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ON THE ZEROS OF THE SOLUTIONS OF THE EQUATION $w^{(k)}+(\mathrm{Re}^{P}+Q)w=0$

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1. Introduction.

It was shown in [3; §5(b), p. 356] that for any polynomial $P(z)$ of degree $r\geqq 1$, there is a polynomial $Q(z)$ of degree $2r-2$ such that the equation,

$$w''+(e^{P}+Q)w=0, \tag{1.1}$$

possesses two linearly independent solutions each having no zeros. This result led to an investigation in [4] of the more general equation of arbitrary order $k\geqq 2$,

$$w^{(k)}+(Re^{P}+Q)w=0, \tag{1.2}$$

where P, Q, and R are any polynomials, with P of degree $r\geqq 1$, and $R\not\equiv 0$. It was shown in [4] that if the degree of Q is less than $kr-k$, then the exponent of convergence (denoted $\lambda(f)$) of the zero-sequence of any solution $f\not\equiv 0$ satisfies $\lambda(f)=\infty$. (We recall (see [12; p. 250] that if $(z_1, z_2, \cdots)$ is the sepuence of zeros of $f(z)$ in $|z|>0$, then $\lambda(f)$ is the infimum of all $\alpha>0$ for which the series $|z_1|^{-\alpha}+|z_2|^{-\alpha}+\cdots$ converges if such an α exists. Otherwise, $\lambda(f)$ is defined to be infinity.)

The above results lead naturally to an investigation of the zero-sequences of solutions of (1.2) in the case when the degree of Q exceeds $kr-k$. One special case has already been successfully treated by the second author in the paper [7] which concerns a problem of M. Ozawa [8]. It was shown in [7] that the conclusion $\lambda(f)=\infty$ holds for all solutions $f\not\equiv 0$ of (1.1) when P is linear and Q is nonconstant. In the present paper, we treat the general equation (1.2) of arbitrary order $k\geqq 2$ where the degree of Q exceeds $kr-k$, and we prove that all solutions $f\not\equiv 0$ satisfy $\lambda(f)=\infty$ except possibly when a very special relation exists between P and Q. Our main result is:

THEOREM 1.1. *Let k be an integer greater than* 1, *and let P, Q, and R be polynomials with $R\not\equiv 0$, $\deg(P)=r\geqq 1$, and $\deg(Q)=n>kr-k$, say $Q(z)=a_nz^n+\cdots$, and $P(z)=b_rz^r+\cdots$. Let $c_1, \cdots, c_k$ be the distinct roots of the equation $c^k+a_n=0$,*

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and assume that there exists a real number θ_0 *in* $(-\pi, \pi]$ *for which,*

$$\cos(r\theta_0+\arg b_r)=0 \tag{1.3}$$

and

$$\cos(((n+k)/k)\theta_0+\arg c_j)\neq 0 \quad \text{for } j=1, \cdots, k. \tag{1.4}$$

Then the exponent of convergence of the zero-sequence of any solution $f\not\equiv 0$ of (1.2) is infinite.

We remark that in the exceptional case when no such point θ_0 exists satisfying (1.3) and (1.4) simultaneously, the situation concerning $\lambda(f)$ is still unclear, but the authors conjecture that the conclusion $\lambda(f)=\infty$ is still valid for all solutions $f\not\equiv 0$. However, our proof will not handle this special case for the following reason: A ray $\arg z=\theta_0$ for which (1.3) holds represents a dividing line between two adjoining sectors where the growth of e^P is changing from very fast to very small. For a solution $f\not\equiv 0$ with $\lambda(f)<\infty$, we can obtain extensive information on the form of f in the sector where e^P is growing fast by using certain techniques from Nevanlinna theory (notably Clunie's lemma (see [6; p. 68])). In the adjoining sector where e^P is decaying, the equation (1.2) is approximated by the equation, $w^{(k)}+Qw=0$, and this fact allows us to asymptotically integrate (1.2) in this sector to yield a fundamental set (see § 3 below). The condition (1.4) is needed in our proof to ensure that the dominant terms in this fundamental set have the same growth pattern in both of the adjoining sectors thus providing a "connection" between the two sectors.

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2. Preliminaries.

(A). [5; pp. 11-12]. An R-set is a countable union of disks whose centers converge to ∞, and whose radii have finite sum. The set of all real θ for which the ray $z=re^{i\theta}$ meets infinitely many disks in an R-set, has measure zero.

(B). Let $f(z)$ be a meromorphic function on the plane. If $H(f)$ is a polynomial in $f, f', f'', \cdots, f^{(k)}$, whose coefficients are meromorphic functions, say $H(f)$ is a sum of terms of the form $a(z)f^{i_0}(f')^{i_1}\cdots(f^{(k)})^{i_k}$, where $a(z)$ is not identically zero, then the maximum of all the numbers, $i_0+i_1+\cdots+i_k$ in the terms of $H(f)$, is called the *total degree* of $H(f)$. In addition, we use the standard notation (see [6; p. 4]) for the *Nevanlinna proximity function* $m(r, f)$ of the function $f(z)$, which is defined as $m(r, f)=(2\pi)^{-1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$, where $\log^+x$ denotes the maximum of $\log x$ and zero. We will need the following slight variant of Clunie's lemma [6; p. 68]. The proof is identical to the proof in [6].

LEMMA 2.1. *Let n be a positive integer, and let f be a transcendental, meromorphic solution of the equation $f^n G(f)=H(f)$, where $G(f)$ and $H(f)$ are polynomials in $f, f', \cdots, f^{(k)}$ (for some $k\geqq 1$), whose coefficients are meromorphic functions. Let Δ denote the set of all coefficients, and assume that the total degree of $H(f)$ is at most n. Then,*

$$m(r, G(f))=0\Big(\Psi(r)+1+\sum_{j=1}^{k} m(r, f^{(j)}/f)\Big) \quad as \quad r \longrightarrow \infty, \tag{2.1}$$

where $\Psi(r)=\max\{m(r, a): a\in\Delta\}$.

(C). We will need the following concepts from [11]:

(a) [11: §94]: *The neighborhood system $F(a, b)$.* Let $-\pi\leqq a<b\leqq\pi$. For each nonnegative real-valued function g on $(0.(b-a)/2)$, let $V(g)$ be the union (over all $\delta\in(0.(b-a)/2))$ of all sectors, $a+\delta<\arg(z-h(\delta))<b-\delta$, where $h(\delta)=g(\delta)e^{i(a+b)/2}$. The set of all $V(g)$ (for all choices of g) is denoted $F(a, b)$, and is a filter base which converges to ∞. Each $V(g)$ is a simply-connected region (see [11; §93]), and we require the following simple fact which is proved in [2; p. 74]:

LEMMA 2.2. *Let V be an element of $F(a, b)$, and let $\varepsilon>0$ be arbitrary. Then there is a constant $R_0(\varepsilon)>0$ such that V contains the set, $a+\varepsilon\leqq\arg z\leqq b-\varepsilon$, $|z|\geqq R_0(\varepsilon)$.*

As in [1], we will say that a statement holds "except in finitely many directions in $F(a, b)$", if there exist finitely many points $r_1<r_2<\cdots<r_q$ in (a, b) such that the statement holds in each of $F(a, r_1), F(r_1, r_2), \cdots, F(r_q, b)$ separately.

(b) [11: §13]: *The relation of asymptotic equivalence.* If $f(z)$ is an analytic function on some element of $F(a, b)$, then $f(z)$ is called *admissible* in $F(a, b)$. If c is a complex number, then the statement $f\rightarrow c$ in $F(a, b)$ means (as is customary) that for any $\varepsilon>0$, there exists an element V of $F(a, b)$ such that $|f(z)-c|<\varepsilon$ for all $z\in V$. The statement $f\ll 1$ in $F(a, b)$, means that in addition to $f\rightarrow 0$, all the functions $\theta_j^k f\rightarrow 0$ in $F(a, b)$, where θ_j denotes the operator $\theta_j f=z(\operatorname{Log} z)\cdots(\operatorname{Log}_{j-1} z)f'(z)$, and where (for $k\geqq 0$), θ_j^k is the kth iterate of θ_j. The statements $f_1\ll f_2$ and $f_1\sim f_2$ in $F(a, b)$ mean respectively $f_1/f_2\ll 1$ and $f_1-f_2\ll f_2$. (This strong relation of asymptotic equivalence is designed to ensure that if M is a non-constant *logarithmic monomial of rank* $\leqq p$ (i.e. a function of the form,

$$M(z)=Kz^{\alpha_0}(\operatorname{Log} z)^{\alpha_1}\cdots(\operatorname{Log}_p z)^{\alpha_p}, \tag{2.2}$$

for real α_j, and complex $K\neq 0$), then $f\sim M$ implies $f'\sim M'$ in $F(a, b)$ (see [11: §28]). As usual, z^α and $\operatorname{Log} z$ will denote the principal branches of these functions on $|\arg z|<\pi$.) We will write $f_1\approx f_2$ to mean $f_1\sim cf_2$ for some nonzero constant c. An admissible function $f(z)$ in $F(a, b)$ is called *trivial* in $F(a, b)$ if $f\ll z^{-\alpha}$ in $F(a, b)$ for every $\alpha>0$. If $f\sim cz^{-1+d}$ in $F(a, b)$, where $c\neq 0$ and

$d>0$, then the *indicial function* of f is the function.

$$IF(f, \varphi)=\cos(d\varphi+\arg c) \quad \text{for} \quad a<\varphi<b. \tag{2.3}$$

If g is any admissible function in $F(a, b)$, we will denote by $\int g$, any primitive of g in an element of $F(a, b)$. We will require the following fact (see [2; p. 75]):

LEMMA 2.3. *Let $f\sim cz^{-1+d}$ in $F(a, b)$, where $c\neq 0$ and $d>0$. If (a_1, b_1) is any subinterval of (a, b) on which $IF(f, \phi)<0$ (respectively, $IF(f, \phi)>0$), then for all real α, $\exp\int f \ll z^\alpha$ (respectively, $\exp\int f\gg z^\alpha$) in $F(a_1, b_1)$.*

(c) The operator θ_1 defined by $\theta_1 f=zf'$, will be denoted simply θ. It is easy to prove by induction that for each $n=1, 2, \cdots$

$$f^{(n)}=z^{-n}\Big(\sum_{j=1}^{n} b_{jn}\theta^j f\Big). \tag{2.4}$$

(d) [11: §49]. A *logarithmic domain* of rank zero (briefly, an LD_0) over $F(a, b)$ is a complex vector space L of admissible functions in $F(a, b)$, which contains the constants, and such that any finite linear combination of elements of L, with coefficients which are logarithmic monomials of rank $\leqq p$ for some $p\geqq 0$, is either trivial in $F(a, b)$ or is $\sim$ to a logarithmic monomial of rank $\leqq p$ in $F(a, b)$. (The simplest examples of such sets L (where we can take (a, b) to be any open subinterval of $(-\pi, \pi)$) are the set of all polynomials, the set of all rational functions, and the set of all rational combinations of logarithmic monomials of rank $\leqq 0$. More extensive examples can be found in [11: §§ 128, 53].)

(e) [1; §3]. If $G(v)$ is a polynomial in v, whose coefficients belong to an LD_0 over $F(a, b)$, then a logarithmic monomial M is called a *critical monomial* of G if there exists an admissible function $h\sim M$ in $F(a, b)$ such that $G(h)$ is not $\sim G(M)$ in $F(a, b)$. The set of critical monomials of G can be produced by using the algorithm in [10; p. 236] which is based on a Newton polygon construction. This algorithm shows that the critical monomials are of rank $\leqq 0$. (In the special case where the coefficients of $G(v)$ are rational functions, the critical monomials are precisely the functions cz^α which form the first term of one of the expansions around $z=\infty$ of the algebraic function defined by $G(v)=0$. This fact follows from [10; §26]).) Associated with a critical monomial M of $G(v)$ is a positive integer which is called its *multiplicity*. The multiplicity is defined as the smallest positive integer j such that M is not a critical monomial of $\partial^j G/\partial v^j$. (See [10; p. 231]). If the multiplicity is equal to 1, the critical monomial M is called *simple*.

(D). We will need the following concepts from [1] and [9]:

(a). Let n be a positive integer, and let $\{R_0(z), \cdots, R_n(z)\}$ be contained in an LD_0 over $F(a, b)$ for some (a, b) with $-\pi\leqq a<b\leqq\pi$, and assume that $R_n(z)$

is non-trivial (see § 2(C) in $F(a, b)$. Using (2.4), rewrite the equation,

(2.5) $$R_n(z)w^{(n)}+R_{n-1}(z)w^{(n-1)}+ \cdots R_0(z)w=0,$$

in the form,

(2.6) $$\Omega(w)=\sum_{j=0}^{n} B_j(z)\theta^j w=0, \qquad \text{where} \quad \theta^0 w=w.$$

By dividing equation (2.6) through by the highest power of z which occurs in the expansions of all the functions $B_j(z)$ for all $j=0, \cdots, n$, we may assume that for each j, we have either $B_j \ll 1$ or $B_j \approx 1$ in $F(a, b)$, and there exists an integer $p \geqq 0$ such that $B_j \ll 1$ for $j > p$, while B_p is $\sim$ to a nonzero constant (denoted $B_p(\infty)$). The integer p is called the *critical degree* of the equation (2.5). The equation,

(2.7) $$F^*(\alpha)=\sum_{j=0}^{n} B_j(\infty)\alpha^j=0,$$

is called the *critical equation* of (2.5). Clearly, $F^*(\alpha)$ is a polynomial in α, of degree p, having constant coefficients.

When (2.5) is written in the form (2.6) we form the algebraic polynomial in v,

(2.8) $$H(v)=\sum_{j=0}^{n} z^j B_j(z)v^j,$$

which we will call the *full factorization polynomial* for (2.5).

Let W belong to an LD_0 over $F(a, b)$, and assume that $W \gg z^{-1}$ in $F(a, b)$. Set $h=\exp\int W$, and let $L(v)$ be the operator defined by $L(v)=\Omega(hv)/h$. Let $H(u)$ and $K(u)$ denote respectively, the full factorization polynomials for $\Omega(w)$ and $L(v)$. In [9; § 10], the following concept is introduced: W is said to have *transform type* (m, q) with respect to H (briefly, $\mathrm{trt}(W, H)=(m, q)$) if $L(v)=0$ has critical degree m, and if q is the minimum multiplicity of all critical monomials M of $K(u)$ which satisfy $z^{-1} \ll M \ll W$ in $F(a, b)$. (If there are no such M, then we set $q=0$.)

3. We will require the following lemma on asymptotic integration.

LEMMA 3.1. *Let k be a positive integer, and let $Q(z)$ be a rational function whose expansion around ∞ is $Q(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots$, where $a_n \neq 0$ and $n/k > -1$. Let $c_1, \cdots, c_k$ be the distinct roots of the equation $c^k+a_n=0$. Let (a, b) be any open subinterval of $(-\pi, \pi)$, and let $G(z)$ be an analytic function in an element of $F(a, b)$ which satisfies $G \ll z^\beta$ over $F(a, b)$, for every real number β. Then, for each integer j in $\{1, \cdots, k\}$, there exist a complex number α_j and a function E_j which is analytic in an element of $F(-\pi, \pi)$, which have the following two properties:*

(a) $E_j \sim c_j z^{n/k}$ over $F(-\pi, \pi)$;

(b) Except in finitely many directions in $F(a, b)$, the equation, $w^{(k)}+(Q+G)w=0$, possesses a fundamental set of solutions $\varphi_1, \cdots, \varphi_k$, where each φ_j is of the form $\varphi_j=h_j \exp\int E_j$ for some analytic function h_j satisfying $h_j \sim z^{\alpha_j}$.

Remark. (1) The proof will show that each function E_j is actually a single-valued branch of an algebraic function (which depends on c_j).

(2) As will be seen from the proof, the lemma will hold if the hypothesis that Q is rational is replaced by the assumption that in some element of $F(-\pi, \pi)$, the function Q is a single-valued analytic branch of an algebraic function, and satisfies $Q \sim a_n z^n$ in $F(-\pi, \pi)$ for some complex $a_n \neq 0$, and some rational number $n > -k$.

4. Proof of Lemma 3.1.

We will apply results from [1] and [9] to the operator $L(w)=w^{(k)}+Qw$. For this purpose, we rewrite the operator L in terms of the operator θ defined by $\theta w=zw'$ (and $\theta^2 w=\theta(\theta w)$ etc.). It is easy to see that we obtain

$$L(w)=z^{-k}(\theta(\theta-1)\cdots(\theta-k+1)w)+Qw. \tag{4.1}$$

Following § 2 (D), we associate with L the full factorization polynomial $H(v)$ defined by,

$$H(v)=z^{-k}(zv(zv-1)\cdots(zv-k+1))+Q. \tag{4.2}$$

In view of the expansion of Q around ∞, it is clear that each function $N_j=c_j z^{n/k}$ is the first term of one of the k expansions around ∞ of the algebraic function defined by $H(v)=0$. (In the terminology of § 2 (C), the functions N_j are the critical monomials of $H(v)$ and each is simple.) Using the terminology of § 2 (D), it follows from [9; Lemmas 10.3, 10.5] that for each j, there is an analytic function E_j satisfying Part (*a*) of the lemma and having the property that its transform type with respect to $H(v)$ is (1, 0). This means that the operator $L_j(u)$ defined by

$$L_j(u)=\left(\exp\left(-\int E_j\right)\right)L\left(\left(\exp\int E_j\right)u\right) \tag{4.3}$$

has critical degree 1 in the terminology of § 2 (D), and therefore from [1; p. 147, Formulas (*a*), (*b*)], there is a real number t such that when the operator $z^{-t}L_j(u)$ is written in terms of θ, say

$$z^{-t}L_j(u)=\sum_{m=0}^{k} B_m(z)\theta^m u, \tag{4.4}$$

the following asymptotic relations hold over $F(-\pi, \pi)$: For each $m>1$, we have $B_m \ll 1$; there is aneonzero constant K_1 such that $B_1 \sim K_1$; either $B_0 \ll 1$ or

$B_0 \sim K_0$ for some nonzero constant K_0.

We now define $M(w)=w^{(k)}+(Q+G)w$, and as above define,

$$M_j(u)=\left(\exp\left(-\int E_j\right)\right)M\left(\left(\exp\int E_j\right)u\right). \tag{4.5}$$

Since $M(w)=L(w)+Gw$, we have,

$$z^{-t}M_j(u)=z^{-t}L_j(u)+z^{-t}Gu=\sum_{m=0}^{k} D_m \theta^m u\,, \tag{4.6}$$

where $D_m=B_m$ for $m>0$, and $D_0=B_0+z^{-t}G$. In view of the hypothesis that $G \ll z^\beta$ over $F(a, b)$ for every real β, it follows from [11; Lemma 53(c)] that with $F(a, b)$ as the neighborhood system, the operator $z^{-t}M_j(u)$ is of the type treated in [1] (i.e. the coefficients belong to a logarithmic domain of rank zero over $F(a, b)$). In addition, the asymptotic relations for the B_m listed above show that the critical degree of $z^{-t}M_j(u)$ is 1. Thus if α_j is the root of the critical equation (see [1; § 3(e)]) for $z^{-t}M_j(u)$, then it follows from [1; Lemma 5 and Theorem 1] that except in finitely many directions in $F(a, b)$, the equation $z^{-t}M_j(u)=0$ possesses a solution $h_j \sim z^{\alpha_j}$. Thus, such functions $\varphi_j=h_j \exp\int E_j$ for $j=1, \cdots, k$, exist except in finitely many directions in $F(a, b)$, and are solutions of $w^{(k)}+(Q+G)w=0$. The fact that they are linearly independent follows immediately by applying Lemma 2.3 to $f=E_i-E_j$ for each pair (i, j) with $i \neq j$.

5. Proof of Theorem 1.1.

We observe first that we can assume that θ_0 belongs to $(-\pi, \pi)$, since if $\theta_0=\pi$, then the change of variable $\rho=-z$ in (1.2) transforms (1.2) into an equation in ρ for which the hypothesis is satisfied for $\theta_0=0$.

We assume contrary to the conclusion that (1.2) possesses a solution $f \not\equiv 0$ satisfying $\lambda(f)<\infty$. Thus we may write $f=Ge^h$, where G and h are entire functions with G of finite order. Thus, from (1.2) we obtain,

$$(h')^k+\Phi_{k-1}(h')+Re^P+Q=0, \tag{5.1}$$

where $\Phi_{k-1}(h')$ is a differential polynomial of total degree at most $k-1$ in $h', h'', \cdots$, whose coefficients are polynomials in $G'/G, G''/G, \cdots, G^{(k)}/G$ having constant coefficients. Applying Lemma 2.1 to (5.1), we find that h' is of finite order.

Now set

$$S=h''-((RP'+R')/kR)h'. \tag{5.2}$$

We now differentiate (5.1) and subtract this new expression from the expression obtained from (5.1) by multiplying it by $(RP'+R')/R$. The terms in e^P cancel and we obtain

(5.3) $$(h')^{k-1}(-kS)=\Psi_{k-1}(h'),$$

where $\Psi_{k-1}(h')$ is a differential polynomial of total degree at most $k-1$ whose coefficients are all meromorphic functions ϕ satisfying $m(r, \phi)=0(\log r)$ as $r\to\infty$. Since h' is of finite order, it follows from Lemma 2.1 that $m(r, S)=0(\log r)$ as $r\to\infty$. However, since R is a polynomial, we see from (5.2) that S can have only finitely many poles. Thus S is a rational function.

We now consider the equation,

(5.4) $$w'-((RP'+R')/kR)w=S.$$

If $S\equiv 0$, set $w_0(z)\equiv 0$. If $S\not\equiv 0$, it follows from [11; Theorem III, Part (b), p. 72] that there exist $\varepsilon_1>0$ and an analytic solution $w_0(z)$ of (5.4) in an element of $F(\theta_0-\varepsilon_1, \theta_0+\varepsilon_1)$, such that,

(5.5) $$w_0\sim -kS/P' \quad \text{over} \quad F(\theta_0-\varepsilon_1, \theta_0+\varepsilon_1).$$

(Thus, in both of the cases, $S\equiv 0$ and $S\not\equiv 0$, the function w_0 solves (5.4).)

Now, let $D(z)$ denote a single-valued analytic branch of $R^{1/k}$ on an element of $F(-\pi, \pi)$. (Thus clearly, if $R(z)=t_pz^p+\cdots$, then $D\sim t_p^{1/k}z^{p/k}$ over $F(-\pi, \pi)$, for some choice of $t_p^{1/k}$.). In view of (5.2) and (5.4), it easily follows that for some constant K_1, we have

(5.6) $$h'=w_0+K_1De^{P/k} \quad \text{in some element of } F(\theta_0-\varepsilon_1, \theta_0+\varepsilon_1).$$

Now the terms of total degree $k-1$ in the differential polynomial $\Phi_{k-1}(h')$ in (5.1) are easily seen to be,

(5.7) $$k(h')^{k-1}(G'/G)+(k(k-1)/2)(h')^{k-2}h''.$$

Hence, when (5.6) is substituted into (5.1), we obtain an equation,

(5.8) $$\sum_{m=0}^{k}\Omega_m(z)e^{mP(z)/k}\equiv 0,$$

where,

(5.9) $$\Omega_k=K_1^kD^k+R=R(K_1^k+1),$$

(5.10) $$\Omega_{k-1}=kK_1^{k-1}D^{k-1}V,$$

where,

(5.11) $$V=w_0+(G'/G)+((k-1)/2)((P'/k)+(D'/D)),$$

and where for each m, the function $\Omega_m(z)$ is a polynomial (with constant coefficients) in the variables $G'/G, \cdots, G^{(k)}/G$, $w_0, w_0', \cdots, D, D', \cdots$, and $Q, P', P'', \cdots$. Standard estimates on the logarithmic derivative of an entire function of finite order, together with the known asymptotic behavior of w_0

and D developed earlier (and the remarks in Parts (A), (B), and (C) in § 2), yield a real number $N>0$ with the following property: For all θ in $(\theta_0-\varepsilon_1, \theta_0+\varepsilon_1)$ outside a set of zero measure, we have for all $m=0, 1, \cdots, k$,

$$|\Omega_m(re^{i\theta})|=0(r^N) \quad \text{as } r\to+\infty. \tag{5.12}$$

Noting that the left side of (1.3) is simply the indicial function $\mathrm{IF}(P', \varphi)$ evaluated at $\varphi=\theta_0$, (see (2.3)) we can choose $\varepsilon_2>0$ so small that $\varepsilon_2<\varepsilon_1$ and such that the only zero of $\mathrm{IF}(P', \varphi)$ on $(\theta_0-\varepsilon_2, \theta_0+\varepsilon_2)$ is at $\varphi=\theta_0$. Since $\mathrm{IF}(P', \varphi)$ must change sign at $\varphi=\theta_0$, this function has constant signs on $(\theta_0-\varepsilon_2, \theta_0)$ and $(\theta_0, \theta_0+\varepsilon_2)$, and these signs are different. Without loss of generality we can assume that

$$\mathrm{IF}(P', \varphi)>0 \quad \text{on } (\theta_0-\varepsilon_2, \theta_0), \tag{5.13}$$

and

$$\mathrm{IF}(P', \varphi)<0 \quad \text{on } (\theta_0, \theta_0+\varepsilon_2), \tag{5.14}$$

since the argument will be symmetric if we interchange the two intervals.

Now from (5.13) and Lemma 2.3, we see that for $m=1, \cdots, k$, we have $e^{-mP/k}\ll z^\beta$ for all real β, over $F(\theta_0-\varepsilon_2, \theta_0)$. Thus, if we divide the equation (5.8) by e^P, and evaluate the resulting relation on any ray $\arg z=\theta$, where θ is an element of $(\theta_0-\varepsilon_2, \theta_0)$ for which (5.12) is valid, we see that $\Omega_k(re^{i\theta})=o(r^\beta)$ for all real β as $r\to+\infty$. Since R in (5.9) is a polynomial which is not identically zero, we must have $K_1^k=-1$. Thus, $\Omega_k(z)\equiv 0$. We now divide the relation (5.8) by $e^{(k-1)P/k}$, and as above we find that for all θ in $(\theta_0-\varepsilon_2, \theta_0)$, with the possible exception of a set H of zero measure where (5.12) fails to hold, we have $\Omega_{k-1}(re^{i\theta})=o(r^\beta)$ for all real β as $r\to+\infty$. Since K_1 is a nonzero constant, and D in (5.10) satisfies $D\sim t_p^{1/k}z^{p/k}$ over $F(-\pi, \pi)$ for certain constants p and t_p, it follows from (5.10) that $V(z)$ has the property that for all θ in $(\theta_0-\varepsilon_2, \theta)_0-H$, we have

$$V(re^{i\theta})=o(r^\beta) \quad \text{for all real } \beta \text{ as } r\to+\infty. \tag{5.15}$$

In view of the asymptotic behavior of D, clearly there exists an analytic branch D_1 of $D^{(k-1)/2}$ in an element T of $F(\theta_0-\varepsilon_2, \theta_0+\varepsilon_2)$, and we can assume that $w_0(z)$ is also analytic in T. We now define a function $W(z)$ on T by,

$$W=GD_1e^{(k-1)P/2k}e^{\int w_0}, \tag{5.16}$$

where $\int w_0$ represents a fixed primitive of w_0 on T. In view of the asymptotic behavior of D and w_0 (see (5.5)) and the fact that G is of finite order, it easily follows (using Lemma 2.2) that if we set $\varepsilon_3=\varepsilon_2/2$, then there is a constant $r_0>0$ such that W is analytic and of finite order in the region,

$$\theta_0-\varepsilon_3<\arg z<\theta_0+\varepsilon_3, \quad |z|>r_0. \tag{5.17}$$

It is clear that $W'/W=V$ on T, and hence from (5.15) it easily follows that for each θ in $(\theta_0-\varepsilon_3, \theta_0)-H$, there is a nonzero constant J_θ such that $W(re^{i\theta})\to J_\theta$ as $r\to+\infty$. It follows from Phragmen-Lindelof principles [12; § 5.61, 5.64, pp. 176-180] that all J_θ are equal to a single number J, and we have,

$$(5.18)\qquad W(re^{i\theta})\to J\neq 0 \quad \text{as } r\to+\infty \quad \text{for } \theta_0-\varepsilon_3<\theta<\theta_0.$$

Since $f=Ge^h$, it follows from (5.6) and (5.16) that we can write on the domain T,

$$(5.19)\qquad W=\psi f D_1 e^{(k-1)P/2k}, \quad \text{where}$$

$$(5.20)\qquad \psi=K_2\exp\left(-K_1\int De^{P/k}\right),$$

for some constant $K_2\neq 0$ and some fixed choice of the primitive in (5.20). In view of (5.14) and Lemma 2.3, we see that $-K_1De^{P/k}\ll z^\beta$ for all real β, over $F(\theta_0, \theta_0+\varepsilon_2)$, and it immediately follows from [2; § 2, Lemma B, Part (b)] that for some constant $K_3\neq 0$, we have

$$(5.21)\qquad \psi\to K_3 \quad \text{over } F(\theta_0, \theta_0+\varepsilon_2).$$

Again in view of (5.14) and Lemma 2.3, we see that in $F(\theta_0, \theta_0+\varepsilon_2)$, the function Re^P satisfies the hypothesis for G in Lemma 3.1. Hence by Lemma 3.1, there exists ε_4 in $(0, \varepsilon_3)$ such that in some element of $F(\theta_0, \theta_0+\varepsilon_4)$, the equation (1.2) possesses a fundamental set of solutions $\varphi_j=h_j\exp\int E_j$ for $j=1, \cdots, k$, where

$$(5.22)\qquad E_j\sim c_j z^{n/k} \quad \text{over } F(-\pi, \pi), \quad \text{and}$$

$$(5.23)\qquad h_j\sim z^{\alpha_j} \quad \text{over } F(\theta_0, \theta_0+\varepsilon_4) \quad \text{for some constant } \alpha_j.$$

Hence, we can write f as a linear combination of $\varphi_1, \cdots, \varphi_k$, say $f=\sum \beta_j\varphi_j$. Since some $\beta_j\neq 0$ (since $f\not\equiv 0$), the set I consisting of all j in $\{1, \cdots, k\}$ for which $\beta_j\neq 0$, is not empty. Since the constants c_j in (5.22) are distinct, the union of the sets of zeros on $(-\pi, \pi)$ of the functions $\mathrm{IF}(E_i-E_j, \theta)$ for all pairs (i, j) in $I\times I$ with $i\neq j$, is a finite set. Letting $\varepsilon_5>0$ be so small that $(\theta_0, \theta_0+\varepsilon_5)$ contains none of these zeros, it follows from Lemma 2.3 (applied to $f=E_i-E_j$) that for each pair (i, j) in $I\times I$ with $i\neq j$, either $\varphi_i\ll\varphi_j$ or $\varphi_j\ll\varphi_i$ in $F(\theta_0, \theta_0+\varepsilon_5)$. Thus, there exists an element m in I such that $\varphi_j\ll\varphi_m$ in $F(\theta_0, \theta_0+\varepsilon_5)$ for all j in I with $j\neq m$. Hence we have,

$$(5.24)\qquad f=\beta_m\varphi_m(1+E) \quad \text{where } E\to 0 \quad \text{over } F(\theta_0, \theta_0+\varepsilon_5).$$

Now for $j=m$, the left side of (1.4) is simply $\mathrm{IF}(E_m, \theta)$ evaluated at $\theta=\theta_0$, and is nonzero by hypothesis. Hence we can assume that ε_5 chosen above is so small that

(5.25) $\quad \mathrm{IF}(E_m, \theta)$ is nowhere zero on $(\theta_0-\varepsilon_5, \theta_0+\varepsilon_5)$.

Now, in view of (5.19), (5.21), (5.24), and the definition of D_1, we can write,

$$W=\psi_1 h_m(1+E)\exp\int U \quad \text{on } F(\theta_0, \theta_0+\varepsilon_5), \tag{5.26}$$

where

$$U=E_m+((k-1)/2k)P'+((k-1)/2)(D'/D), \tag{5.27}$$

and where for some constant $K_4\neq 0$,

$$\psi_1\to K_4\neq 0 \quad \text{over } F(\theta_0, \theta_0+\varepsilon_5). \tag{5.28}$$

Since D is a branch of an algebraic function and satisfies $D\sim t_p^{1/k}z^{p/k}$ over $F(-\pi, \pi)$, it follows from § 2 (C) that either $D'/D\ll z^{-1}$ or $D'/D\sim(p/k)z^{-1}$ over $F(-\pi, \pi)$. In addition, by hypothesis, we have $n/k>r-1$ where r is the degree of P, and hence from (5.22), we see that

$$U\sim E_m \quad \text{over } F(-\pi, \pi). \tag{5.29}$$

We now distinguish the two cases given by (5.25).

Case I. $\mathrm{IF}(E_m, \theta)<0$ on $(\theta_0-\varepsilon_5, \theta_0+\varepsilon_5)$. In this case, it follows from (5.29) and Lemma 2.3 that $\exp\int U\ll z^{\beta}$ for all real β, over $F(\theta_0-\varepsilon_5, \theta_0+\varepsilon_5)$. Hence, in view of the representation (5.26) (and the asymptotic relations (5.23), (5.24) and (5.28)), we see that $W\to 0$ over $F(\theta_0, \theta_0+\varepsilon_5)$. This fact, together with (5.18) provides a contradiction of the Phragmen-Lindelof principle [12; § 5.64, p. 179].

Case II. $\mathrm{IF}(E_m, \theta)>0$ on $(\theta_0-\varepsilon_5, \theta_0+\varepsilon_5)$. In this case, we define,

$$W_0=z^{-\alpha_m}W\exp\left(-\int U\right). \tag{5.30}$$

Since W is analytic on the region (5.17), and since $z^{-\alpha_m}$ and $\exp(-\int U)$ are analytic on an element of $F(-\pi, \pi)$, clearly W_0 is analytic on the region (5.17) (possibly with a larger r_0). Since W is of finite order on the region (5.17), the same is clearly true of W_0 (by (5.22) and (5.29)). From Lemma 2.3, we have that $\exp(-\int U)\ll z^{\beta}$ for all real β over $F(\theta_0-\varepsilon_5, \theta_0+\varepsilon_5)$ and so from (5.18), we see that

$$W_0(re^{i\theta})\to 0 \quad \text{as } r\to+\infty \quad \text{for } \theta_0-\varepsilon_5<\theta<\theta_0. \tag{5.31}$$

However, by (5.26) we have,

$$W_0=\psi_1(h_m/z^{\alpha_m})(1+E) \quad \text{on } F(\theta_0, \theta_0+\varepsilon_5), \tag{5.32}$$

and in view of (5.23), (5.24), (5.28) and Lemma 2.2, we have $W_0(re^{i\theta})\to K_4\neq 0$ as $r\to+\infty$ for $\theta_0<\theta<\theta_0+\varepsilon_5$. This fact, together with (5.31), again provides a contradiction to the same Phragmen-Lindelof principle as in Case I.

This contradiction establishes the theorem.

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