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A CLASSIFICATION OF 3-DIMENSIONAL CONTACT METRIC MANIFOLDS WITH $Q\varphi=\varphi Q$

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1. Introduction

The assumption that $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a contact metric manifold is very weak, since the set of metrics associated to the contact form η is huge. Even if the structure is η-Einstein we do not have a complete classification. Also for $m=1$, we know very little about the geometry of these manifolds [8]. On the other hand if the structure is Sasakian, the Ricci operator Q commutes with φ ([1], p. 76), but in general $Q\varphi\neq\varphi Q$ and the problem of the characterization of contact metric manifolds with $Q\varphi=\varphi Q$ is open. In [13] Tanno defined a special family of contact metric manifolds by the requirement that ξ belong to the k-nullity distribution of g. We also know very little about these manifolds (see [13] and [9]). In § 3 of this paper we first prove that on a 3-dimensional contact metric manifold the conditions, i) the structure is η-Einstein, ii) $Q\varphi=\varphi Q$ and iii) ξ belongs to the k-nullity distribution of g are equivalent. We then show that a 3-dimensional contact metric manifold on which $Q\varphi=\varphi Q$ is either Sasakian, flat or of constant ξ-sectional curvature k and constant φ-sectional curvature $-k$. Finally we give some auxiliary results on locally φ-symmetric contact metric 3-manifolds and on contact metric 3-manifolds immersed in a 4-dimensional manifold of contant curvature $+1$.

2. Preliminaries

A C^∞ manifold M^{2m+1} is said to be a *contact manifold,* if it carries a global 1-form η such that $\eta\wedge(d\eta)^m\neq 0$ everywhere. We assume throughout that all manifolds are connected. Given a contact form η, it is well known that there exists a unique vector field ξ, called the *characteristic vector field* of η, satisfying $\eta(\xi)=1$ and $d\eta(\xi, X)=0$ for all vector fields X. *A* Riemannian metric g is said to be an *associated metric* if there exists a tensor field φ of type (1, 1) such that

$$d\eta(X, Y)=g(X, \varphi Y),\ \eta(X)=g(X, \xi),\ \varphi^2=-I+\eta\otimes\xi. \tag{2.1}$$

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From these conditions one can easily obtain

$$\varphi\xi=0, \quad \eta\circ\varphi=0, \quad g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y). \tag{2.2}$$

The structure (φ, ξ, η, g) is called a *contact metric structure*, and a manifold M^{2m+1} with a contact metric structure (φ, ξ, η, g) is said to be a *contact metric manifold*.

Denoting by $\mathcal{L}$ and R Lie differentiation and the curvature tensor respectively, we define the operators l and h by

$$lX=R(X, \xi)\xi, \qquad h=\frac{1}{2}\mathcal{L}_\xi\varphi. \tag{2.3}$$

The (1, 1)-type tensors h and l are symmetric and satisfy

$$h\xi=0, \quad l\xi=0, \quad Trh=0, \quad Trh\varphi=0 \quad \text{and} \quad h\varphi=-\varphi h. \tag{2.4}$$

We also have the following formulas for a contact metric manifold:

$$\nabla_X\xi=-\varphi X-\varphi hX \quad (\text{and hence } \nabla_\xi\xi=0) \tag{2.5}$$

$$\nabla_\xi\varphi=0 \tag{2.6}$$

$$Trl=g(Q\xi, \xi)=2m-Trh^2 \tag{2.7}$$

$$\varphi l\varphi-l=2(\varphi^2+h^2) \tag{2.8}$$

$$\nabla_\xi h=\varphi-\varphi l-\varphi h^2 \tag{2.9}$$

where Q is the Ricci operator and ∇ the Riemannian connection of g. Formulas (2.5)-(2.8) occur in [1] and (2.9) in [3].

A contact metric manifold for which ξ is Killing is called a *K-contact manifold*. A contact structure on M^{2m+1} naturally gives rise to an almost complex structure on the product $M^{2m+1}\times \boldsymbol{R}$. If this almost complex structure is integrable, the given contact metric manifold is said to be *Sasakian*. Equivalently, (see [1, p. 75] or [3, pp. 534-535]) a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi=\eta(Y)X-\eta(X)Y \tag{2.10}$$

for all vector fields X and Y.

It is easy to see that a 3-dimensional contact metric manifold is Sasakian if and only if $h=0$. For details we refer the reader to [1].

A contact metric structure is said to be *η-Einstein* if

$$Q=aI+b\eta\otimes\xi \tag{2.11}$$

where a, b are smooth functions on M^{2m+1}. We also recall that the k-nullity distribution (see Tanno [13]) of a Riemannian manifold (M, g), for a real number k, is a distribution

$$N(k): p \to N_p(k) = \{Z \in T_pM : R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y) \text{ for any } X, Y \in T_pM\}.$$

Finally the sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a *ξ-sectional curvature* and the sectional curvature $K(X, \varphi X)$ of a plane section spanned by vectors X and φX with X orthogonal to ξ is called a *φ-sectional curvature.*

We close this paragraph with two examples of 3-dimensional η-Einstein contact metric manifolds:

1) $\boldsymbol{R}^3(x^1, x^2, x^3)$ with the contact form $\eta = 1/2(dx^3 - x^2 dx^1)$ and associated metric $g = 1/4(\eta \otimes \eta + (dx^1)^2 + (dx^2)^2)$, is an η-Einstein Sasakian manifold (see [1] or [6] for more details).

2) $\boldsymbol{R}^3$ or T^3 (torus) with $\eta = 1/2(\cos x^3 dx^1 + \sin x^3 dx^2)$ and $g_{ij} = (1/4)\delta_{ij}$, is an η-Einstein (non-Sasakian) contact metric manifold.

3. Main results

Before we state our first result we need the following lemma which was proved in [4], but we include its proof here for completeness and because we will use many of the formulas which will appear in the proof.

LEMMA 3.1. *Let M^3 be a contact metric manifold with a contact metric structure (φ, ξ, η, g) such that $\varphi Q = Q\varphi$. Then the function Trl is constant everywhere on M^3.*

Before we give the proof of the Lemma we recall that the curvature tensor of a 3-dimensional Riemannian manifold is given by

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{S}{2}(g(Y, Z)X - g(X, Z)Y) \tag{3.1}$$

where S is the scalar curvature of the manifold.

Proof of the Lemma 3.1. Using $\varphi Q = Q\varphi$, (2.7) and $\varphi\xi = 0$ we have that

$$Q\xi = (Trl)\xi . \tag{3.2}$$

From (3.1), using (2.3) and (3.2) we have for any X,

$$lX = QX + \left(Trl - \frac{S}{2}\right)X + \eta(X)\left(\frac{S}{2} - 2Trl\right)\xi \tag{3.3}$$

and hence $Q\varphi = \varphi Q$ and $\varphi\xi = 0$ give

$$\varphi l = l\varphi . \tag{3.4}$$

By virtue of (3.4), (2.8) and (2.9) we obtain

$$-l = \varphi^2 + h^2 \tag{3.5}$$

and $\nabla_\xi h = 0$. Differentiating (3.5) along ξ and using (2.6) and $\nabla_\xi h = 0$ we find that $\nabla_\xi l = 0$ and therefore $\xi Trl = 0$. If at a point $P \in M^3$ there exists $X \in T_p M^3$, $X \neq \xi$ such that $lX = 0$, then $l = 0$ at P. In fact if Y is the projection of X on the contact subbundle, $\eta = 0$, we have $lY = 0$, since $l\xi = 0$. Using (3.4) we have $l\varphi Y = 0$. So $l = 0$ at P (and thus $Trl = 0$ at P). We now suppose that $l \neq 0$ on a neighborhood U of a point P. Using (3.4) and that φ is antisymmetric we get $g(\varphi X, lX) = 0$. So lX is parallel to X for any X orthogonal to ξ. It is not hard to see that $lX = 1/2(Trl)X$ for any X orthogonal to ξ. Thus for any X, we have

$$lX = -\frac{1}{2}(Trl)\varphi^2 X . \tag{3.6}$$

Substituting (3.6) in (3.3) we get

$$QX = aX + b\eta(X)\xi \tag{3.7}$$

where $a = \frac{1}{2}(S - Trl)$ and $b = \frac{1}{2}(3Trl - S)$. Differentiating (3.7) with respect to Y and using (3.7) and $\nabla_\xi \xi = 0$ we find

$$(\nabla_Y Q)X = (Ya)X + ((Yb)\eta(X) + bg(X, \nabla_Y \xi))\xi + b\eta(X)\nabla_Y \xi . \tag{3.8}$$

So using $\xi Trl = 0$ and $\nabla_\xi \xi = 0$ we have from (3.8) with $X = Y = \xi$, $(\nabla_\xi Q)\xi = 0$. Also using $h\varphi = -\varphi h$, (2.5) and (2.2) we get from (3.8) with $Y = X$ orthogonal to ξ

$$g((\nabla_X Q)X + (\nabla_{\varphi X} Q)\varphi X, \xi) = 0 .$$

But it is well known that

$$(\nabla_X Q)X + (\nabla_{\varphi X} Q)\varphi X + (\nabla_\xi Q)\xi = \frac{1}{2} \operatorname{grad} S$$

for any unit X orthogonal to ξ. Hence we easily get from the last two equations that $\xi S = 0$, and thus $\nabla_\xi Q = 0$, since $S = TrQ$. Therefore differentiating (3.1) with respect to ξ and using $\nabla_\xi Q = 0$ we have $\nabla_\xi R = 0$. So from the second identity of Bianchi we get

$$(\nabla_X R)(Y, \xi, Z) = (\nabla_Y R)(X, \xi, Z) \tag{3.9}$$

Now, substituting (3.7) in (3.1) we obtain

$$\begin{aligned} R(X, Y)Z = &\{\gamma g(Y, Z) + b\eta(Y)\eta(Z)\}X \\ &- \{\gamma g(X, Z) + b\eta(X)\eta(Z)\}Y \\ &+ b\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\xi . \end{aligned} \tag{3.10}$$

where $\gamma=S/2-Trl$. For $Z=\xi$, (3.10) gives

$$R(X, Y)\xi=\frac{Trl}{2}(\eta(Y)X-\eta(X)Y) \tag{3.11}$$

Using (3.11) we obtain $(\nabla_X R)(Y, \xi, \xi)=\frac{1}{2}(XTrl)Y$, for X, Y orthogonal to ξ. From this and (3.9) for $Z=\xi$ we get $(XTrl)Y=(YTrl)X$. Therefore $XTrl=0$ for X orthogonal to ξ, but $\xi Trl=0$, so the function Trl is constant and this completes the proof of the Lemma.

Remark 3.1. When $l=0$ everywhere, then using (3.1), (3.2) and (3.3) we get $R(X, Y)\xi=0$. So by Theorem B of [2], M^3 is flat.

PROPOSITION 3.2. *Let M^3 be a contact metric manifold with contact metric structure (φ, ξ, η, g). Then the following conditions are equivalent:*

i) *M^3 is η-Einstein*
ii) *$Q\varphi=\varphi Q$*
iii) *ξ belongs to the k-nullity distribution*

Proof. i→ii. This follows immediately from (2.11) and $\varphi\xi=0$.
ii→iii. This follows from (3.11) and $Trl=const.$
iii→i. By the assumption we have

$$R(X, Y)\xi=k(\eta(Y)X-\eta(X)Y) \tag{3.12}$$

where k is a constant$\leqq 1$ [13]. From (3.12) we have $Q\xi=2k\xi$ and so from (3.1) we find

$$R(X, Y)\xi=\eta(Y)QY-\eta(X)QX+\left(2k-\frac{S}{2}\right)(\eta(Y)X-\eta(X)Y) \tag{3.13}$$

Comparing (3.12) and (3.13) we get

$$\eta(Y)\left\{QX+\left(k-\frac{S}{2}\right)X\right\}-\eta(X)\left\{QY+\left(k-\frac{S}{2}\right)Y\right\}=0.$$

Taking Y orthogonal to ξ and $X=\xi$ we have $QY=((S/2)-k)Y$ and so for any Z

$$QZ=\left(\frac{S}{2}-k\right)Z+\left(3k-\frac{S}{2}\right)\eta(Z)\xi.$$

This completes the proof.

Remark 3.2. Because $a+b=Trl$ (see formula (3.7)), using Lemma 3.1 and Proposition 3.2 we have the following. On any η-Einstein ($Q=aI+b\eta\otimes\xi$) contact metric manifold M^3, $a+b=const.$ ($=Trl$). It is known that for any η-Einstein K-contact manifold M^{2m+1} ($m>1$) we have $a=const.$, $b=const.$

THEOREM 3.3. *Let M^3 be a contact metric manifold on which $Q\varphi=\varphi Q$.*

Then M^3 is either Sasakian, flat or of constant ξ-sectional curvature $k<1$ and constant φ-sectional curvature $-k$.

Proof. We can easily see from the proof of Lemma 3.1 and Remark 3.1 that if $Trl=0$, $l=0$ and in turn that M^3 is flat. If $Trl=2$, (2.7) gives $Trh^2=0$ and hence, since h is symmetric, $h=0$; thus M^3 is Sasakian.

If $Trl\neq 0$ and 2 then from Proposition 3.2 and (3.12) we have

$$R(X, Y)\xi=k(\eta(Y)X-\eta(X)Y) \tag{3.14}$$

where $k=Trl/2$ is now <1. This implies that

$$(\nabla_X\varphi)Y=g(X+hX, Y)\xi-\eta(Y)(X+hX) \tag{3.15}$$

as was pointed out by Tanno ([13] pp. 446-447, cf. Olszak [7] p. 251); in fact this is true for any 3-dimensional contact metric manifold (Tanno [14] p. 353.). Computing $R(X, Y)\xi$ from (2.5) we have

$$\begin{aligned} R(X, Y)\xi &= -(\nabla_X\varphi)Y+(\nabla_Y\varphi)X-(\nabla_X\varphi h)Y+(\nabla_Y\varphi h)X \\ &= -(\nabla_X\varphi)Y+(\nabla_Y\varphi)X-(\nabla_X\varphi)hY-\varphi(\nabla_X h)Y \\ &\quad +(\nabla_Y\varphi)hX+\varphi(\nabla_Y h)X . \end{aligned}$$

Then using (3.14) and (3.15) we have

$$\begin{aligned} k(\eta(Y)X-\eta(X)Y) &= -\eta(X)(Y+hY)+\eta(Y)(X+hX) \\ &\quad -\varphi((\nabla_X h)Y-(\nabla_Y h)X) \end{aligned}$$

or

$$\begin{aligned} &\eta(Y)hX-\eta(X)hY-\varphi((\nabla_X h)Y-(\nabla_Y h)X) \\ &\quad =(k-1)(\eta Y)X-\eta(X)Y) \end{aligned} \tag{3.16}$$

Now let X be a unit eigenvector of h, say $hX=\lambda X$, $X\perp\xi$. Since $Trh^2=2(1-k)$, $\lambda=\pm\sqrt{1-k}$ and hence is a constant. Setting $Y=\varphi X$, (3.16) yields

$$\varphi((\nabla_X h)\varphi X-(\nabla_{\varphi X} h)X)=0$$

from which

$$\varphi(-\lambda\nabla_X\varphi X-h\nabla_X\varphi X-\lambda\nabla_{\varphi X}X+h\nabla_{\varphi X}X)=0 . \tag{3.17}$$

Taking the inner product of (3.17) with X and recalling that $\varphi h+h\varphi=0$, we have

$$\lambda g(\nabla_{\varphi X}X, \varphi X)=0 .$$

Since $\lambda\neq 0$ ($k\neq 1$) and X is unit, $\nabla_{\varphi X}X$ is orthogonal to both X and φX and hence collinear with ξ. Now

$$\eta(\nabla_{\varphi X}X)=g(\nabla_{\varphi X}X,\ \xi)=-g(\nabla_{\varphi X}\xi,\ X)=g(-X+hX,\ X)=\lambda-1\,.$$

Therefore

$$\nabla_{\varphi X}X=(\lambda-1)\xi\,.$$

Similarly taking the inner product of (3.17) with φX yields

$$\nabla_X\varphi X=(\lambda+1)\xi$$

and in turn $\nabla_X X=0$ and

$$[X,\ \varphi X]=2\xi\,.$$

Now from the form of the curvature tensor (3.10), we have

$$R(X,\ \varphi X)X=-\Big(\frac{S}{2}-Trl\Big)\varphi X$$

and by direct computation using $\nabla_X\xi=-(1+\lambda)\varphi X$,

$$\begin{aligned}R(X,\ \varphi X)X&=\nabla_X\nabla_{\varphi X}X-\nabla_{\varphi X}\nabla_X X-\nabla_{[X,\varphi X]}X\\&=(\lambda-1)\nabla_X\xi-2\nabla_\xi X\\&=(1-\lambda^2)\varphi X-2\nabla_\xi X\,.\end{aligned}$$

Thus

$$\nabla_\xi X=\Big(\frac{S}{4}+\frac{\lambda^2-1}{2}\Big)\varphi X$$

and hence

$$[\xi,\ X]=\Big(\frac{S}{4}+\frac{(\lambda+1)^2}{2}\Big)\varphi X\,.$$

Now computing $R(\xi,\ X)\xi$ by (3.14) and by direct computation we have

$$\begin{aligned}(\lambda^2-1)X&=\nabla_\xi(-\varphi X-\varphi hX)-\nabla_{(S/4+(\lambda+1)^2/2)\varphi X}\xi\\&=-(1+\lambda)\varphi\nabla_\xi X-\Big(\frac{S}{4}+\frac{(\lambda+1)^2}{2}\Big)(X-hX)\\&=\Big[(1+\lambda)\Big(\frac{S}{4}+\frac{\lambda^2-1}{2}\Big)-(1-\lambda)\Big(\frac{S}{4}+\frac{(\lambda+1)^2}{2}\Big)\Big]X\end{aligned}$$

from which

$$S=2(1-\lambda^2)=2k\,.$$

From (3.14) and (3.10) we see that

$$K(X,\ \xi)=k \quad\text{and}\quad K(X,\ \varphi X)=-k$$

as desired.

Remark 3.3. We also note for $k\neq 0$ and 1 that from (3.7) the Ricci operator is given by $QX=2k\eta(X)\xi$ and that the scalar curvature is constant, viz., $2k$.

DEFINITION. A contact metric stracture (φ, ξ, η, g) is said to be *locally φ-symmetric* if $\varphi^2(\nabla_W R)(X, Y, Z)=0$, for all vector fields W, X, Y, Z orthogonal to ξ.

This notion was introduced for Sasakian manifolds by Takahashi [11]. The next theorem generalizes Theorem 4.1 of Watanabe [15].

THEOREM 3.4. *Let M^3 be a contact metric manifold with $Q\varphi=\varphi Q$. Then M^3 is locally φ-symmetric if and only if the scalar curvature S of M^3 is constant.*

Proof. From the proof of Lemma 3.1 we see that either $l=0$ everywhere (and hence by Remark 3.1, that M^3 is flat) or $Trl=const. \neq 0$ and in this case all the formulas in Lemma 3.1 are valid. Differentiating (3.10) with respect to W and using Lemma 3.1 we obtain

$$
\begin{aligned}
(3.18)\quad 2(\nabla_W R)(X,Y,Z)=&\,g(Y,Z)\{-(WS)\eta(X)\xi+2b(g(X,\nabla_W\xi)\xi+\eta(X)\nabla_W\xi)\}\\
&-g(X,Z)\{-(WS)\eta(Y)\xi+2b(g(Y,\nabla_W\xi)\xi+\eta(Y)\nabla_W\xi)\}\\
&-\{(WS)g(\varphi^2Y,Z)-2bg(g(Y,\nabla_W\xi)\xi+\eta(Y)\nabla_W\xi,Z)\}X\\
&+\{(WS)g(\varphi^2X,Z)-2bg(g(X,\nabla_W\xi)\xi+\eta(X)\nabla_W\xi,Z)\}Y\,.
\end{aligned}
$$

Taking W, X, Y, Z orthogonal to ξ and using (2.1) and $\varphi\xi=0$ we get from (3.18)

$$2\varphi^2(\nabla_W R)(X, Y, Z)=(WS)(g(X, Z)Y-g(Y, Z)X)$$

The rest of the proof follows immediately from this and $\xi S=0$ (again see the proof of Lemma 3.1).

Remark 3.4. Using (3.8) with $Trl=const.$, (2.5), (3.5) and (3.6) we obtain the following formula

$$(3.19)\qquad 2|\nabla Q|^2=|gradS|^2+(3Trl-S)^2(4-Trl)$$

which is valid on any contact metric manifold M^3 with $Q\varphi=\varphi Q$.

Furthermore Blair and Sharma [5] recently proved that a locally symmetric contact metric manifold M^3 has constant curvature 0 or 1. Thus using (3.19), $Trl\leqq 2$ and the result of [5] we easily obtain the following. A locally φ-symmetric contact metric manifold M^3 with $Q\varphi=\varphi Q$ is a space form (with curvature 0 or 1) if and only if $S=3Trl$.

Before we state our next Theorem we need the following Lemma.

LEMMA 3.5. *Let M^3 be a contact metric manifold with $Q\varphi=\varphi Q$, isometrically immersed in a Riemannian manifold M^4 of constant curvature* 1. *If ξ is not an eigenvector of the Weingarten map A at a point p of M^3, then $Trl=2$.*

The proof of Lemma 3.5 is similar to the proof of Lemma 2.1 of Takahashi and Tanno [10].

THEOREM 3.6. *Let M^3 be a contact metric manifold with $Q\varphi=\varphi Q$. If M^3 is isometrically immersed in a Riemannian manifold M^4 of constant sectional curvature* 1, *then M^3 is Sakakian.*

Proof. Because M^3 is isometrically immersed in a space of constant sectional curvaturel 1 the following equations of Gauss and Codazzi are valid, for any vector fields X, Y, Z on M^3:

$$R(X, Y)Z=g(Y, Z)X-g(X, Z)Y+g(AY, Z)AX-g(AX, Z)AY \tag{3.20}$$

$$(\nabla_X A)Y=(\nabla_Y A)X \tag{3.21}$$

Combining (3.11) and (3.20) for $Z=\xi$ we get

$$\left(1-\frac{Trl}{2}\right)(\eta(Y)X-\eta(X)Y)+g(A\xi, Y)AX-g(A\xi, X)AY=0 \tag{3.22}$$

For M^3 to be Sasakian it is sufficient to prove, by (2.10) and (3.11), that $Trl=2$. Suppose $Trl\neq 2$ and hence $Trl<2$. According to the Lemma 3.5, ξ must be an eigenvector of A everywhere on M^3. Let

$$A\xi=\nu\xi \tag{3.23}$$

where ν is a smooth function on M^3. From (3.22) with $Y=\xi$ and (3.23) we have

$$\left(1-\frac{Trl}{2}\right)X+\nu AX=0$$

with $\nu\neq 0$ for any X orthogonal to ξ. So

$$AX=\rho X, \qquad \rho=\nu^{-1}\left(\frac{Trl}{2}-1\right). \tag{3.24}$$

Using (3.21) with $Y=\xi$ and X orthogonal to ξ the equation (3.24) and the fact that $\nabla_\xi X$ and $\nabla_X \xi$ are also orthogonal to ξ, we find

$$\nabla_X A\xi-A\nabla_X\xi=\nabla_\xi AX-A\nabla_\xi X$$

or

$$(X\nu)\xi+(\nu-\rho)\nabla_X\xi=(\xi\rho)X$$

or using (2.5)

$$(X\nu)\xi+(\nu-\rho)(-\varphi X-\varphi hX)=(\xi\rho)X.$$

From this we get $X\nu=0$ and so

$$(\nu-\rho)(-\varphi X-\varphi hX)=(\xi\rho)X. \tag{3.25}$$

Applying φ to (3.25) and using (2.1) and $\varphi\xi=h\xi=0$ we obtain $(\nu-\rho)(X+hX)=(\xi\rho)\varphi X$. Now replacing X by φX in (3.25) and using $\varphi h=-h\varphi$ we have $(\nu-\rho)(X-hX)=(\xi\rho)\varphi X$. Adding the last two equations we get $\nu=\rho$, i.e. $(Trl/2)-1=\nu^2\geqq 0$, which is a contradiction. This completes the proof.

Our last Theorem generalizes the Theorems (3.6) and (3.8) of Tanno [12] for 3-dimensional manifolds.

THEOREM 3.7. *Let M^3 be a contact metric manifold with $Q\varphi=\varphi Q$. If M^3 is isometrically immersed in a Riemannian manifold M^4 of constant curvature* 1, *then M^3 is of constant curvature* 1 *if and only if the scalar curvature of M^3 is equal to* 6.

Proof. By the assumption and Theorem 3.6 we have $Trl=2$. Supposing M^3 is of constant curvature 1 and using (3.10) with $Z=Y$ orthogonal to X, $|X|=|Y|=1$ and X, Y orthogonal to ξ, we have $1=g(R(X,Y)Y,X)=\gamma=(S/2)-2$, i.e. $S=6$. Now if $S=6$ then $b=(1/2)(3Trl-S)=0$ and $\gamma=(S/2)-Trl=1$ and hence from (3.10) we get $R(X,Y)Z=g(Y,Z)X-g(X,Z)Y$ completing the proof of the theorem.

REFERENCES

[1] D.E. BLAIR, Contact manifolds in Riemannian Geometry, Lecture Notes in Mathematics, 509, Springer-Verlag, Berlin, 1976.

[2] D.E. BLAIR, Two remarks on contact metric structures, Tôhoku Math. J., **29** (1977), 319-324.

[3] D.E. BLAIR AND J.N. PATNAIK, Contact manifolds with characteristic vector field annihilated by the curvature, Bull. Inst. Math. Acad. Sinica, **9** (1981), 533-545.

[4] D.E. BLAIR AND T. KOUFOGIORGOS, Conformally flat contact metric manifolds, submitted.

[5] D.E. BLAIR AND R. SHARMA, Three dimensional locally symmetric contact metric manifolds, to appear in Boll. Un. Mat. Ital.

[6] M. OKUMURA, On infinitesimal conformal and projective transformations of normal contact spaces, Tôhoku Math. J., **14** (1962), 398-412.

[7] Z. OLSZAK, On contact metric manifolds, Tôhoku Math. J., **31** (1979), 247-253.

[8] D. PERRONE, Torsion and critical metrics on contact three manifolds, Kōdai Math. J. **13** (1990), 88-100.

[9] R. SHARMA AND T. KOUFOGIORGOS, Locally symmetric and Ricci-symmetric contact metric manifolds, submitted.

[10] T. TAKAHASHI AND S. TANNO, *K*-contact Riemannian manifolds isometrically immersed in a space of constant curvature, Tôhoku Math. J., **23** (1971), 535-539.

[11] T. TAKAHASHI, Sasakian φ-symmetric spaces, Tôhoku Math. J., **29** (1977), 91-113.

[12] S. TANNO, Isometric immersions of Sasakian manifolds in spheres, Kōdai Math. Sem. Rep., **21** (1969), 448-458.

[13] S. TANNO, Ricci curvatures of contact Riemannian manifolds, Tôhoku Math. J., **40** (1988), 441-448.

[14] S. TANNO, Variational problems on contact Riemannian manifolds, Trans. A.M.S. **314** (1989), 349-379.

[15] Y. WATANABE, Geodesic symmetries in Sasakian locally φ-symmetric spaces, Kōdai Math. J., **3** (1980), 48-55.

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