=\sum_{|\bar{I}|=odd\geq 1} f_{od,1,\bar{I}}(y_B)\sigma^{\bar{I}}, \\ f_{od,0}(y_B)=f_{od,0,B}(y_B)+\sum_{|I|=even\geq 2} f_{od,0,I}(y_B)\sigma^I. \end{cases}$$

As $F(G(x_B,\theta))=(x_B,\theta)$, we should have the relations

$$f_{ev}(g_{ev}(x_B,\theta), g_{od}(x_B,\theta))=x_B, \qquad f_{od}(g_{ev}(x_B,\theta), g_{od}(x_B,\theta))=\theta. \tag{2.38}$$

From the first equation in (2.38) and the supersmoothness, we have

$$\begin{aligned} x_B &= f_{ev,0}(y_B+y_S+\bar{y}\theta)+f_{ev,1}(y_B+y_S+\bar{y}\theta)(\omega+\bar{\omega}\theta) \\ &= f_{ev,0}(y_B)+\sum_{|k|\geq 1}\frac{1}{k\,!}f^{(k)}_{ev,0}(y_B)(y_S^k+k y_S^{k-1}\bar{y}\theta) \\ &\qquad\qquad + \sum_{|\ell|\geq 0}\frac{1}{\ell\,!}f^{(\ell)}_{ev,0}(y_B)(y_S^\ell+\ell y_S^{\ell-1}\bar{y}\theta)(\omega+\bar{\omega}\theta) \\ &= f_{ev,0}(y_B)+\sum_{|k|\geq 1}\frac{1}{k\,!}f^{(k)}_{ev,1}(y_B)y_S^k+\sum_{|\ell|\geq 0}\frac{1}{\ell\,!}f^{(\ell)}_{ev,1}(y_B)y_S^\ell\omega \\ &\quad +\Big\{\sum_{|k|\geq 1}\frac{1}{(k-1)\,!}(f^{(k)}_{ev,0}(y_B)+f^{(k)}_{ev,1}(y_B)\omega)y_S^{k-1}\bar{y}+\sum_{|\ell|\geq 0}\frac{1}{\ell\,!}f^{(\ell)}_{ev,1}(y_B)y_S^\ell\bar{\omega}\Big\}\theta. \end{aligned}$$

Therefore

$$x_B=f_{ev,0,B}(y_B)+f_{ev,0,S}(y_B)+\sum_{|k|\geq 1}\frac{1}{k\,!}f^{(k)}_{ev,0}(y_B)y_S^k+\sum_{|\ell|\geq 0}\frac{1}{\ell\,!}f^{(\ell)}_{ev,1}(y_B)y_S^\ell\omega \tag{2.38}$$

and

$$0=\sum_{|k|\geq 1}\frac{1}{(k-1)\,!}(f^{(k)}_{ev,0}(y_B)+f^{(k)}_{ev,1}(y_B)\omega)y_S^{k-1}\bar{y}+\sum_{|\ell|\geq 0}\frac{1}{\ell\,!}f^{(\ell)}_{ev,1}(y_B)y_S^\ell(\bar{\omega}_B+\bar{\omega}_S). \tag{2.40}$$

As $g'_{ev,0,B}(\tilde{x}_B)\neq 0$ by (2.37), using the standard inverse function theorem, there exists a function $f_{ev,0,B}(y_B)$ such that

$$f_{ev,0,B}(g_{ev,0,B}(x_B))=x_B \tag{2.41}$$

near $x_B=\tilde{x}_B$. Therefore, we get from (2.39),

$$f_{ev,0,S}(y_B)+\sum_{|k|\geq 1}\frac{1}{k!}f_{ev,0}^{(k)}(y_B)y_S^k+\Big(f_{ev,1}(y_B)+\sum_{|k|\geq 1}\frac{1}{k!}f_{ev,1}^{(k)}(y_B)y_S^k\Big)\omega=0. \tag{2.42}$$

For each I satisfying $|I|=1$, we pick up the term of degree 1 from (2.40) to get

$$f_{ev,1,I}(y_B)g_{od,0,B}(x_B)+f'_{ev,0,B}(g_{ev,0,B}(x_B))g_{ev,1,I}(x_B)=0\,. \tag{2.43}$$

As $g'_{ev,0,B}(x_B)g_{od,0,B}(x_B)\neq 0$ by (2.37), there exists a function $f_{ev,1,I}(y_B)$ such that the above equation is satisfied when $y_B=g_{ev,0,B}(x_B)$. Equations (2.41) and (2.42) correspond to the degree 0 and 1 part of (2.39) and (2.40), respectively.

Using these, we may solve the degree 2 part of (2.39) and then the degree 3 part of (2.40). Doing recursively, we may construct functions $f_{ev,0}$ and $f_{ev,1}$.

From the second equation of (2.38), we get

$$\begin{aligned}\theta&=f_{od,1}(y_B+y_S+\bar{y}\theta)+f_{od,0}(y_B+y_S+\bar{y}\theta)(\omega+\bar{\omega}\theta)\\&=\sum_{|\ell|\geq 0}\frac{1}{\ell!}f_{od,1}^{(\ell)}(y_B)y_S^\ell-\sum_{|\ell|\geq 0}\frac{1}{\ell!}f_{od,0}^{(\ell)}(y_B)y_S^\ell\omega\\&\quad+\Big\{\sum_{|k|\geq 1}\frac{1}{(k-1)!}(f_{od,1}^{(k)}(y_B)+f_{od,0}^{(k)}(y_B)\omega)y_S^{k-1}\bar{y}+\sum_{|k|\geq 1}\frac{1}{k!}f_{od,0}^{(k)}(y_B)\,y_S^k\bar{\omega}\Big\}\theta.\end{aligned}$$

That is,

$$0=f_{od,1,S}(y_B)+\sum_{|k|\geq 1}\frac{1}{k!}f_{od,1}^{(k)}(y_B)y_S^k+\sum_{|\ell|\geq 0}\frac{1}{\ell!}f_{od,0}^{(\ell)}(y_B)y_S^\ell\omega \tag{2.44}$$

and

$$1=\sum_{|k|\geq 1}\frac{1}{(k-1)!}(f_{od,1}^{(k)}(y_B)+f_{od,0}^{(k)}(y_B)\omega)y_S^{k-1}\bar{y}+\sum_{|k|\geq 1}\frac{1}{k!}f_{od,0}^{(k)}(y_B)y_S^k\bar{\omega}\,. \tag{2.45}$$

By the same arguments as above, we may construct functions $f_{od,1}(y_B)$ and $f_{od,0}(y_B)$ which satisfy the desired properties.

(3) For general m, n, we do analogously as above but with more patience. ■

Moreover, we have

PROPOSITION 2.18 (Implicit Function Theorem). *Let $\Phi(X,Y):U\times U'\to\mathfrak{C}^{m'|n'}$ be a supersmooth mapping and $(\tilde{X},\tilde{Y})\in U\times U'$, where U and U' are superdomains of $\mathfrak{R}^{m|n}$ and $\mathfrak{R}^{m'|n'}$, respectively. Suppose $\Phi(\tilde{X},\tilde{Y})=0$ and $\partial_Y\Phi=[\partial_{y_i}\Phi,\vec{\partial}_{\omega_r}\Phi]$ is a continuous and invertible supermatrix at $(\tilde{X}_B,\tilde{Y}_B)\in\pi_B(U)\times\pi_B(U')$. Then, there exist a superdomain $V\subset U$ satisfying $\tilde{X}_B\in\pi_B(V)$ and a unique supersmooth mapping $Y=f(X)$ on V such that $\tilde{Y}=f(\tilde{X})$ and $\Phi(X,f(X))=0$ in V. Moreover, we have*

$$\partial_X f(X)=-[\partial_Y\Phi(X,Y)]^{-1}[\partial_X\Phi(X,Y)]|_{Y=f(X)}\,. \tag{2.46}$$

Proof. (2.46) is easily obtained by

$$0=\partial_X\Phi(X, f(X))=(\partial_X\Phi(X, Y)+\partial_Y\Phi(X, Y)\partial_X f(X))|_{Y=f(X)}.$$

The existence proof is omitted here because the arguments in proving Proposition 2.16 work well in this situation. ■

§ 3. Integration

Integration (even case). Now, we define the integration of a supersmooth function $u(x)$ on an even superdomain $U_{ev}\subset\mathfrak{R}^{m|0}$, which is similar to the integral of holomorphic functions on a complex domain. (See, Rogers [24] or [27].)

DEFINITION 3.1. Let $u(x)$ be a supersmooth function defined on a even superdomain $U_{ev}\subset\mathfrak{R}^{1|0}$. Let $\lambda=\lambda_B+\lambda_S$, $\mu=\mu_B+\mu_S\in U_{ev}$ and let a continuous and piecewise C^1-curve $c:[\lambda_B, \mu_B]\to U_{ev}$ be given such that $c(\lambda_B)=\lambda$, $c(\mu_B)=\mu$. We define

$$\int_c dx\, u(x)=\int_{\lambda_B}^{\mu_B}dt\, u(c(t))\dot{c}(t)\in\mathfrak{C} \tag{3.1}$$

and call it the integral of u along the curve c.

Using the integration by parts, we get the following fundamental result (see [6]).

PROPOSITION 3.2. *Let $u(t)\in C^\infty([\lambda_B, \mu_B]; \mathfrak{C})$ and let $u(x)$ be the Grassmann continuation of $u(t)$. Suppose that there exists a function $U(t)\in C^\infty([\lambda_B, \mu_B]; \mathfrak{C})$ satisfying $U'(t)=u(t)$ on $[\lambda_B, \mu_B]$. Then, for any continuous and piecewise C^1-curve $c:[\lambda_B, \mu_B]\to U_{ev}\subset\mathfrak{R}^{1|0}$ such that $c(\lambda_B)=\lambda$, $c(\mu_B)=\mu$, we have*

$$\int_c dx\, u(x)=U(\lambda)-U(\mu). \tag{3.2}$$

Proof. By definition, we get

$$\begin{aligned}\int_{\lambda_B}^{\mu_B}dt\, u(c(t))\dot{c}(t)&=\int_{\lambda_B}^{\mu_B}dt \sum_{\ell\geq 0}\frac{1}{\ell!}u^{(\ell)}(c_B(t))c_S(t)^\ell(\dot{c}_B(t)+\dot{c}_S(t))\\
&=\int_{\lambda_B}^{\mu_B}dt\, u(c_B(t))\dot{c}_B(t)+\int_{\lambda_B}^{\mu_B}dt\sum_{k\geq 1}\frac{1}{k!}u^{(k)}(c_B(t))\dot{c}_B(t)c_S(t)^k\\
&\quad+\int_{\lambda_B}^{\mu_B}dt\sum_{\ell\geq 0}\frac{1}{\ell!}u^{(\ell)}(c_B(t))c_S(t)^\ell\dot{c}_S(t)\\
&=U(\mu_B)-U(\lambda_B)+\sum_{\ell\geq 0}\frac{1}{(\ell+1)!}\{U^{(\ell+1)}(\mu_B)(\mu_S)^{\ell+1}-U^{(\ell+1)}(\lambda_B)(\lambda_S)^{\ell+1}\}\\
&=U(\mu)-U(\lambda).\quad ■\end{aligned}$$

COROLLARY 3.3. *Let $u(x)$ be a supersmooth function defined on a even super-*

domain $U_{ev}\subset\mathfrak{R}^{1|0}$ *into* $\mathfrak{C}$. *Let* c_1, c_2 *be continuous and piecewise* C^1*-curves from* $[\lambda_B, \mu_B]\to U_{ev}$ *such that* $\lambda=c_1(\lambda_B)=c_2(\lambda_B)$ *and* $\mu=c_1(\mu_B)=c_2(\mu_B)$. *If* c_1 *is homotopic to* c_2, *then*

$$\int_{c_1} dx\, u(x)=\int_{c_2} dx\, u(x). \tag{3.3}$$

Thus, if $[\lambda_B, \mu_B]\subset\pi_B(U_{ev})$, *we have*

$$\int_{\lambda}^{\mu} dx\, u(x)=\int_{\lambda_B}^{\mu_B} dt\, u(t). \tag{3.4}$$

Because of (3.4), we have

DEFINITION 3.4. (1) Let I_{ev} be a even superdomain in $\mathfrak{R}^{m|0}$ such that $\pi_B(I_{ev})=\prod_{j=1}^{m}(a_j, b_j)\subset\boldsymbol{R}^m$ with $-\infty<a_j<b_j<\infty$, which is called a even supercube. For $u\in\mathcal{C}_{SS}(I_{ev};\mathfrak{C})$, we define

$$\int_{I_{ev}} dx\, u(x)=\int_{a_1}^{b_1} dq_1\cdots\int_{a_m}^{b_m} dq_m\, u(q_1, \cdots, q_m)=\int_{\pi_B(I_{ev})} dx_B\, u(x_B). \tag{3.5}$$

(2) For any even superdomain $U_{ev}\subset\mathfrak{R}^{m|0}$ such that $\pi_B(U_{ev})$ is of definite area, we may put

$$\int_{U_{ev}} dx\, u(x)=\int_{\pi_B(U_{ev})} dx_B\, u(x_B) \tag{3.6}$$

for $u\in\mathcal{C}_{SS}(U_{ev};\mathfrak{C})$.

Remarks. (1) The formula (3.6) stemms easily from the well-known procedures to define multiple integrals in Riemannian integration. (2) The reason why we should use 'contour integration' is explained precisely in [24]. As we treat only even superdomains here, her arguments there are simplified considerably. But we should change the role of the 'body' in our treatment, if we need to catch up all arguments of Rogers, which is noted in the remark after Proposition 2.5.

Integration (odd and mixed case). Let v be a polynomial of odd variables $\theta=(\theta_1, \cdots, \theta_n)\in\mathfrak{R}_{od}^n$ such that

$$v(\theta_1, \cdots, \theta_n)=\sum_{|b|\leq n} v_b\theta^b \text{ with homogeneous } v_b\theta^b\in\mathfrak{C} \text{ for each } b.$$

Denote by $P_n(\mathfrak{C})$ the set of all v as above.

DEFINITION 3.5. For $v\in P_n(\mathfrak{C})$, we put

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta)=\int_{\mathfrak{R}^{0|n}} d\theta_n\cdots d\theta_1 v(\theta_1, \cdots, \theta_n)=(\vec{\partial}_{\theta_n}\cdots\vec{\partial}_{\theta_1} v)(0) \tag{3.7}$$

and we call it the integral of v on $\mathfrak{R}^{0|n}$.

Above definition yields readily that

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, \theta_1 \cdots \theta_n = 1. \tag{3.8}$$

Moreover, we have

PROPOSITION 3.6. *Given $v, w \in P_n(\mathfrak{C})$, we have the following:*

(1) (*$\mathfrak{C}$-linearity*) *For any homogeneous $\lambda, \mu \in \mathfrak{C}$,*

$$\int_{\mathfrak{R}^{0|n}} d\theta(\lambda v + \mu w)(\theta) = (-1)^{np(\lambda)}\lambda\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) + (-1)^{np(\mu)}\mu\int_{\mathfrak{R}^{0|n}} d\theta\, w(\theta). \tag{3.9}$$

(2) (*Translational invariance*) *For any $\rho \in \mathfrak{R}^{0|n}$, we have*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta + \rho) = \int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta). \tag{3.10}$$

(3) (*Integration by parts*) *For $v \in P_n(\mathfrak{C})$ such that $p(v)=1$ or 0, we have*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta)\vec{\partial}_{\theta_s} w(\theta) = -(-1)^{p(v)}\int_{\mathfrak{R}^{0|n}} d\theta(\vec{\partial}_{\theta_s} v(\theta)) w(\theta). \tag{3.11}$$

(4) (*Linear change of variables*) *Let $A = (A_{jk})$ with $A_{jk} \in \mathfrak{R}_{ev}$ be invertible. Then,*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = (\det A)^{-1}\int_{\mathfrak{R}^{0|n}} d\omega\, v(A\cdot\omega). \tag{3.12}$$

(5) (*Interation of integrals*)

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n-k}} d\theta_n \cdots d\theta_{k+1}\Big(\int_{\mathfrak{R}^{0|k}} d\theta_k \cdots d\theta_1 v(\theta_1, \cdots, \theta_k, \theta_{k+1}, \cdots, \theta_n)\Big). \tag{3.13}$$

(6) (*Odd change of variables*) *Let $\theta = \theta(\omega)$ be an odd change of variables such that $\theta(0)=0$ and $\det(\vec{\partial}\theta(\omega)/\partial\omega|_{\omega=0}) \neq 0$. Then, for any $v \in P_n(\mathfrak{C})$,*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n}} d\omega\, v(\theta(\omega)) \det{}^{-1}\Big(\frac{\vec{\partial}\theta(\omega)}{\partial\omega}\Big). \tag{3.14}$$

(7) *For $v \in P_n(\mathfrak{C})$ and $\omega \in \mathfrak{R}^{0|n}$,*

$$\int_{\mathfrak{R}^{0|n}} d\theta(\theta_1 - \omega_1) \cdots (\theta_n - \omega_n) v(\theta) = v(\omega). \tag{3.15}$$

Remarks. (1) All above assertions are easily obtained by following the arguments in pp. 755-757 of [27], so proofs are omitted here. (2) (3.15) allows us to put $\delta(\theta - \omega) = (\theta_1 - \omega_1) \cdots (\theta_n - \omega_n)$, though $\delta(-\theta) = (-1)^n \delta(\theta)$.

Finally, we define

DEFINITION 3.7. Let $U=U_{ev}\times\mathfrak{R}_{od}^{n}\subset\mathfrak{R}^{m|n}$ be a superdomain and let $u\in\mathcal{C}_{SS}(U;\mathfrak{C})$, that is, $u(x,\theta)=\sum u_a(x)\theta^a$ with $u_a(x)\in\mathcal{C}_{SS}(U_{ev};\mathfrak{C})$. Then, we define

$$\int_U dxd\theta\, u(x,\theta)=\int_{U_{ev}}dx\left\{\int_{\mathfrak{R}^{0|n}}d\theta\, u(x,\theta)\right\}$$
$$=\int_{\pi_B(U_{ev})}dx_B u_{\tilde{1}}(x_B)\ \text{with}\ \tilde{1}=(1,\cdots,1)$$
$$=\int_{\mathfrak{R}^{0|n}}d\theta\left\{\int_{U_{ev}}dx\, u(x,\theta)\right\}.$$

Change of variables under integral sign.

THEOREM 3.8. *Let*

$$x=x(y,\omega),\quad \theta=\theta(y,\omega) \tag{3.17}$$

be a supersmooth diffeomorphism from $\mathfrak{R}_Y^{m|n}$ *to* $\mathfrak{R}_X^{m|n}$. *Putting*

$$M=\begin{bmatrix} A & C \\ D & B\end{bmatrix},\quad \begin{cases} A=\dfrac{\partial x}{\partial y}, & C=\dfrac{\vec{\partial} x}{\partial\omega}, \\ D=\dfrac{\partial\theta}{\partial y}, & B=\dfrac{\vec{\partial}\theta}{\partial\omega}, \end{cases} \tag{3.18}$$

we assume that either $\det A|_{\omega=0}$ *and* $\det(B-DA^{-1}C)|_{\omega=0}$ *or* $\det B|_{\omega=0}$ *and* $\det(A-CB^{-1}D)|_{\omega=0}$, *are invertible for all y. Then, for any function* $f\in\mathcal{C}_{SS}(\mathfrak{R}_X^{m|n};\mathfrak{C})$ *which is integrable on* $\mathfrak{R}_X^{m|n}$, *we have the change of variables formula*

$$\int_{\mathfrak{R}_X^{m|n}}dxd\theta\, f(x,\theta)=\int_{\mathfrak{R}_Y^{m|n}}dyd\omega\, f(x(y,\omega),\theta(y,\omega))(\operatorname{sdet}M)(y,\omega). \tag{3.19}$$

For the proof, do as same as in pp. 759–760, [26] where their super Euclidean space is modelled on Λ_L^{R} and $\mathfrak{R}_X^{m|n}$ and $\mathfrak{R}_Y^{m|n}$ are replaced by suitable 'singular manifolds' in $(\Lambda_{L,ev}^{R})^m\times(\Lambda_{L,od}^{R})^n$. Here, Λ_L^{R} is defined as similar as Λ_L^{C} in (1.5).

§ 4. A Hamilton equation on super Euclidean space

Super Hamiltonian flows. Let a function $H(x;\xi,\theta;\pi)$ on $\mathfrak{R}^{2m|2n}$ be given which satisfies the following where $\mathrm{proj}_I(\cdot)$ is defined just before (1.16):

Assumption A.

(A.1) $H(x;\xi,\theta;\pi)\in\mathcal{C}_{SS}(\mathfrak{R}^{2m|2n};\mathfrak{R}_{ev})$.
(A.2) $H(x_B;\xi_B,0;0)\in C^\infty(R^{2m};\boldsymbol{R})$.
(A.3) For any multi-indeces α, β, a and b satisfying $|\alpha|+|\beta|+|a|+|b|\geqq 2$ and any $I\in\mathfrak{I}$, there exists a positive constant $C_{\alpha,\beta,a,b}$, independent of $I\in\mathfrak{I}$, such that

$$|\mathrm{proj}_I(\partial_x^\alpha\partial_\xi^\beta\vec{\partial}_\theta^a\vec{\partial}_\pi^b H(x_B;\xi_B,0;0))|\leqq C_{\alpha,\beta,a,b}.$$

Or, we consider more specially that

Assumption AS.
(AS.1) $H(x;\xi,\theta;\pi)\in\mathcal{C}_{SS}(\mathfrak{R}^{2m|2n};\mathfrak{R}_{ev})$.
(AS.2) $H(x_B;\xi_B,0;0)\in C^\infty(R^{2m};\boldsymbol{R})$ and $\partial_\beta^a\vec{\partial}_\pi^b H(x_B;\xi_B,0;0)\in C^\infty(R^{2m};\boldsymbol{C})$.
(AS.3) For any multi-indeces α, β, a and b satisfying $|\alpha|+|\beta|+|a|+|b|\geqq 2$, there exists a positive constant $C_{\alpha,\beta,a,b}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta\vec{\partial}_\theta^a\vec{\partial}_\pi^b H(x_B;\xi_B,0;0)|\leqq C_{\alpha,\beta,a,b}.$$

Example. We take, as the simplest example, the following Schrödinger equation with spin (called, Pauli or more precisely Pauli type equation) on $\boldsymbol{R}^m$:

$$\frac{\hbar}{i}\frac{\partial\psi(q,t)}{\partial t}=(H\psi)(q,t) \tag{4.1}$$

with

$$H=-\sum_{j=1}^m\left(\frac{\hbar}{i}\partial_{q_j}-A_j(q)\right)^2+\frac{\hbar}{2i}\sum_{j,k=1}^m F_{jk}(q)\gamma^j\gamma^k+\Phi(q).$$

Here, $F_{jk}=\partial_{q_j}A_k-\partial_{q_k}A_j$ is the field strength of an external smooth gauge potential $A=\sum_{j=1}^m A_j(q)dq_j$ on $\boldsymbol{R}^m$ with $\sum_{j=1}^m\partial_{q_j}A_j(q)=0$ and $\Phi(q)$ is a smooth potential function on $\boldsymbol{R}^m$. $\{\gamma^j\}_{j=1}^m$ stand for the Hermitian $r\times r$-matrices, called the (Euclidean) Dirac matrices, satisfying $\gamma^j\gamma^k+\gamma^k\gamma^j=-2\delta_{jk}$ and $\psi(q,t)\in\boldsymbol{C}^r$ for each $(q,t)\in\boldsymbol{R}^m\times\boldsymbol{R}$ with $r=2^l$ where $l=[m/2]=$the largest integer not exceeding $m/2$. Using the procedures introduced in [11], we get the 'full symbol' $H=H(x;\xi,\theta;\pi)$ of (4.1) as follows:

$$H(x;\xi,\theta;\pi)=H_B+H_S \quad\text{with}\quad H_B=H_B(x;\xi,\theta;\pi)=\sum_{\mu=1}^m(\xi_\mu-A_\mu(x))^2+\Phi(x).$$

Here $H_S=H_S(x;\xi,\theta;\pi)$ is given by, for $m=2l$,

$$\begin{aligned}H_S=\frac{1}{2}\sum_{j,k=1}^l\{&(F_{2j\,2k}(x)-F_{2j-1\,2k-1}(x)-2iF_{2j\,2k-1}(x))\theta_j\theta_k\\&+(F_{2j\,2k}(x)-F_{2j-1\,2k-1}(x)+2iF_{2j\,2k-1}(x))\pi_j\pi_k\\&-2(F_{2j\,2k}(x)+F_{2j-1\,2k-1}(x)-iF_{2j-1\,2k}(x)+iF_{2j\,2k-1}(x))\theta_j\pi_k\}\end{aligned}$$

and for $m=2l+1$,

$$H_S=\sum_{k=1}^{l}\{(F_{1\,2k+1}(x)+iF_{1\,2k}(x))\theta_0\theta_k+(F_{1\,2k+1}(x)-iF_{1\,2k}(x))\pi_0\pi_k$$
$$-(F_{1\,2k+1}(x)-iF_{1\,2k}(x))\theta_0\pi_k+(F_{1\,2k+1}(x)+iF_{1\,2k}(x))\theta_k\pi_0\}$$
$$+\frac{1}{2}\sum_{j,k=1}^{l}\{-(F_{2j\,2k}(x)-F_{2j+1\,2k+1}(x)-2iF_{2j\,2k+1}(x))\theta_j\theta_k$$
$$-(F_{2j\,2k}(x)-F_{2j+1\,2k+1}(x)+2iF_{2j\,2k+1}(x))\pi_j\pi_k$$
$$-2(F_{2j\,2k}(x)+F_{2j+1\,2k+1}(x)-iF_{2j+1\,2k}(x)+iF_{2j\,2k+1}(x))\theta_j\pi_k\}.$$

This Hamiltonian satisfies Assumption AS if $\Phi(q)$ is real-valued and satisfies $|\partial_q^\alpha\Phi(q)|\leqq C_\alpha$ for $|\alpha|\geqq 2$ and $A_j(q)=\sum_{j=1}^m a_{jk}q_k$ with a_{jk} and $\sum_{j=1}^m a_{jj}=0$. Moreover, this Hamiltonian has the real body and the complex soul which is the main reason why we introduced our supernumber algebra $\mathfrak{R}$ as in § 1.

Let $T>0$ be fixed arbitrarily.

For $t, s\in[-T, T]$, we want to construct a solution $(x;\xi, \theta;\pi)$ of the super Hamiltonian equation given by

$$(4.2)\qquad \begin{cases} \dfrac{d}{dt}x(t)=\partial_\xi H(x;\xi,\theta;\pi), \\ \dfrac{d}{dt}\xi(t)=-\partial_x H(x;\xi,\theta;\pi), \\ \dfrac{d}{dt}\theta(t)=-\vec{\partial}_\pi H(x;\xi,\theta;\pi), \\ \dfrac{d}{dt}\pi(t)=-\vec{\partial}_\theta H(x;\xi,\theta;\pi). \end{cases}$$

with the initial condition at $t=s$ given by

$$(4.3)\qquad (x(s);\xi(s), \theta(s);\pi(s))=(y;\eta, \omega;\rho)\in\mathfrak{R}^{2m|2n}.$$

Remark. Above equations are introduced to describe a classical spinning particle in [3] and [5] independently. See also the paper [18]. But there has been no paper treating the existence of the solution though Assumption A above should be weakened for physical applications.

To solve (4.2) with (4.3), we first observe the body part of (4.2). That is, putting $H_B(x_B, \xi_B)=H(x_B;\xi_B, 0;0)$, we consider the following differential equation:

$$(4.2\text{B})\qquad \begin{cases} \dfrac{d}{dt}x_B(t)=\partial_{\xi_B}H_B(x_B(t), \xi_B(t)), \\ \dfrac{d}{dt}\xi_B(t)=-\partial_{x_B}H_B(x_B(t), \xi_B(t)) \end{cases}$$

with the initial condition at $t=s$ given by

$$(x_B(s), \xi_B(s))=(y_B, \eta_B)\in \boldsymbol{R}^{2m}=T^*\boldsymbol{R}^m. \tag{4.3B}$$

By successive approximation, we easily obtain the following (cf. Fujiwara [7]):

PROPOSITION 4.1. *Let Assumption A hold. For any $T>0$ and any $t, s\in[-T, T]$, there exists a unique solution of* (4.2B) *with* (4.2B) *which is C^∞* in *$(t, s: y_B, \eta_B)$. Moreover, there exists a constant $\delta_0(T)>0$ with the following properties: If $|t-s|<\delta_0(T)$, there exist positive constants C_0 and $C^{(0)}_{\alpha,\beta}$ for $|\alpha|+|\beta|\geqq 1$, independent of $(t, s: y_B, \eta_B)$ such that*

$$\begin{cases} |x_B(t, s: y_B, \eta_B)-y_B|\leqq C_0(1+|y_B|+|\eta_B|)|t-s|, \\ |\xi_B(t, s: y_B, \eta_B)-\eta_B|\leqq C_0(1+|y_B|+|\eta_B|)|t-s|, \end{cases} \tag{4.4}$$

$$\begin{cases} |\partial^\alpha_{y_B}\partial^\beta_{\eta_B}(x_B(t, s: y_B, \eta_B)-y_B)|\leqq C^{(0)}_{\alpha,\beta}|t-s|, \\ |\partial^\alpha_{y_B}\partial^\beta_{\eta_B}(\xi_B(t, s: y_B, \eta_B)-\xi_B)|\leqq C^{(0)}_{\alpha,\beta}|t-s|, \end{cases} \tag{4.5}$$

$$\begin{cases} |(\partial_s x_B)(t, s: y_B, \eta_B)+\partial_{\xi_B}H_B(y_B, \eta_B)|\leqq C_0(1+|y_B|+|\eta_B|)|t-s|, \\ |(\partial_s \xi_B)(t, s: y_B, \eta_B)-\partial_{x_B}H_B(y_B, \eta_B)|\leqq C_0(1+|y_B|+|\eta_B|)|t-s|. \end{cases} \tag{4.6}$$

PROPOSITION 4.2. *Under Assumption A, there exists a unique solution of* (4.2) *with* (4.3), *for any $T>0$, and any $t, s\in[-T, T]$.*

Proof. For notational simplicity, we put $z=(x, \xi)$ and $\phi=(\theta, \pi)$. Decomposing

$$x(t)=x_B(t)+x_S(t), \quad \xi(t)=\xi_B(t)+\xi_S(t), \tag{4.7}$$

we write $(z, \phi)=(z(t), \phi(t))$, with $z(t)=(x(t), \xi(t))$ and $\phi(t)=(\theta(t), \pi(t))$. Moreover, $z(t)=z_B(t)+z_S(t)$, with $z_B(t)=(x_B(t), \xi_B(t))$ being given in Proposition 4.1. Using this, (4.2) can be rewritten by

$$\frac{d}{dt}\begin{bmatrix} z(t) \\ \phi(t) \end{bmatrix}=\begin{bmatrix} X_{ev}(z(t), \phi(t)) \\ X_{od}(z(t), \phi(t)) \end{bmatrix} \tag{4.8}$$

where $X_{ev}(z, \phi)=(\partial_\xi H(z, \phi), -\partial_x H(z, \phi))$, and $X_{od}(z, \phi)=(-\partial_\pi H(z, \phi), -\partial_\theta H(z, \phi))$. By Proposition 4.1, we need to consider only the soul part $(z_S(t), \phi(t))=(x_S(t); \xi_S(t), \theta(t); \pi(t))$. So, we have

$$\frac{d}{dt}\begin{bmatrix} z_S(t) \\ \phi(t) \end{bmatrix}=\begin{bmatrix} \partial_z X_{ev}(z_B(t), 0) & 0 \\ 0 & \vec{\partial}_\phi X_{od}(z_B(t), 0) \end{bmatrix}\begin{bmatrix} z_S(t) \\ \phi(t) \end{bmatrix} + \begin{bmatrix} \sum_{\substack{2\leq|\alpha|+|a| \\ |a|=even}} X_{ev,\alpha,a}(z_B(t))z_S^\alpha(t)\phi^a(t) \\ \sum_{\substack{1\leq|\alpha|+|a| \\ |a|=odd}} X_{od,\alpha,a}(z_B(t))z_S^\alpha(t)\phi^a(t) \end{bmatrix} \tag{4.9}$$

where $X_{ev,\alpha,a}(z_B)=(1/\alpha!)\partial_z^\alpha\vec{\partial}_\psi^a X_{ev}(z_B, 0)$ and $X_{od,\alpha,a}(z_B)=(1/\alpha!)\partial_z^\alpha\vec{\partial}_\psi^a X_{od}(z_B, 0)$.

To calculate (4.9) more concretely, we decompose (z_S, ψ) by

$$z_S=\sum_{j\geq 1} z_{[2j]} \quad \text{and} \quad \psi=\sum_{j\geq 1} \psi_{[2j-1]} \tag{4.10}$$

where $z_{[2j]}$, $\psi_{[2j-1]}$ are degree $[2j]$ and $[2j-1]$ component. Then, for $\psi_{[1]}$, we have

$$\frac{d}{dt}\psi_{[1]}(t)=\vec{\partial}_\psi X_{od}(z_B(t), 0)\psi_{[1]}(t) \quad \text{with} \quad \psi_{[1]}(0)=(\omega_{[1]}, \rho_{[1]}). \tag{4.11}$$

Using the degree, (4.11) can be solved easily for $|t-s|<\delta_0(T)$, because $\partial_\psi X_{od}(z_B, 0)$ is uniformly bounded on R^{2m} by Assumption A.

Now, consider (4.9) for $(z_{[2j]}(t), \psi_{[2j+1]}(t))$. Then, we get the following explicit form:

$$\frac{d}{dt}\begin{bmatrix} z_{[2j]}(t) \\ \psi_{[2j+1]}(t) \end{bmatrix}=\begin{bmatrix} \partial_z X_{ev}(z_B(t), 0) & 0 \\ 0 & \vec{\partial}_\psi X_{odd}(z_B(t), 0) \end{bmatrix}\begin{bmatrix} z_{[2j]}(t) \\ \psi_{[2j+1]}(t) \end{bmatrix} + \begin{bmatrix} P_{[2j]}(t, s : z_{[2]}, \cdots, z_{[2j-2]}, \psi_{[1]}, \cdots, \psi_{[2j-1]}) \\ Q_{[2j+1]}(t, s : z_{[2]}, \cdots, z_{[2j-2]}, \psi_{[1]}, \cdots, \psi_{[2j-1]}) \end{bmatrix} \tag{4.12}$$

where

$$\begin{aligned} &P_{[2j]}(t, s : z_{[2]}, \cdots, z_{[2j-2]}, \psi_{[1]}, \cdots, \psi_{[2j-1]}) \\ &= \sum_{[2j]} \prod_{u=1}^{2m} \frac{\alpha_u!}{k_0(\alpha_u)! \cdots k_{j-1}(\alpha_u)!} z_{1,[2]}^{k_0(\alpha_1)} \cdots z_{1,[2j-2]}^{k_{j-1}(\alpha_1)} \cdots z_{2m,[2]}^{k_0(\alpha_{2m})} \cdots z_{2m,[2j-2]}^{k_{j-1}(\alpha_{2m})} \\ &\quad \times \psi_{1,[1]}^{m_1(a_1)} \cdots \psi_{1,[2j-1]}^{m_j(a_1)} \cdots \psi_{2n,[1]}^{m_1(a_{2n})} \cdots \psi_{2n,[2j-1]}^{m_j(a_{2n})} X_{ev,\alpha,a}(z_B(t)) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &Q_{[2j+1]}(t, s : z_{[2]}, \cdots, z_{[2j-2]}, \psi_{[1]}, \cdots, \psi_{[2j-1]}) \\ &= \sum_{(2j+1)} \prod_{u=1}^{2m} \frac{\alpha_u!}{k_0(\alpha_u)! \cdots k_{j-1}(\alpha_u)!} z_{1,[2]}^{k_0(\alpha_1)} \cdots z_{1,[2j-2]}^{k_{j-1}(\alpha_1)} \cdots z_{2m,[2]}^{k_0(\alpha_{2m})} \cdots z_{2m,[2j-2]}^{k_{j-1}(\alpha_{2m})} \\ &\quad \times \psi_{1,[1]}^{m_1(a_1)} \cdots \psi_{1,[2j-1]}^{m_j(a_1)} \cdots \psi_{2n,[1]}^{m_1(a_{2n})} \cdots \psi_{2n,[2j-1]}^{m_j(a_{2n})} X_{odd,\alpha,a}(z_B(t)). \end{aligned} \tag{4.14}$$

Here, $\sum_{(2j)}$ (resp. $\sum_{(2j+1)}$) stands for the sum of all partitions $(k_u(\cdot), m_\mu(\cdot))$ satisfying the following:

(1) $\sum_{u=0}^{j-1} k_u(\alpha_i)=\alpha_i$ $(i=1, \cdots, 2m)$, $\sum_{\mu=1}^{j} m_\mu(a_r)=a_r$ $(r=1, \cdots, 2n)$,

(2) $\sum_{i=1}^{2m}\sum_{u=0}^{j-1} 2uk_u(\alpha_i)+|a|=2j$ (resp. $2j+1$),

(3) $2\leq|a|+|\alpha|$.

If we assume that $(z_{[2k]}(t), \psi_{[2k-1]}(t))$, $k=0, \cdots, j-1$ are solved, then, $P_{[2j-2]}(t, s : \cdots)$ and $Q_{[2j-1]}(t, s : \cdots)$ are the known data. So, we get $(z_{[2j]}, \psi_{[2j+1]})$ from (4.12) by using the variation of constant. Thus, inductively we get a unique solution $(z_S(t), \psi(t))$ of (4.9) with the initial condition at $t=s$ given by $(z_S(s), \psi(s))=(ys ; \eta, \omega ; \rho)$. ■

Next, we investigate the smoothness of $(z(t), \phi(t))$ with respect to the initial data.

PROPOSITION 4.3. *Fix $T>0$ arbitrarily. Under Assumption A, the solution $(z(t), \phi(t))$ of (4.2) is 'smooth' in $(t, s: y; \eta, \omega; \rho)$, that is, smooth in (t, s) for fixed $(y; \eta, \omega; \rho)$ and supersmooth in $(y; \eta, \omega; \rho)$ for fixed (t, s). Moreover, there exists $\delta_1(T)>0$ such that the following properties hold: If $|t-s|\leqq\delta_1(T)$ and $I\in\mathfrak{I}$, there exist C_1 and $C_k^{(0)}$ independent of I such that*

$$
(4.15)\quad \left\{\begin{array}{l} |\mathrm{proj}_I(z(t, s: y_B; \eta_B, 0; 0)-(y_B, \eta_B))|\leqq C_1(1+|y_B|+|\eta_B|)|t-s|, \\ |\mathrm{proj}_I(\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b(z(t, s: y; \eta, \omega; \rho)-(y, \eta))(y_B; \eta_B, 0; 0))|\leqq C_k^{(0)}|t-s| \end{array}\right.
$$

for $k=|\alpha|+|\beta|+|a|+|b|\geqq 1$. Analogously, we have

$$
(4.16)\quad |\mathrm{proj}_I(\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b(\phi(t, s: y; \eta, \omega; \rho)-(\omega, \rho))(y_B; \eta_B, 0; 0))|\leqq C_k^{(0)}|t-s|.
$$

Proof. Remark that the first and the second estimates of (4.15) with $|a|+|b|=0$ are already given in Proposition 4.1. In oder to prove the smoothness in $(y; \eta, \omega; \rho)$, we differentiate (4.2) formally in $(y; \eta, \omega; \rho)$, which gives us the following differential equation:

$$
(4.17)\qquad \frac{d}{dt}J^{(1)}(t)=A^{(1)}(t)J^{(1)}(t) \quad \text{with} \quad J^{(1)}(s)=Id.
$$

Here

$$
(4.18)\qquad J^{(1)}(t)=\begin{bmatrix} \partial_y x & \partial_\eta x & \vec{\partial}_\omega x & \vec{\partial}_\rho x \\ \partial_y \xi & \partial_\eta \xi & \vec{\partial}_\omega \xi & \vec{\partial}_\rho \xi \\ \partial_y \theta & \partial_\eta \theta & \vec{\partial}_\omega \theta & \vec{\partial}_\rho \theta \\ \partial_y \pi & \partial_\eta \pi & \vec{\partial}_\omega \pi & \vec{\partial}_\rho \pi \end{bmatrix}
$$

and

$$
(4.19)\qquad A^{(1)}(t)=\begin{bmatrix} \partial_x\partial_\xi H & \partial_\xi\partial_\xi H & \vec{\partial}_\theta\partial_\xi H & {}_\pi\vec{\partial}\partial_\xi H \\ -\partial_x\partial_x H & -\partial_\xi\partial_x H & -\vec{\partial}_\theta\partial_x H & -\vec{\partial}_\pi\partial_x H \\ -\partial_x\vec{\partial}_\pi H & -\partial_\xi\vec{\partial}_\pi H & -\vec{\partial}_\theta\vec{\partial}_\pi H & -\vec{\partial}_\pi\vec{\partial}_\pi H \\ -\partial_x\vec{\partial}_\theta H & -\partial_\xi\vec{\partial}_\theta H & -\vec{\partial}_\theta\vec{\partial}_\theta H & -\vec{\partial}_\pi\vec{\partial}_\theta H \end{bmatrix}
$$

Remarking that the each component of $A^{(1)}(t)$ is supersmooth and bounded independently of $I\in\mathfrak{I}$ by (A.3) and using a similar method as in the proof of Proposition 4.2, we get the unique solution of (4.17) for $[s, t]$. Thus, we have that the solution of (4.2) is supersmooth with respect to $(y; \eta, \omega; \rho)$. Moreover, we easily get the following estimate

$$
(4.20)\qquad |\mathrm{proj}_I(J^{(1)}(t, s: y_B; \eta_B, 0; 0)-I)|\leqq C_1^{(0)}|t-s|e^{C_1^{(0)}|t-s|}.
$$

Furthermore, for each positive integer k, putting

$$
(4.21)\qquad J^{(k)}=(\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b(x; \xi, \theta; \pi))_{(|\alpha|+|\beta|+|a|+|b|=k)},
$$

we have the following differential equation:

$$\frac{d}{dt}J^{(k)}(t)=A^{(k)}J^{(k)}(t)+B^{(k)} \quad \text{with} \quad J^{(k)}(0)=0 \tag{4.22}$$

where the each component of $A^{(k)}(t)$ and $B^{(k)}$ is supersmooth and bounded. So, we get also

$$|\mathrm{proj}_I(J^{(k)}(t, s: y_B; \eta_B, 0; 0))| \leqq C_k^{(0)}|t-s|e^{C_k^{(0)}|t-s|}. \tag{4.23}$$

It is easily seen that $z_{ab}(t, s)=\vec{\partial}_\omega^a\vec{\partial}_\rho^b z(t, s: y; \eta, 0; 0)$ and $\phi_{ab}(t, s)=\vec{\partial}_\omega^a\vec{\partial}_\rho^b\phi(t, s: y; \eta, 0; 0)$ are supersmooth functions on $\mathfrak{R}^{2m|0}$ by using the uniqueness of the solution for (4.22). Thus, putting

$$z(t)=\sum z_{ab}(t, s)\omega^a\rho^b \quad \text{and} \quad \phi(t)=\sum\phi_{ab}(t, s)\omega^a\rho^b, \tag{4.24}$$

we have proved Proposition 4.3, again by the uniqueness of the solution of (4.2). ■

Remark. It follows readily from the above arguments that if H satisfies Assumption AS, then $\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b z(t, s: y_B; \eta_B, 0; 0)$ and $\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b\phi(t, s: y_B; \eta_B, 0; 0)$ are complex valued for $k=|\alpha|+|\beta|+|a|+|b|\geqq 1$. Therefore, in this case, (4.15) and (4.16) hold with $|\mathrm{proj}_I(\cdot)|$ by $|\cdot|$.

PROPOSITION 4.4. *Let $\delta_1=\delta_1(T)$ be fixed so as to $0\leqq\delta_1<1$ and $C_k^{(0)}\delta_1<1/2$ ($k=1, 2, 3$) where $C_k^{(0)}$ are the constants in Proposition 4.3. Let $|t-s|<\delta_1$. Then, we have the following:*

(i) *For any fixed (t, s, η, ρ), the mapping*

$$\mathfrak{R}^{m|n}\ni(y, \omega)\longmapsto(x=x(t, s: y; \eta, \omega; \rho), \quad \theta=\theta(t, s: y; \eta, \omega; \rho))\in\mathfrak{R}^{m|n} \tag{4.25}$$

is supersmooth. We denote the inverse mapping defined by

$$\mathfrak{R}^{m|n}\ni(x, \theta)\longmapsto(y=y(t, s: x; \theta, \eta; \rho), \quad \omega=\omega(t, s: x, \theta, \eta, \rho))\in\mathfrak{R}^{m|n}, \tag{4.26}$$

which is supersmooth in (x, θ, η, ρ) for fixed t, s.

(ii) *For any fixed (t, s, y, ω), the mapping*

$$\mathfrak{R}^{m|n}\ni(\eta, \rho)\longmapsto(\xi=\xi(t, s: y; \eta, \omega; \rho), \quad \pi=\pi(t, s: y; \eta, \omega; \rho))\in\mathfrak{R}^{m|n} \tag{4.27}$$

is supersmooth. The inverse mapping defined by

$$\mathfrak{R}^{m|n}\ni(\xi, \pi)\longmapsto(\eta=\eta(t, s: y; \xi, \omega; \pi), \quad \rho=\rho(t, s: y; \xi, \omega; \pi))\in\mathfrak{R}^{m|n}, \tag{4.28}$$

is supersmooth in $(y; \xi, \omega; \pi)$ for fixed t, s.

Proof. (i) To prove the bijectivity of the mapping (4.25), for each fixed $t, s\in[-T, T]$ and given $(x; \eta, \theta; \rho)$, we want to solve the following equations with respect to (y, ω):

$$
(4.29)\qquad \begin{cases} x=x(t, s: y; \eta, \omega; \rho), \\ \theta=\theta(t, s: y; \eta, \omega; \rho). \end{cases}
$$

Following the arguments in Kitada & Kumano-go [14] or [7], we can solve the body part of (4.29). Namely, we put, for given (x_B, η_B),

$$
(4.30)\qquad T^{(B)}_{(x_B, \eta_B)}(y_B)=x_B+y_B-x_B(t, s: y_B, \eta_B).
$$

Then, by using (4.15) and the mean value theorem, we have

$$
(4.31)\qquad |T^{(B)}_{(x_B, \eta_B)}(y_B)-T^{(B)}_{(x_B, \eta_B)}(y'_B)|
$$
$$
=\left|\int_0^1\{I-\partial_{y_B}x_B(t, s: y'_B+\tau(y_B-y'_B), \eta_B)\}d\tau\right| |y_B-y'_B|\leqq\frac{1}{2}|y_B-y'_B|
$$

for $|t-s|\leqq\delta_1$. Thus, the mapping (4.30) is contractive and has a unique fixed point. That is, for any given (x_B, η_B), there exists a unique $\bar{y}_B$ such that $x_B=x_B(t, s: \bar{y}_B, \eta_B)$. So, we write it as $\bar{y}_B=y_B(t, s: x_B, \eta_B)$. Now, decomposing $x=\sum x_{[2j]}$ and $\theta=\sum\theta_{[2j-1]}$, we consider (4.29) at each degree. First, looking at $\theta_{[1]}$, we get by the supersmoothness of θ with respect to its arguments.

$$
(4.32)\qquad \theta_{[1]}=\sum_{r=1}^{n}\vec{\partial}_{\omega_r}\theta(t, s: y_B; \eta_B, 0; 0)\omega_{r,[1]}+\sum_{r=1}^{n}\vec{\partial}_{\rho_r}\theta(t, s: y_B; \eta_B, 0; 0)\rho_{r,[1]}.
$$

Since $\vec{\partial}_{\omega}\theta(t, s: y_B; \eta_B, 0; 0)$ is invertible by (4.20) for small $|t-s|$, we can solve $\omega_{[1]}$ from (4.32). Similarly, we consider (4.29) for general j:

$$
(4.33)\qquad \begin{cases} x_{[2j]}=\sum_{k=1}^{m}(\partial_{y_k}x_B(t, s: y_B, \eta_B)y_{k,[2j]}+\partial_{\eta_k}x_B(t, s: y_B, \eta_B)\eta_{k,[2j]}) \\ \qquad +X_{[2j]}(t, s: y_{[2]}, \cdots, y_{[2j-2]}; \eta_{[2]}, \cdots, \eta_{[2j-2]}, \omega_{[1]}, \cdots, \\ \qquad\qquad \omega_{[2j-1]}; \rho_{[1]}, \cdots, \rho_{[2j-1]}), \\ \theta_{[2j+1]}=\sum_{r=0}^{l}(\vec{\partial}_{\omega_r}\theta(t, s: y_B; \eta_B, 0; 0)\omega_{r,[2j+1]} \\ \qquad +\vec{\partial}_{\rho_r}\theta(t, s: y_B; \eta_B, 0; 0)\rho_{r,[2j+1]})+\Theta_{[2j+1]}(t, s: y_{[2]}, \cdots, \\ \qquad y_{[2j-2]}; \eta_{[2]}, \cdots, \eta_{[2j-2]}, \omega_{[1]}, \cdots, \omega_{[2j-1]}; \rho_{[1]}, \cdots, \rho_{[2j-1]}) \end{cases}
$$

where

$$
(4.34)\qquad X_{[2j]}(t, s: y_{[2]}, \cdots, y_{[2j-2]}; \eta_{[2]}, \cdots, \eta_{[2j-2]}, \omega_{[1]}, \cdots, \omega_{[2j-1]}; \rho_{[1]}, \cdots, \rho_{[2j-1]})
$$
$$
=\sum_{[2j]}\frac{1}{\alpha!}\frac{1}{\beta!}\prod_{u=1}^{m}\frac{\alpha_u!}{k_0(\alpha_u)!\cdots k_{j-1}(\alpha_u)!}\frac{\beta_u!}{k_0(\beta_u)!\cdots k_{j-1}(\beta_u)!}
$$
$$
\times y_{1,[2]}^{k_0(\alpha_1)}\cdots y_{1,[2j-2]}^{k_{j-1}(\alpha_1)}\cdots y_{m,[2]}^{k_0(\alpha_m)}\cdots y_{m,[2j-2]}^{k_{j-1}(\alpha_m)}
$$

$$\times\eta_{1,[2]}^{k_0(\beta_1)}\cdots\eta_{1,[2j-2]}^{k_{j-1}(\beta_1)}\cdots\eta_{m,[2]}^{k_0(\beta_m)}\cdots\eta_{m,[2j-2]}^{k_{j-1}(\beta_m)}$$

$$\times\omega_{1,[1]}^{m_1(a_1)}\cdots\omega_{1,[2j-1]}^{m_j(a_1)}\cdots\omega_{n,[1]}^{m_1(a_n)}\cdots\omega_{n,[2j-1]}^{m_j(a_n)}$$

$$\times\rho_{1,[1]}^{m_1(b_1)}\cdots\rho_{1,[2j-1]}^{m_j(b_1)}\cdots\rho_{n,[1]}^{m_1(b_n)}\cdots\rho_{n,[2j-1]}^{m_j(b_n)}$$

$$\times\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b x(t,s:y_B;\eta_B,0;0),$$

(4.35) $$\Theta_{[2j+1]}(t,s:y_{[2]},\cdots,y_{[2j-2]};\eta_{[2]},\cdots,\eta_{[2j-2]},\omega_{[1]},\cdots,\omega_{[2j-1]};\rho_{[1]},\cdots,\rho_{[2j-1]})$$

$$=\sum_{[2j+1]}\frac{1}{\alpha!}\frac{1}{\beta!}\prod_{u=1}^{m}\frac{\alpha_u!}{k_0(\alpha_u)!\cdots k_{j-1}(\alpha_u)!}\frac{\beta_u!}{k_0(\beta_u)!\cdots k_{j-1}(\beta_u)!}$$

$$\times y_{1,[2]}^{k_0(\alpha_1)}\cdots y_{1,[2j-2]}^{k_{j-1}(\alpha_1)}\cdots y_{m,[2]}^{k_0(\alpha_m)}\cdots y_{m,[2j-2]}^{k_{j-1}(\alpha_m)}$$

$$\times\eta_{1,[2]}^{k_0(\beta_1)}\cdots\eta_{1,[2j-2]}^{k_{j-1}(\beta_1)}\cdots\eta_{m,[2]}^{k_0(\beta_m)}\cdots\eta_{m,[2j-2]}^{k_{j-1}(\beta_m)}$$

$$\times\omega_{1,[1]}^{m_1(a_1)}\cdots\omega_{1,[2j-1]}^{m_j(a_1)}\cdots\omega_{n,[1]}^{m_1(a_n)}\cdots\omega_{n,[2j-1]}^{m_j(a_n)}$$

$$\times\rho_{1,[1]}^{m_1(b_1)}\cdots\rho_{1,[2j-1]}^{m_j(b_1)}\cdots\rho_{n,[1]}^{m_1(b_n)}\cdots\rho_{n,[2j-1]}^{m_j(b_n)}$$

$$\times\partial_y^\alpha\partial_\eta^\beta\vec{\partial}_\omega^a\vec{\partial}_\rho^b\theta(t,s:y_B;\eta_B,0;0).$$

Here, $\sum_{[2j]}$ (resp. $\sum_{[2j+1]}$) stands for the sum of all partitions $(k_u(\cdot), m_\mu(\cdot))$ satisfying the following:

(1) $\sum_{u=0}^{j-1}k_u(\alpha_i)=\alpha_i$, $\sum_{u=0}^{j-1}k_u(\beta_i)=\beta_i$ $(i=1,\cdots,m)$,

(2) $\sum_{i=1}^{m}\sum_{u=0}^{j-1}2uk_u(\alpha_i)+\sum_{i=1}^{m}\sum_{u=0}^{j-1}2uk_u(\beta_i)+|a|+|b|=2j$ (resp. $=2j+1$),

(3) $\sum_{\mu=1}^{j-1}m_\mu(a_r)=a_r$, $\sum_{\mu=1}^{j-1}m_\mu(b_r)=b_r$ $(r=1,\cdots,n)$,

(4) $|a|+|b|+|\alpha|+|\beta|\geqq 2$.

So using (4.21), we can solve $(y_{[2j]}, \omega_{[2j+1]})$ for general j. Using the Proposition 2.16 (inverse function theorem), we get the supersmoothness of the mapping (4.26). The other assertion is proved similarily. ■

The mappings defined in Proposition 4.4 satisfy the following:

PROPOSITION 4.5. *Let* $|t-s|<\delta_1$ *then we have*

(4.36) $$\begin{cases} x(t,s:y(t,s:x;\xi,\theta;\pi);\xi,\omega(t,s:x;\xi,\theta;\pi);\pi)=x, \\ \theta(t,s:y(t,s:x;\xi,\theta;\pi);\xi,\omega(t,s:x;\xi,\theta;\pi);\pi)=\theta, \end{cases}$$

(4.37) $$\begin{cases} \xi(t,s:x;\eta(t,s:x;\xi,\theta;\pi),\theta;\rho(t,s:x;\xi,\theta;\pi))=\xi, \\ \pi(t,s:x;\eta(t,s:x;\xi,\theta;\pi),\theta;\rho(t,s:x;\xi,\theta;\pi))=\pi. \end{cases}$$

(4.38) $$\begin{cases} x(t,s:x;\eta(t,s:x;\xi,\theta;\pi),\theta;\rho(t,s:x;\xi,\theta;\pi))=y(t,s:x;\xi,\theta;\pi), \\ \theta(t,s:x;\eta(t,s:x;\xi,\theta;\pi),\theta;\rho(t,s:x;\xi,\theta;\pi))=\omega(t,s:x;\xi,\theta;\pi), \end{cases}$$

$$(4.39)\quad \begin{cases} \xi(t, s: y(t, s: x; \xi, \theta; \pi); \xi, \omega(t, s: x; \xi, \theta; \pi); \pi)=\eta(t, s: x; \xi, \theta; \pi), \\ \pi(t, s: y(t, s: x; \xi, \theta; \pi); \xi, \omega(t, s: x; \xi, \theta; \pi); \pi)=\rho(t, s: x; \xi, \theta; \pi). \end{cases}$$

$(y(t); \eta(t), \omega(t); \rho(t))$ is 'smooth' in $(t, s: x; \xi, \theta; \pi)$ with the following estimates: There exist constants $C^{(1)}_{\alpha,\beta}$ and C_2, independent of $(t, s: x; \xi, \theta; \pi)$ and $I \in \mathfrak{J}$, such that for $|\alpha|+|\beta|+|a|+|b| \geqq 1$ and $p, q=0$ or 1 with $p+q \leqq 1$

$$(4.40)\quad \begin{cases} |\mathrm{proj}_I(\partial_t^p \partial_s^q \partial_x^\alpha \partial_\xi^\beta \vec{\partial}_\theta^a \vec{\partial}_\pi^b (y(t, s: x; \xi, \theta; \pi)-x)(y_B; \eta_B, 0; 0))| \leqq C^{(1)}_{\alpha,\beta}|t-s|, \\ |\mathrm{proj}_I(\partial_t^p \partial_s^q \partial_x^\alpha \partial_\xi^\beta \vec{\partial}_\theta^a \vec{\partial}_\pi^b (\eta(t, s: x; \xi, \theta; \pi)-\xi)(y_B; \eta_B, 0; 0))| \leqq C^{(1)}_{\alpha,\beta}|t-s|, \\ |\mathrm{proj}_I(\partial_t^p \partial_s^q \partial_x^\alpha \partial_\xi^\beta \vec{\partial}_\theta^a \vec{\partial}_\pi^b (\omega(t, s: x; \xi, \theta; \pi)-\theta)(y_B; \eta_B, 0; 0))| \leqq C^{(1)}_{\alpha,\beta}|t-s|, \\ |\mathrm{proj}_I(\partial_t^p \partial_s^q \partial_x^\alpha \partial_\xi^\beta \vec{\partial}_\theta^a \vec{\partial}_\pi^b (\rho(t, s: x; \xi, \theta; \pi)-\pi)(y_B; \eta_B, 0; 0))| \leqq C^{(1)}_{\alpha,\beta}|t-s|, \end{cases}$$

$$(4.41)\quad \begin{cases} |y_B(t, s: x_B, \xi_B)-x_B| \leqq C_2|t-s|(1+|x_B|+|\xi_B|), \\ |\eta_B(t, s: x_B, \xi_B)-\xi_B| \leqq C_2|t-s|(1+|x_B|+|\xi_B|). \end{cases}$$

Proof. (4.36-37) and (4.38-39) follow from Proposition 4.4. We get easily the first two inequalities of (4.40) for $a=b=0$ by differentiating (4.36) and using (4.16). Then, we write

$$(4.42)\quad y_B(t, s: x_B, \xi_B)-x_B=\xi_B\int_0^1 \partial_{\xi_B} y_B(t, s: x_B, \tau\xi_B)d\tau + x_B\int_0^1 (\partial_{x_B} y_B(t, s: \tau x_B, 0)-I)d\tau + y_B(t, s: 0, 0).$$

Using the first inequality of (4.40) for $a=b=0$, we have

$$|y_B(t, s: x_B, \xi_B)-x_B| \leqq C_0(1+|x_B|+|\xi_B|)|t-s|+|y_B(t, s: 0, 0)|.$$

By (4.3), (4.38), we get

$$|y_B(t, s: 0, 0)|=|x_B(t, s: 0, \eta_B(t, s: 0, 0))-0| \leqq C_0|t-s|,$$

which proves the first inequality of (4.41). Similarly, we have the second inequality of (4.41). To prove other inequalities in (4.40), we do as we did in proving Proposition 4.4 but omit the details. ∎

Action integral. We construct the action integral along the Hamiltonian flow given above. First, we remark

LEMMA 4.6. *Let $(x; \xi, \theta; \pi)$ be the Hamiltonian flow defined by* (4.2). *Then we have*

$$(4.43)\quad \frac{d}{dt}H(x; \xi, \theta; \pi)=0.$$

Proof. By using the composition rule of derivatives, we get

$$\frac{d}{dt}H(x\,;\xi,\theta\,;\pi)=\sum_{j=1}^{m}\Big(\frac{\partial H}{\partial x_j}\frac{dx_j}{dt}+\frac{\partial H}{\partial \xi_j}\frac{d\xi_j}{dt}\Big)+\sum_{r=1}^{n}\Big(\frac{\vec{\partial} H}{\partial \theta_r}\frac{d\theta_r}{dt}+\frac{\vec{\partial} H}{\partial \pi_r}\frac{d\pi_r}{dt}\Big).$$

Substituting (4.2) in the above equation, we get (4.43). ■

Now, we define

$$(4.44)\qquad \begin{aligned} u(t, s)&=u(t, s: y\,;\eta,\omega\,;\rho)\\ &=\langle\eta\,|\,y\rangle-\langle\rho\,|\,\omega\rangle+\int_s^t L(x(\tau, s), \theta(\tau, s), \xi(\tau, s), \pi(\tau, s))d\tau \end{aligned}$$

where

$$(4.45)\quad L(x\,;\xi,\theta\,;\pi)=\langle\xi\,|\,\partial_\xi H(x\,;\xi,\theta\,;\pi)\rangle+\langle\pi\,|\,\vec{\partial}_\pi H(x\,;\xi,\theta\,;\pi)\rangle-H(x\,;\xi,\theta:\pi).$$

Here, we put

$$\langle\eta\,|\,y\rangle=\sum_{j=1}^{m}\eta_j y_j,\quad \langle\rho\,|\,\omega\rangle:=\sum_{r=0}^{l}\rho_r\omega_r,\quad etc.$$

$$x(t, s)=x(t)=x(t, s: y\,;\eta,\omega\,;\rho),\quad \xi(t, s)=\xi(t)=\xi(t, s: y\,;\eta,\omega\,;\rho),\quad etc.$$

LEMMA 4.7. *Let* $|t-s|<\delta_1$. *Then,* $u(t, s)=u(t, s: x\,;\xi,\theta\,;\pi)$ *is 'smooth' in* $(t, s: x\,;\xi,\theta\,;\pi)$, *and it satisfies:*

$$(4.46)\qquad \begin{cases} \partial_t u(t, s)=\langle\xi(t, s)\,|\,\partial_t x(t, s)\rangle-\langle\pi(t, s)\,|\,\partial_t\theta(t, s)\rangle\\ \qquad\qquad -H(x(t, s)\,;\xi(t, s), \theta(t, s)\,;\pi(t, s)),\\ \partial_s u(t, s)=-\langle\xi(t, s)\,|\,\partial_s x(t, s)\rangle-\langle\pi(t, s)\,|\,\partial_s\theta(t, s)\rangle\\ \qquad\qquad +H(x(t, s)\,;\xi(t, s), \theta(t, s)\,;\pi(t, s)). \end{cases}$$

$$(4.47)\qquad \begin{cases} \partial_y u(t, s)=\langle\xi(t, s)\,|\,\partial_y x(t, s)\rangle-\langle\pi(t, s)\,|\,\partial_y\theta(t, s)\rangle,\\ \partial_\eta u(t, s)=y+\langle\xi(t, s)\,|\,\partial_\eta x(t, s)\rangle-\langle\pi(t, s)\,|\,\partial_\eta\theta(t, s)\rangle,\\ \vec{\partial}_\omega u(t, s)=\langle\xi(t, s)\,|\,\vec{\partial}_\omega x(t, s)\rangle+\langle\pi(t, s)\,|\,\vec{\partial}_\omega\theta(t, s)\rangle,\\ \vec{\partial}_\rho u(t, s)=-\omega+\langle\xi(t, s)\,|\,\vec{\partial}_\rho x(t, s)\rangle+\langle\pi(t, s)\,|\,\vec{\partial}_\rho\theta(t, s)\rangle. \end{cases}$$

Proof. As is readily seen the 'smoothness' of u in $(t, s: x\,;\xi,\theta\,;\pi)$ by composition rule of differentiable functions, we have (4.46). To prove the first equality of (4.47), we put

$$(4.48)\quad W_j(t, s)=\partial_{y_j}u-\langle\xi(t, s)\,|\,\partial_{y_j}x(t, s)\rangle+\langle\pi(t, s)\,|\,\partial_{y_j}\theta(t, s)\rangle\quad (j=1, \cdots, m).$$

Then, $W_j'(t, s)=0$ and $W_j(s, s)=0$ by easy computations, which gives the desired equation. The other equations of (4.47) can be similarly obtained. ■

Putting

(4.49) $\phi(t, s: x; \eta, \theta; \rho)=u(t, s: y(t, s: x; \eta, \theta; \rho); \omega(t, s: x; \eta, \theta; \rho), \eta; \rho)$,

we have:

PROPOSITION 4.8 (Hamilton-Jacobi equation). *Let* $|t-s|<\delta_1$, *then*

(i) $\phi(t, s: x; \eta, \theta; \rho)$ *is 'smooth' in* $(t, s: x; \xi, \theta; \pi)$.

(ii) $\phi(s, s, x; \eta, \theta; \rho)=\langle\eta|x\rangle-\langle\rho|\theta\rangle$.

(iii) $$\begin{cases} \partial_x\phi(t, s: x; \eta, \theta; \rho)=\xi(t, s: x; \eta, \theta; \rho), \\ \partial_\eta\phi(t, s: x; \eta, \theta; \rho)=y(t, s: x; \eta, \theta; \rho), \\ \vec{\partial}_\theta\phi(t, s: x; \eta, \theta; \rho)=\pi(t, s: x; \eta, \theta; \rho), \\ \vec{\partial}_\rho\phi(t, s: x; \eta, \theta; \rho)=-\omega(t, s: x; \eta, \theta; \rho). \end{cases}$$

(iv) $$\begin{cases} \partial_t\phi(t, s: x; \eta, \theta; \rho)+H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)=0, \\ \partial_s\phi(t, s: x; \eta, \theta; \rho)-H(\partial_\eta\phi; \eta, -\vec{\partial}_\rho\phi; \rho)=0. \end{cases}$$

(v) $\phi(t, s: x; \eta, \theta; \rho)$ *satisfies the following estimates for any* $I\in\mathfrak{J}$:

$$(4.50)\quad \begin{cases} |\mathrm{proj}_I(\partial_x^\alpha\partial_\eta^\beta\phi(t, s: x_B; \eta_B, 0; 0))|\leqq C_3(1+|x_B|+|\eta_B|)^{2-|\alpha|-|\beta|} \\ \qquad\qquad\qquad\qquad for\quad |\alpha|+|\beta|\leqq 2, \\ |\mathrm{proj}_I(\partial_x^\alpha\partial_\eta^\beta\vec{\partial}_\theta^a\vec{\partial}_\rho^b\phi(t, s: x_B; \eta_B, 0; 0))|\leqq C_3 \\ \qquad\qquad\qquad\qquad for\quad |\alpha|+|\beta|+|a|+|b|\geqq 2. \end{cases}$$

$$(4.51)\quad \begin{aligned} &|\mathrm{proj}_I(\phi(t', s', x_B; \eta_B, 0; 0)-\phi(t, s: x_B; \eta_B, 0; 0))| \\ &\qquad\leqq C_3(1+|x_B|+|\eta_B|)^2(|t-t'|+|s-s'|), \end{aligned}$$

and for $|a|+|b|\geqq 2$,

$$(4.52)\quad \begin{aligned} &|\mathrm{proj}_I(\vec{\partial}_\theta^a\vec{\partial}_\rho^b\phi(t', s', x_B; \eta_B, 0; 0)-\vec{\partial}_\theta^a\vec{\partial}_\rho^b\phi(t, s: x_B; \eta_B, 0; 0))| \\ &\qquad\leqq C_3(|t-t'|+|s-s'|). \end{aligned}$$

Proof. (i)-(ii) are directly obtained by using (4.46), (4.47) and the expression (4.48). To show the first part of (iii), we differentiate (4.49) with respect to x. Then, using (4.35) and (4.47), we have

$$(4.53)\quad \begin{aligned} &\partial_{x_j}\phi(t, s: x; \eta, \theta; \rho) \\ &\quad=\partial_{x_j}y\partial_y u(t, s: y(t, s: x; \eta, \theta; \rho); \eta, \omega(t, s: x; \eta, \theta; \rho); \rho) \\ &\qquad+\partial_{x_j}\omega\vec{\partial}_\omega u(t, s: y(t, s: x; \eta, \theta; \rho); \eta, \omega(t, s: x; \eta, \theta; \rho); \rho) \\ &\quad=\xi(t, s)[\partial_{x_j}y\partial_y x(t, s)+\partial_{x_j}\omega\vec{\partial}_\omega x(t, s)] \end{aligned}$$

$$+\pi(t, s)[\partial_{x_j} y \partial_y \theta(t, s)+\partial_{x_j}\omega\vec{\partial}_\omega \theta(t, s)]$$
$$=\xi_j(t, s: y(t, s: x; \eta, \theta; \rho); \eta, \omega(t, s: x; \eta, \theta; \rho); \rho).$$

The other equations of (iii) can be obtained by similar computations. (iv) is a directly consequence of (4.46), (4.47), (iii) and (4.36), (4.37). Using (4.40) and computing straightfowardly, we get (4.50)-(4.52). ■

Continuity equation. Put

$$(4.54)\quad J(t, s: x; \eta, \theta; \rho)=\operatorname{sdet}\begin{bmatrix}\partial_x y(t, s: x; \eta, \theta; \rho) & \vec{\partial}_\theta y(t, s: x; \eta, \theta; \rho)\\ \partial_x \omega(t, s: x; \eta, \theta; \rho) & \vec{\partial}_\theta \omega(t, s: x; \eta, \theta; \rho)\end{bmatrix}$$

which is well-defined for $|t-s|\leqq\delta_1(T)$, $t, s\in[-T, T]$, because of Proposition 4.4.

PROPOSIITON 4.9 (Continuity equation). *For* $|t-s|\leqq\delta_1(T)$, $J(t, s: x; \eta, \theta; \rho)$ *satisfies the following:*

$$(4.54)\qquad J(s, s, x; \eta, \theta; \rho)=1$$

$$(4.55)\qquad \begin{cases}\partial_t J(t, s: x; \eta, \theta; \rho)\\ \qquad =-\sum_{j=1}^m \partial_{x_j}\{J\partial_{\xi_j}H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)\}\\ \qquad\quad -\sum_{r=1}^n \vec{\partial}_{\theta_r}\{J\vec{\partial}_{\pi_r}H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)\},\\ \partial_s J(t, s: x; \eta, \theta; \rho)\\ \qquad =\sum_{j=1}^m \partial_{\xi_j}\{J\partial_{x_j}H(\partial_\eta\phi; \eta, -\vec{\partial}_\rho\phi; \rho)\}\\ \qquad\quad -\sum_{r=1}^n \vec{\partial}_{\pi_r}\{J\vec{\partial}_{\theta_r}H(\partial_\xi\phi; \eta, -\vec{\partial}_\rho\phi; \rho)\}.\end{cases}$$

Proof. (4.54) is an easy consequence of (4.18). To obtain (4.55), we use the similar argument stated in Appendix A, [18]. Differentiating the Hamilton-Jacobi equation with respect to η and ρ and using (iii) of Proposition 4.8, we have

$$(4.56)\qquad \begin{cases}\partial_t y_j+\sum_{h=1}^m \partial_{x_h} y_j \partial_{\xi_h} H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)\\ \qquad +\sum_{w=1}^n \vec{\partial}_{\theta_\omega} y_j \vec{\partial}_{\pi_\omega} H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)=0,\\ \partial_t \omega_u+\sum_{h=1}^m \partial_{x_h}\omega_u \partial_{\xi_h} H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)\\ \qquad -\sum_{w=1}^n \vec{\partial}_{\theta_\omega}\omega_u \vec{\partial}_{\pi_\omega} H(x; \partial_x\phi, \theta; \vec{\partial}_\theta\phi)=0.\end{cases}$$

Define a matrix $M=(M_{BA})(A, B=1, \cdots, m+n)$ by

$$M=\begin{bmatrix} M_{jk} & M_{\bar{v}k} \\ M_{j\bar{u}} & M_{\bar{v}\bar{u}} \end{bmatrix} \quad j, k=1, \cdots, m, \tag{4.57}$$

$$\bar{u}=u+m, \quad \bar{v}=v+m \quad \text{and} \quad u, v=1\cdots, n$$

where

$$M_{jk}=\partial_{x_j}y_k, \quad M_{j\bar{u}}=\partial_{x_j}\omega_u, \quad M_{\bar{v}k}=\vec{\partial}_{\theta_v}y_k, \quad M_{\bar{v}\bar{u}}=\vec{\partial}_{\theta}v\omega_u. \tag{4.58}$$

Also, we denote by $N=M^{-1}=[N_{BA}]=\begin{bmatrix} N_{jk} & N_{\bar{v}k} \\ N_{j\bar{u}} & N_{\bar{v}\bar{u}} \end{bmatrix}$. Then, we get

$$\begin{cases} \sum_{h=1}^{m} M_{hj}N_{kh}+\sum_{w=1}^{n} M_{\bar{w}j}N_{k\bar{w}}=\delta_{jk} \\ \qquad \sum_{h=1}^{m} M_{h\bar{u}}N_{jh}+\sum_{w=1}^{n} M_{\bar{w}\bar{u}}N_{j\bar{w}}=0, \\ \sum_{h=1}^{m} M_{hk}N_{\bar{u}h}+\sum_{w=1}^{n} M_{\bar{w}k}N_{\bar{u}\bar{w}}=0, \\ \qquad \sum_{h=1}^{m} M_{h\bar{v}}N_{\bar{u}h}+\sum_{w=1}^{n} M_{\bar{w}\bar{v}}N_{\bar{u}\bar{w}}=\delta_{\bar{v}\bar{u}}. \end{cases} \tag{4.59}$$

Differentiate the each equation of (4.56) with respect to x and θ, we get

$$\begin{cases} \partial_t M_{kj}+\sum_{h=1}^{m}(\partial_{\xi_h}H\partial_{x_h}M_{kj}+\partial_{x_k}(\partial_{\xi_h}H)M_{hj}) \\ \qquad -\sum_{w=1}^{n}(\vec{\partial}_{\pi_w}H\vec{\partial}_{\theta_w}M_{kj}+\partial_{x_k}(\vec{\partial}_{\pi_w}H)M_{\bar{w}j}=0, \\ \partial_t M_{\bar{v}j}+\sum_{h=1}^{m}(\partial_{\xi_h}H\partial_{x_h}M_{\bar{v}j}+\vec{\partial}_{\theta_v}(\partial_{\xi_h}H)M_{hj}) \\ \qquad -\sum_{w=1}^{n}(\vec{\partial}_{\pi_w}H\vec{\partial}_{\theta_w}M_{\bar{v}j}+\vec{\partial}_{\theta_v}(\vec{\partial}_{\pi_w}M)M_{\bar{w}j}=0, \\ \partial_t M_{k\bar{u}}+\sum_{h=1}^{m}(\partial_{\xi_h}H\partial_{x_h}M_{k\bar{u}}+\partial_{x_k}(\partial_{\xi_h}H)M_{h\bar{u}}) \\ \qquad -\sum_{w=1}^{n}(\vec{\partial}_{\pi_w}H\vec{\partial}_{\theta_w}M_{k\bar{u}}+\partial_{x_k}(\vec{\partial}_{\pi_w}H)M_{\bar{w}\bar{u}}=0, \\ \partial_t M_{\bar{v}\bar{u}}+\sum_{h=1}^{m}(\partial_{\xi_h}H\partial_{x_h}M_{\bar{v}\bar{u}}+\vec{\partial}_{\theta_v}(\partial_{\xi_h}H)M_{h\bar{u}}) \\ \qquad -\sum_{w=1}^{n}(\vec{\partial}_{\pi_w}H\vec{\partial}_{\theta_w}M_{\bar{v}\bar{u}}+\vec{\partial}_{\theta_v}(\vec{\partial}_{\pi_w}H)M_{\bar{w}\bar{u}})=0. \end{cases} \tag{4.60}$$

Substituting (4.60) into (2.19) and using (4.59), we have easily (4.55). ■

Also, by a direct computations combined with Proposition 4.2, we have

PROPOSITION 4.10. *Under Assumption A, we have, for any* $I\in\mathfrak{I}$, $|t-s|\leqq\delta_1(T)$ *and for* $p, q=0$ *or* 1 *with* $p+q\leqq 1$ *and* $|\alpha|+|\beta|+|a|+|b|\geqq 0$, *there exists a constant* $C_{p,q,\alpha,\beta,a,b}$ *such that*

$$(4.61)\qquad |\mathrm{proj}_I(\partial_t^p\partial_s^q\partial_x^\alpha\partial_\eta^\beta\vec{\partial}_\theta^a\vec{\partial}_\rho^b(J(t, s:x;\eta, \theta;\rho)-1)(t, s:x_B;\eta_B, 0;0))|$$
$$\leqq C_{p,q,\alpha,\beta,a,b}|t-s|.$$

Now, we put

$$(4.62)\qquad \mu(t, s:x;\eta, \theta;\rho)=J(t, s:x;\eta, \theta;\rho)^{1/2}$$

which is a super version of the van Vleck determinant for the classical mechanics (see, [11], [18]). By using Proposition 4.9, we get easily the following:

PROPOSITION 4.11. *For* $|t-s|\leqq\delta_1(T)$, $\mu(t, s:x;\eta, \theta;\rho)$ *satisfies the following:*

$$(4.63)\quad \mu(s, s, x;\eta, \theta;\rho)=1.$$

$$(4.64)\quad \partial_t\mu+\sum_{j=1}^m\partial_{x_j}\mu\partial_{\xi_j}H+\sum_{u=1}^n\vec{\partial}_{\theta_u}\mu\vec{\partial}_{\pi_u}H$$
$$+\frac{1}{2}\mu\Big\{\sum_{j=1}^m\partial_{x_j}\partial_{\xi_j}H+\sum_{j,k=1}^m\partial_{x_j}\partial_{x_k}\phi\partial_{\xi_k}\partial_{\xi_j}H+\sum_{v=1}^n\sum_{j=1}^m\partial_{x_j}\vec{\partial}_{\theta_v}\phi\vec{\partial}_{\pi_v}\partial_{\xi_j}H\Big\}$$
$$+\frac{1}{2}\mu\Big\{\sum_{u=1}^n\vec{\partial}_{\theta_u}\vec{\partial}_{\pi_u}H+\sum_{u=1}^n\sum_{k=1}^m\vec{\partial}_{\theta_u}\partial_{x_k}\phi\partial_{\xi_k}\vec{\partial}_{\pi_u}H+\sum_{u,v=1}^n\vec{\partial}_{\theta_u}\vec{\partial}_{\theta_v}\phi\vec{\partial}_{\pi_v}\vec{\partial}_{\pi_u}H\Big\}=0,$$

where arguments of μ *and* ϕ *are* $(t, s:x;\eta, \theta;\rho)$ *and those of* H *are* $(x;\partial_x\phi, \theta;\vec{\partial}_\theta\phi)$. *Moreover, we have, for any* $I\in\mathfrak{I}$, *any* $p, q=0$ *or* 1 *with* $p+q\leqq 1$ *and* $|\alpha|+|\beta|+|a|+|b|\geqq 0$, *there exists a constant* $C_{p,q,\alpha,\beta,a,b}$ *such that*

$$(4.65)\qquad |\mathrm{proj}_I(\partial_t^p\partial_s^q\partial_x^\alpha\partial_\eta^\beta\vec{\partial}_\theta^a\vec{\partial}_\rho^b(\mu(t, s:x;\eta, \theta;\rho)-1)(t, s:x_B;\eta_B, 0;0))|$$
$$\leqq C_{p,q,\alpha,\beta,a,b}|t-s|.$$

Remark. It seems not necessary to consider the van Vleck determinant if we stay only in classical mechanics. But, if we want to 'quantize' such classical mechanics, it is natural to take it into account (see, Inoue & Maeda [9] and references therein).

References

[1] F. A. Berezin, The Method of Second Quantization, Academic Press, New York-London, 1966.

[2] F. A. Berezin, Introduction to Superanalysis (ed. A. A. Kirillov), Mathematical Physics and Spplied Mathematics, vol. 9, D. Reidel Publ. Comp., Dordrecht-Boston-Lancaster-Tokyo, 1987.

[3] F. A. Berezin and M. S. Marinov, Particle spin dynamics as the Grassmann variant of classical mechanics, Ann. Physics., **104** (1977), 336-362.

[4] P. Bryant, DeWitt supermanifolds and infinite dimensional ground rings, J. London Math. Soc., **39** (1989), 347-368.

[5] R. Casalbuoni, On the quantization of systems with anticommuting variables, II Nuovo Cimento **33A** (1976), 116-125.

[6] B. DeWitt, Supermanifolds, Cambridge Univ. Press, London, 1984.

[7] D. Fujiwara, A construction of the fundamental solution for the Schrödinger equation, J. D'Analyse Math., **35** (1979), 41-96.

[8] J, Hoyos, M. Quiros, J. Lamirez Mittelbrunn and F. J. de Urries, Generalized supermanifolds, I, II, III, J. Math. Phys., **25** (1984), 833-854.

[9] A. Inoue and Y. Maeda, On integral transformations associated with a certain Lagrangian- as a prototype of quantization, J. Math. Soc. Japan, **37** (1985), 219-244.

[10] A. Inoue and Y. Maeda, Super oscillatory integrals and a path integral for a non-relativistic spinning particle, Proc. Japan Acad. Ser. A, **63** (1987), 1-3.

[11] A. Inoue and Y. Maeda, Studies of Schrödinger equations with spin using superanalysis, I. A. general scheme of constructing a fundamental solutions, Warwick, preprint series (1988).

[12] A. Jadczyk and K. Pilch, Superspaces and supersymmetries, Commun. Math. Phys., **78** (1981), 373-390.

[13] H. H. Keller, Differential Calculus in Locally Convex Spaces, Lecture Notes in Math. vol. 417, Springer-verlag, Berlin-New York-Tokyo-Heidelberg, 1974.

[14] H. Kitada and H. Kumano-go, A family of Fourier integral operators and the fundamental solution for a Schrödinger equation, Osaka J. Math., **18** (1981), 291-360.

[15] B. Kostant and S. Sternberg, Symplectic reduction, BRS cohomology and infinite-dimensional Clifford algebras, Ann. Math., **176** (1987), 49-113.

[16] G. Kothe, Topological Linear Spaces I, Springer, Berlin-New York-Tokyo-Heidelberg, 1969.

[17] D. A. Leites, Introduction to the theory of supermanifolds, Russian Math. Surveys, **35** (1980), 1-64.

[18] J. Manes and B. Zumino, WKB method, susy quantum mechanics and the index theorem, Nuclear Phys. B (FS 16), **270** (1986), 651-686.

[19] Y. I. Manin, New dimensions in geometry, Lecture Notes in Math. vol. 1111, pp. 59-101, Springer-verlag, Berlin-New York-Tokyo-Heidelberg.

[20] J. L. Martin, Generalized classical dynamics and the classical analogue of a Fermi oscillator, Proc. Roy. Soc. A, **251** (1959), 536-542.

[21] J. L. Martin, The Feynman principle for a Fermi system, Proc. Roy. Soc. A, **251** (1959), 543-549.

[22] S. Matsumoto and K. Kakazu, A note on topology of supermanifolds, J. Math.

Phys., **27** (1986), 2690-2692.
[23] A. ROGERS, A global theory of supermanifolds, J. Math. Phys., **21** (1980), 1352-1365.
[24] A. ROGERS, Consistant superspace integration, J. Math. Phys., **26** (1985), 385-392.
[25] A. ROGERS, Graded manifolds, supermanifolds and infinite-dimensional Grassmann algebra, Commun. Math. Phys., **105** (1986), 375-384.
[26] V. S. VLADIMIROV AND I. V. VOLOVICH, Superanalysis I. Differential calculas, Theor. Math. Phys., **59** (1983), 317-335.
[27] V. S. VLADIMIROV, Superanalysis II. Integral calculus, Theor. Math. Phys. **60** (1984), 743-765.
[28] K. YAGI, Super-differential calculus, Osaka J. Math., (1988), 243-257.
[29] S. YAMAMURO, A Theory of Differentiation in Locally Convex Spaces, Memoirs Amer. Math. Soc., Amer. Math. Soc., Providence, Rhode Island, USA, 1979.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU,
TOKYO, 152, JAPAN

DEPARTMENT OP MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA
223, JAPAN