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RECURRENCE AND TRANSIENCE OF GAUSSIAN DIFFUSION PROCESSES

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, P_x, X_t)$ be a standard Markov process taking values in a locally compact separable metric space (S, ρ), that is, for each $x \in S$, X_t is an S-valued, right continuous and quasi-left continuous strong Markov process starting at x defined on a probability space with filtrations $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$ (c.f. [1]). Let $\mathcal{B}(S)$ be the topological Borel field of S. For an $E \in \mathcal{B}(S)$, we denote by σ_E the hitting time of E, *i.e.*,

$$\sigma_E = \begin{cases} \inf\{t>0 : X_t \in E\}. & \text{if } \{t>0 : X_t \in E\} \neq \phi \\ \infty, & \text{if } \{t>0 : X_t \in E\} = \phi . \end{cases}$$

The Markov process $(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, P_x, X_t)$ (henceforth we will write (X_t, P_x)) is called irreducible if

(1.1) $\quad P_x(\sigma_U < \infty) > 0$ for every $x \in S$ and every open subset $U \neq \phi$ of S.

For the Markov process (X_t, P_x), an $a \in S$ is called a recurrent point if

$$P_a(\underline{\lim}_{t\to\infty} \rho(X_t, a) = 0) = 1. \tag{1.2}$$

If every $a \in S$ is a recurrent point for (X_t, P_x), we say that the Markov process (X_t, P_x) is recurrent. On the other hand, an $a \in S$ is a transient point for (X_t, P_x) if

$$P_a(\underline{\lim}_{t\to\infty} \rho(X_t, a) > 0) = 1. \tag{1.3}$$

Furthermore, if every $a \in S$ is a transient point for (X_t, P_x), we say that the Markov process (X_t, P_x) is transient.

For the Markov process (X_t, P_x), one of most important problems is to determine whether it is recurrent or transient, and this problem has been extensively studied by many probabilists under various settings. ([1], [2], [3], [5], [8], [9], $\cdots$). Nevertheless there are a few classes of Markov processes for which complete criteria of recurrence are known.

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The purpose of the present paper is firstly to discuss recurrence criteria in terms of mean sojourn times for general Markov processes, secondly to give an explicit necessary and sufficient condition for irreducibility of Gaussian diffusion processes, and thirdly to determine recurrence and transience completely for irreducible Gaussian diffusion processes.

Let $\boldsymbol{R}^d$ ($d\geq 1$) be the d-dimensional Euclidean space. For $x\in\boldsymbol{R}^d$ and $y\in\boldsymbol{R}^d$, let $\langle x, y\rangle$ be the inner product and $\|x\|=\langle x, x\rangle^{1/2}$ the norm. We denote by $M(\boldsymbol{R}^d)$ the totality of real $d\times d$ matrices and by $M_+(\boldsymbol{R}^d)$ the totality of real, symmetric and non-negative definite $d\times d$ matrices.

For each $B\in M_+(\boldsymbol{R}^d)$ there corresponds a d-dimensional Brownian motion W_t with the diffusion matrix B, which is defined on a complete probability space with filtrations $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, namely, W_t is a continuous Gaussian process on $\boldsymbol{R}^d$ with the mean vector 0 and the covariance matrix

$$E(W_t^i W_s^j)=b_{ij}\min\{t, s\} \qquad (1\leq i, j\leq d) \quad \text{for } B=(b_{ij}). \tag{1.4}$$

For arbitrarily given $A\in M(\boldsymbol{R}^d)$ and $B\in M_+(\boldsymbol{R}^d)$, let us consider the following equation

$$X_t=x+\int_0^t AX_s ds+W_t, \tag{1.5}$$

where $x\in\boldsymbol{R}^d$ and W_t is a d-dimensional Brownian motion with the diffusion matrix B defined on a complete probability space with filtrations $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

As easily seen, for every $x\in\boldsymbol{R}^d$, the equation (1.5) has a unique solution, which defines a diffusion process (a continuous process having the strong Markov property) $(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, P_x, X_t)$ (shortly we write (X_t, P_x)) taking values in $\boldsymbol{R}^d$. Such Markov processes are called Gaussian diffusion processes, following [6], since the transition probabilities are Gaussian distributions as seen in Lemma 3.1. From now on, we say the diffusion process (X_t, P_x) governed by the equation (1.5) *the Gaussian diffusion process associated with* $A\in M(\boldsymbol{R}^d)$ *and* $B\in M_+(\boldsymbol{R}^d)$.

This paper is organized as follows. In section 2, we give sufficient conditions for a point to be recurrent or transient in terms of mean sojourn time at neighbourhoods of the point for general Markov processes. The conditions turn to be necessary and sufficient if the process satisfies the strong Feller property. In section 3, we obtain a necessary and sufficient condition for a Gaussian diffusion process to be irreducible. It will also be shown that non-irreducible Gaussian diffusion processes can be reduced to irreducible ones. In section 4, we completely determine recurrence and transience of all irreducible Gaussian diffusion processes.

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the present paper.

2. Criteria of recurrence and transience via mean sojourn times.

Let (X_t, P_x) be a standard Markov process taking values in a locally compact separable metric space (S, ρ). Let us denote by $P_t(x, dy)$ and $R_\lambda(x, dy)$ the transition probability and the resolvent kernel of (X_t, P_x) respectively, so that

$$R_\lambda(x, B)=\int_0^\infty e^{-\lambda t}P_t(x, B)dt$$

$$=E_x\left(\int_0^\infty e^{-\lambda t}I_B(X_t)dt\right),$$

where $\lambda\geq 0$, $x\in S$, $B\in\mathcal{B}(S)$, E_x denotes the expectation with respect to P_x, and

$$I_B(x)=\begin{cases}1, & \text{if } x\in B,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, $R_0(x, B)$ is called the mean sojourn time in B.

Let $B(S)$ be the totality of bounded measurable functions defined on S, and let $C_b(S)$ be the totality of bounded continuous functions defined on S. For an $f\in B(S)$, set

$$P_tf(x)=\int_S P_t(x, dy)f(y).$$

If $P_t(C_b(S))\subset C_b(S)$ holds for every $t>0$, we say that the Markov process (X_t, P_x) satisfies the Feller property. Furthermore, if $P_t(B(S))\subset C_b(S)$ holds for every $t>0$, we say that the Markov process (X_t, P_x) satisfies the strong Feller property,

We here give criteria of recurrence and transience of the Markov process (X_t, P_x) in terms of mean sojourn times.

THEOREM 2.1. *Suppose that the Markov process (X_t, P_x) satisfies the Feller property. Let $a\in S$ be fixed. If for some open set U containing a, $R_0(a, U)<\infty$, then a is a transient point for (X_t, P_x).*

Proof. Clearly $R_0(a, U)>0$. We first claim that there exists an open set V containing a such that

$$\inf_{b\in\bar{V}} R_0(b, U)>0, \tag{2.1}$$

To see this, choose an $f\leq 0$ from $C_b(S)$ satisfying $f(a)>0$ and $f=0$ outside of U. Then $R_0(a, U)>0$ implies that for some $t>0$

$$\int_0^t P_sf(a)ds>0.$$

Using the Feller property we have an open subset V containing a such that $\bar{V}\subset U$ and

$$\inf_{x\in\bar{V}}\int_0^t P_s f(x)ds>0,$$

which implies (2.1). For each $r>0$, let

$$\sigma_V^r=\sigma_V\circ\Theta_r=\inf\{t>r\,;\,X_t\in V\}.$$

By the strong Markov property we have

$$\begin{aligned}\int_r^\infty dt\,P_t(a,U)&\geq E_a\Big(\int_{\sigma_V^r}^\infty dt\,I_U(X_t):\sigma_V^r<\infty\Big)\\&=E_a(R_0(X_{\sigma_V^r},U):\sigma_V^r<\infty)\\&\geq\inf_{b\in\bar{V}}R_0(b,U)P_a(\sigma_V^r<\infty).\end{aligned}$$

Hence letting $r\to\infty$ and using (2.1) and the assumption we get

$$\lim_{r\to\infty}P_a(\sigma_V^r<\infty)=P_a(\sigma_V^r<\infty\text{ for every }r>0)=0,$$

which yields (1.3). Therefore, a is a transient point. Q.E.D.

THEOREM 2.2. *Let $a\in S$ be fixed. Suppose that the Markov process (X_t,P_x) satisfies the following property: for every open subset U containing a,*

$$\lim_{\varepsilon\to 0}\sup_{x\in B_\varepsilon(a)}P_x(\sigma^r_{B_\varepsilon(a)}<\infty)\leq P_a(\sigma_U^r<\infty)\qquad\textit{for every } r>0, \tag{2.2}$$

where $B_\varepsilon(a)$ denotes the closed ball centered at a with the radius $\varepsilon>0$. If $R_0(a,U)=\infty$ for every open subset U containing a, then a is a recurrent point for (X_t,P_x).

Proof. For any fixed $r>0$ and $\varepsilon>0$, set $B=B_\varepsilon(a)$ and define a sequence of stopping times τ_n by

$$\tau_0=0,$$
$$\tau_1=\sigma_B^r,\ \cdots,$$
$$\tau_n=\begin{cases}\tau_1\circ\Theta_{\tau_{n-1}}+\tau_{n-1}, & \text{if } \tau_{n-1}<\infty\\ \infty, & \text{if } \tau_{n-1}=\infty.\end{cases}$$
$$\cdots.$$

By using the strong Markov property repeatedly, we have

$$\sup_{x\in B}P_x(\tau_n<\infty)\leq(\sup_{x\in B}P_x(\tau_1<\infty))^n\qquad(n=1,2,\cdots), \tag{2.3}$$

so that

$$
\begin{aligned}
R_0(a, B) &= \sum_{n=0}^{\infty} E_a\left(\int_{\tau_n}^{\tau_{n+1}} I_B(X_t)dt : \tau_n<\infty\right) \\
&= \sum_{n=0}^{\infty} E_a\left(E_{X_{\tau_n}}\left(\int_0^{\tau_1} I_B(X_t)dt\right) : \tau_n<\infty\right) \\
&\leq r\sum_{n=0}^{\infty} P_a(\tau_n<\infty) \\
&\leq r\sum_{n=0}^{\infty} (\sup_{x\in B} P_x(\tau_1<\infty))^n. \qquad \text{(by (2.3))}
\end{aligned}
$$

Since $R_0(a, B)=\infty$ by the assumption, we have

$$
\sup_{x\in B} P_x(\sigma_B^r<\infty)=1. \tag{2.4}
$$

Noting that (2.4) holds for $B=B_\varepsilon(a)$ with an arbitrary small $\varepsilon>0$, by the assumption (2.2) we obtain that

$$
P_a(\sigma_U^r<\infty)=1
$$

holds for every $r>0$ and every open subset U containing a, which yields (1.1). Therefore a is a recurrent point. Q. E. D.

We here discuss about the condition (2.2).

Remark 1. Suppose that the Markov process (X_t, P_x) satisfies the strong Feller property. Then the condition (2.2) is fullfilled.

Proof. If $B\subset U$,

$$
P_x(\sigma_B^r<\infty)\leq P_x(\sigma_U^r<\infty). \tag{2.5}
$$

By the strong Feller property,

$$
P_x(\sigma_U^r<\infty)=P_r\phi(x)
$$

is continuous in $x\in S$, where $\phi(x)=P_x(\sigma_U^0<\infty)$. Hence (2.2) follows immediately.

Remark 2. Suppose that (X_t, P_x) is a Lévy process (a process with independent increments) on $\boldsymbol{R}^d$. Then the condition (2.2) is fullfilled.

Proof. By the translation invariance of the Lévy process

$$
\begin{aligned}
\sup_{x\in B_\varepsilon(a)} P_x(\sigma_{B_\varepsilon(a)}^r<\infty) &= \sup_{x\in B_\varepsilon(a)} P_0(\sigma_{B_\varepsilon(a)-x}^r<\infty) \\
&\leq P_0(\sigma_{B_{2\varepsilon}(0)}^r<\infty) \\
&\leq P_a(\sigma_U^r<\infty), \qquad \text{if } B_{2\varepsilon}(a)\subset U.
\end{aligned}
$$

Thus we obtain (2.2).

Remark 3. It might be expected that even though the condition (2.2) fails, Theorem 2.2 holds under the assumption of the Feller property of (X_t, P_x). However, it is not true. We indeed have the following counter example, which is due to K. Uchiyama. Let

$$S_0=\{(m, n): m\in \boldsymbol{Z}_+, n\in \boldsymbol{Z}_+ \text{ and } m\geq n\}, \qquad S_1=\{(\infty, n): n\in \boldsymbol{Z}_+\},$$

and set

$$S=S_0\cup S_1,$$

where $\boldsymbol{Z}_+$ stands for the totality of nonnegative integers. We equip S_0 with the discrete topology, and each element (∞, n) of S_1 is regarded as a limit point of the sequence $\{(m, n): m\in\boldsymbol{Z}_+\}$ as $m\to\infty$, so that if U is an open set containing (∞, n), U contains $\{(m, n): m\geq m_0 \text{ or } m=\infty\}$ for some $m_0\geq n$. Clearly S is a locally compact separable metrizable space with the topology.

Now we choose a $1/2<p<1$, and a decreasing sequence $0<q_m<1-p$ vanishing as $m\to\infty$. Let us define a generator A of a Markov process (X_t, P_x) on S as follows. For each $f\in C_b(S)$

$$\begin{aligned}
Af(m, n)&=p(f(m, n+1)-f(m, n))+(1-p)(f(m, n-1)-f(m, n)) \\
&\qquad\qquad\qquad\qquad \text{if } \ 1\leq n<m \text{ or } m=\infty, \\
Af(m, m)&=f(m-1, m-1)-f(m, m) \qquad \text{if } \ m\geq 1, \\
Af(m, 0)&=(p-q_m)(f(m, 1)-f(m, 0))+q_m(f(m+1, 0)-f(m, 0)) \\
&\qquad +(1-p)(f(0, 0)-f(m, 0)) \qquad \text{if } \ m\geq 1, \\
Af(\infty, 0)&=p(f(\infty, 1)-f(\infty, 0))+(1-p)(f(0, 0)-f(\infty, 0)), \\
Af(0, 0)&=f(1, 0)-f(0, 0).
\end{aligned}$$

Then it is easy to see that A is a bounded operator from $C_b(S)$ into itself.

It is obvious that A generates a Markov process (X_t, P_x) on the state space S that satisfies the Feller property. Moreover, the Markov process is irreducible in the sense of (1.1).

On the other hand, if we equip the state space S with the discrete topology, then the Markov process (X_t, P_x) is a continuous time Markov chain that is not irreducible. In fact, S_0 is an irreducible class and S_1 is a transient class in the sense of Markov chain theory, because for every $(\infty, n)\in S_1$, X_t starting at (∞, n) goes far away along the line $\{(\infty, n'): n'=0, 1, 2, \cdots\}$ with positive probability.

We next claim that the Markov chain (X_t, P_x) restricted on the irreducible class S_0 is recurrent.

Let us define a function $h: S_0\to\boldsymbol{R}$ by $h((m, n))=m$ for each $(m, n)\in S_0$. Obviously, h is nonnegative, $h(x)\to\infty$ as $x\to\infty$, and satisfies $Ah(x)\leq 0$ for every $x\in S_0\setminus\{(0, 0)\}$. This is a well-known criterion of recurrence for Markov chains (c.f. [7]), so that the restricted process $(X_t, P_x)_{x\in S_0}$ is recurrent. Therefore, by the general theory of Markov chains, we have

$$R_0(x, \{(m, n)\})=\infty$$

for every $x\in S$ and every $(m,n)\in S_0$. From this it follows that for every $(\infty, n)\in S_1$ and every open set U containing (∞, n) with respect to the topology specified in the beginning,

$$R_0((\infty, n), U)=\infty . \tag{2.6}$$

Accordingly, we have shown that the condition (2.6) does not imply that (∞, n) is a recurrent point.

THEOREM 2.3. *Suppose that the Markov process (X_t, P_x) is irreducible and satisfies the strong Feller property. If there is at least one recurrent point for (X_t, P_x), then the process (X_t, P_x) is recurrent.*

Proof. Suppose that a be a recurrent point. Let U be an arbitrarily fixed open set containing a with compact closure $\bar{U}$. Then obviously

$$P_a(\sigma_U^r<\infty)=1 \qquad \text{for every } r>0. \tag{2.7}$$

We first claim that

$$P_x(\sigma_U^1<\infty)=1 \qquad \text{for every } x\in S. \tag{2.8}$$

By the strong Feller property $\phi(x)=P_x(\sigma_U^1<\infty)$ is a continuous function of $x\in S$. Using (2.7) and the strong Markov property, we see that for every $\lambda>0$

$$R_\lambda\phi(a)=\int_0^\infty e^{-\lambda t}P_a(\sigma_U^{1+t}<\infty)dt=\frac{1}{\lambda}. \tag{2.9}$$

Since the irreducibility of (X_t, P_x) implies that $R_\lambda(a, \cdot)$ is everywhere dense, it follows from (2.9) and the continuity of ϕ that $\phi(x)=1$ for every $x\in S$, thus we get (2.8). We next show that

$$R_0(x, U)=\infty \qquad \text{for every } x\in S. \tag{2.10}$$

Let us define a sequence of stopping times $\{\tau_n\}$ by

$$\tau_0=0 \qquad \tau_1=\sigma_U^1, \cdots, \tau_n=\tau_1\circ\Theta_{\tau_{n-1}}+\tau_{n-1}, \cdots.$$

By (2.8), $\tau_n<\infty$ $P_x-a.s.$ for every $n\geq 1$. Noting that $E_x\left(\int_{\tau_1}^{\tau_2}I_U(X_s)ds\right)>0$ and it is continuous in $x\in S$, by the strong Feller property, we get for every $n\geq 1$

$$\begin{aligned} E_x\left(\int_{\tau_n}^{\tau_{n+1}}I_U(X_s)ds\right)&=E_x\left(E_{X_{\tau_{n-1}}}\left(\int_{\tau_1}^{\tau_2}I_U(X_s)ds\right)\right) \\ &\geq \inf_{y\in\bar{U}}E_y\left(\int_{\tau_1}^{\tau_2}I_U(X_s)ds\right) \\ &>0, \end{aligned} \tag{2.11}$$

which yields (2.10).

Finally it follows from the resolvent equation that for every $\lambda>0$

$$(2.12)\qquad \begin{aligned} R_0 &= R_\lambda + \lambda R_0 R_\lambda \\ &\geq \lambda R_0 R_\lambda . \end{aligned}$$

Hence for every $b\in S$ and every open set V containing b we have

$$(2.13)\qquad \begin{aligned} R_0(b, V) &\geq \lambda \int_S R_0(b, dx) R_\lambda(x, V) \\ &\geq \lambda R_0(b, U) \inf_{x\in U} R_\lambda(x, V). \end{aligned}$$

Since $\inf_{x\in U} R_\lambda(x, V)>0$ follows from the irreducibility and the Feller property, by (2.10), we obtain $R_0(b, V)=\infty$. Therefore by virtue of Theorem 2.2 b is a recurrent point. Q.E.D.

3. On irreducibility of Gaussian diffusion processes

Let $A\in M(\boldsymbol{R}^d)$ and $B\in M_+(\boldsymbol{R}^d)$, and let (X_t, P_x) be a Gaussian diffusion process associated with A and B, governed by the equation (1.5).

We first find the transition probability $P_t(x, dy)$ explicitly.

LEMMA 3.1. *$P_t(x, dy)$ is a Gaussian distribution on $\boldsymbol{R}^d$ with the mean vector $e^{tA}x$ and the covariance matrix*

$$(3.1)\qquad V(t)=\int_0^t e^{sA} B e^{sA^*} ds,$$

where A^* is the transposed matrix of A. In particular, the characteristic function $\hat{P}_t(x, z)$ of $P_t(x, dy)$ is

$$(3.2)\qquad \hat{P}_t(x, z)=\exp\left\{i\langle e^{tA}x, z\rangle-\frac{1}{2}\langle V(t)z, z\rangle\right\}.$$

The proof is quite routine, so we omit it.

We first present a necessary and sufficient condition for the process (X_t, P_x) to be irreducible. However, we notice that this result was already discussed in the introduction of [4] in a different setting. We set

$$N=\bigcap_{n=0}^{\kappa} Ker\, BA^{*n}, \qquad \kappa=dim(Ker\, B),$$

where

$$Ker\, C=\{x\in\boldsymbol{R}^d : Cx=0\}.$$

THEOREM 3.2. *The Gaussian diffusion process (X_t, P_x) associated with $A\in M(\boldsymbol{R}^d)$ and $B\in M_+(\boldsymbol{R}^d)$ is irreducible on $\boldsymbol{R}^d$ if and only if*

(3.3) $$\bigcap_{n=0}^{\kappa} Ker\ BA^{*n}=\{0\},$$

where $\kappa=dim(Ker\ B)$. *In this case, for every* $t>0$, $V(t)$ *is a regular matrix and the transition probability* $P_t(x, dy)$ *has a Gussian density with respect to the Lebesgue measure on* $\boldsymbol{R}^d$

(3.4) $$P_t(x, y)=(2\pi)^{-d/2}(\det V(t))^{-1/2}\exp\left\{-\frac{1}{2}\langle V^{-1}(t)(y-e^{tA}x), y--e^{tA}x\rangle\right\}.$$

In order to make the paper self-contained we will here give the proof, which is based on the following

LEMMA 3.3. *For every* $t>0$, $N=Ker\ V(t)$.

Proof. Suppose that $V(t)x=0$. Then we see by (3.1) that

$$\int_0^t \|\sqrt{B}e^{sA^*}x\|^2 ds=0,$$

so that

(3.5) $$\sqrt{B}e^{sA^*}x=0, \qquad 0\leq\forall s\leq t.$$

Noting that $e^{tA^*}x$ is analytic in $t\geq 0$, we see that (3.5) is equivalent to

$$\sqrt{B}A^{*n}x=0, \qquad \forall n\geq 0.$$

Since for every $C\in M(\boldsymbol{R}^d)$

$$Ker\ \sqrt{B}C=Ker\ BC,$$

we get

$$Ker\ V(t)\subset\bigcap_{n=0}^{\infty} Ker\ BA^{*n}.$$

The inverse inclusion is clear, hence we have shown

(3.6) $$Ker\ V(t)=\bigcap_{n=0}^{\infty} Ker\ BA^{*n}.$$

It remains to show that

(3.7) $$\bigcap_{n=0}^{\infty} Ker\ BA^{*n}=\bigcap_{n=0}^{\kappa} Ker\ BA^{*n}.$$

Noticing that

$$N=\bigcap_{n=0}^{\kappa}\{x : A^{*n}x\in Ker\ B\},$$

we find it sufficient for (3.7) to show

(3.8) $$A^*N\subset N$$

(this is another expression for $N\subset\bigcap_{n=1}^{\kappa+1} Ker\, BA^{*n}$). But if $z\in N$, the $\kappa+1$ vectors z, $A^*z, \cdots, A^{*\kappa}z$ must be linearly dependent, or what amount to the same, there must be a real sequence $c_0, c_1, \cdots, c_n$ with $n\leq\kappa$ and $c_n\neq 0$ such that

$$c_0 z+c_1 A^*z+\cdots+c_n A^{*n}z=0.$$

Operating $A^{*(\kappa+1-n)}$ to the left-hand side and thereby finding that $A^{*(\kappa+1)}z\in Ker\, B$, we see $A^*z\in N$ and hence (3.8). The proof of Lemma 3.3 is completed.
Q. E. D.

Proof of Theorem 3.2.

Suppose that (3.3) holds. Then by Lemma 3.3, $V(t)$ is a regular matrix for every $t>0$. Hence for every $x\in\boldsymbol{R}^d$, $P_t(x, dy)$ is a non-degenerate Gaussian distribution on $\boldsymbol{R}^d$, so it follows immediately that (X_t, P_x) is irreducible. Conversely, suppose that (3.3) fails. Then by Lemma 3.3 $N=Ker\, V(t)\neq\{0\}$ for every $t>0$. Let $E=(\bigcap_{n=0}^{\kappa} Ker\, BA^{*n})^{\perp}$ be the orthogonal compliment of $\bigcap_{n=0}^{\kappa} Ker\, BA^{*n}$, which is a proper subspace of $\boldsymbol{R}^d$. Then it is easy to see

$$\operatorname{Supp} P_t(0,\cdot)=(Ker\, V(t))^{\perp}=E \qquad \text{for every } t>0,$$

which implies

$$P_0(X_t\in E \text{ for all } t\geq 0)=1, \tag{3.9}$$

because E is a closed subset of $\boldsymbol{R}^d$ and X_t is continuous in $t\geq 0$ $P_0-a.s.$. Therefore (X_t, P_x) is not irreducible. Q. E. D.

Remark. The generator of the Gaussian diffusion process associated with $A\in M(\boldsymbol{R}^d)$ and $B\in M_+(\boldsymbol{R}^d)$ is given by

$$L=\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} b_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}x_j\frac{\partial}{\partial x_i}. \tag{3.10}$$

Note that L can be rewritten as follows:

$$L=\sum_{i=1}^{d} V_i^2+V_0, \tag{3.11}$$

where

$$V_i=\sum_{j=1}^{d} c_{ij}\frac{\partial}{\partial x_j} \quad \text{with} \quad C=(c_{ij})=\sqrt{\frac{1}{2}B} \quad (1\leq i\leq d)$$

and

$$V_0=\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}x_j\frac{\partial}{\partial x_i}.$$

Then it is easy to show that the Lie algebra $\mathcal{L}[V_0, V_1, \cdots, V_d]$ generated by vector fields $V_0, V_1, \cdots, V_d$ is a linear space that is spanned by

$$\sum_{j=1}^{d}(CA^{*n})_{ij}\frac{\partial}{\partial x_j} \qquad (1\leq i\leq d,\ n\geq 0) \text{ and } V_0.$$

Hence by (3.7), the condition (3.3) is equivalent to

(3.12) $$\text{rank of } \mathcal{L}[V_0, V_1, \cdots, V_d](x)=d \qquad \text{for every } x\in \boldsymbol{R}^d,$$

which is the Hörmander condition of the hypoellipticity of L (c.f. [4]).

Example 1. Suppose that $B\in M_+(\boldsymbol{R}^d)$ is a regular matrix. Then (3.3) clearly holds since $\bigcap_{n=0}^{\kappa} Ker\, BA^{*n}=Ker\, B=\{0\}$. Hence for every $A\in M(\boldsymbol{R}^d)$ the Gaussian diffusion process (X_t, P_x) associated with A and B is irreducible.

Example 2. Let us consider the following $A\in M(\boldsymbol{R}^d)$ and $B\in M_+(\boldsymbol{R}^d)$

$$A=\begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{pmatrix}.$$

In this case, B is extremely degenerate. Nevertheless, (3.3) is verified and the associated Gaussian diffusion process is irreducible. In fact we have

$$\kappa=dim\,(Ker\, B)=d-1,$$

but for $0\leq n\leq d-1$

$$BA^{*n}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & & & & & & & \\ \vdots & & & & & & 0 & \\ 0 & & & & & & & \end{pmatrix},$$

$$Ker\, BA^{*n}=\{x=(x_1, \cdots, x_d): x_{n+1}=0\}\ ;$$

hence (3.3).

We will next show how to reduce a non-irreducible Gaussian diffusion process to an irreducible one. Suppose that we are given $A\in M(\boldsymbol{R}^d)$ and $B\in M_+(\boldsymbol{R}^d)$ such that

(3.10) $$N=\bigcap_{n=0}^{\kappa} Ker\, BA^{*n}\neq\{0\} \qquad \text{with} \quad \kappa=dim\,(Ker\, B).$$

Let (X_t, P_x) be a Gaussian diffusion process on $\boldsymbol{R}^d$ associated with A and B, governed by the equation

(1.5) $$X_t=x+\int_0^t AX_s ds+W_t,$$

where $x\in\boldsymbol{R}^d$ and W_t is a d-dimensional Brownian motion with the diffusion matrix B. Then by (3.10) and Theorem 3.2 the diffusion process (X_t, P_x) is not irreducible. Let $E=N^{\perp}$ be the same one as in the proof of Theorem 3.2. Note by (3.10) that E is a proper subspace of $\boldsymbol{R}^d$. Furthermore, we can restrict the diffusion process (X_t, P_x) into the state space E so that it may be an E-valued irreducible Gaussian diffusion process as seen in the following

THEOREM 3.4. (i) $AE\subset E$, (ii) W_t *is an E-valued Brownian motion with the diffusion matrix* $B|_E$ (*the restriction on E of the linear transformation B*), (iii) $(X_t, P_x)_{x\in E}$ *is an irreducible Gaussian diffusion process on E.*

Proof. (i) follows from (3.8). For $y\in N$, we get

$$E(\langle W_t, y\rangle^2)=\langle By, y\rangle t=0,$$

which implies that

$$P(W_t\in E \quad \text{for all } t\geq 0)=1.$$

Since B is symmetric and $N\subset Ker\, B$, $BE\subset E$. Thus W_t is an E-valued Brownian motion with the diffusion matrix $B|_E$. Finally, by Lemma 3.1 and (i) just verified, we see

$$\text{(3.11)} \qquad \operatorname{Supp} P_t(x, \cdot)=E+e^{tA}x=E \qquad \text{for every } x\in E \text{ and every } t>0,$$

completing the proof of (iii). Q.E.D.

4. Determination of recurrence and transience of irreducible Gaussian diffusion processes

Let (X_t, P_x) be a Gaussian diffusion process on $\boldsymbol{R}^d$ associated with an $A\in M(\boldsymbol{R}^d)$ and a $B\in M_+(\boldsymbol{R}^d)$, governed by the equation (1.5). In this section, assuming that (X_t, P_x) is irreducible, we determine whether it is recurrent or transient in terms of A and B.

Recall that by Theorem 3.2 the irreducibility assumption is equivalent to

$$\text{(4.1)} \qquad \bigcap_{n=0}^{\kappa} Ker\, BA^{*n}=\{0\} \qquad \text{with} \quad \kappa=dim(Ker\, B).$$

Under the condition (4.1), we will give criteria for recurrence properties of (X_t, P_x) by means of the real Jordan canonical form $\tilde{A}$ of A:

$$\tilde{A}=\begin{pmatrix} J_1 & & & & \boldsymbol{0} \\ & \ddots & & & \\ & & J_r & & \\ & & & K_1 & \\ & & & & \ddots \\ \boldsymbol{0} & & & & & K_s \end{pmatrix}.$$

where J_i $(1\leq i\leq r)$ are $m_i\times m_i$ Jordan blocks with real eigen-values λ_i and the size m_i, *i.e.*;

$$J_i=\begin{pmatrix} \lambda_i & & \boldsymbol{0} \\ 1 & \ddots & \\ \boldsymbol{0} & \ddots 1 & \lambda_i \end{pmatrix},$$

K_j $(1 \leq j \leq s)$ are $2n_j \times 2n_j$ Jordan blocks with complex eigen-values $\lambda_j = a_j + \sqrt{-1}b_j$ and the size n_j, *i.e.*;

$$K_j = \begin{pmatrix} a_j & -b_j & & & & & \mathbf{0} \\ b_j & a_j & \ddots & & & & \\ 1 & 0 & \ddots & & & & \\ 0 & 1 & \ddots & \ddots & & & \\ & & \ddots & a_j & -b_j & & \\ & & \ddots & b_j & a_j & & \\ & & & 1 & 0 & a_j & -b_j \\ \mathbf{0} & & & 0 & 1 & b_j & a_j \end{pmatrix}$$

and

$$\sum_{i=1}^{r} m_i + \sum_{j=1}^{s} 2n_j = d .$$

Now we can state a main result of this section.

THEOREM 4.1. *Let (X_t, P_x) be the Gaussian diffusion process on $\boldsymbol{R}^d$ associated with $A \in M(\boldsymbol{R}^d)$ and $B \in M_+(\boldsymbol{R}^d)$. Let $\tilde{A}$ be the real Jordan cononical form of A. Suppose that the process (X_t, P_x) is irreducible. Then*

(I) *If every eigen-value λ of A has negative real part (i.e. $Re\,\lambda < 0$), then (X_t, P_x) has a unique stationary distribution, hence it is positively recurrent.*

(II) *If the following* (a) *or* (b) *holds, then (X_t, P_x) is null recurrent (i.e. (X_t, P_x) is recurrent but has no stationary distribution).*

(a) *$\tilde{A}$ contains a unique Jordan block of the form $\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ $(b \neq 0)$ and all other Jordan blocks have eigen-values with negative real parts.*

(b) *$\tilde{A}$ contains one or two Jordan block with size* 1 *of the form* (0) *and all other Jordan blocks have eigen-values with negative real parts.*

(III) *In all remaining cases, the process (X_t, P_x) is transient.*

In what follows (X_t, P_x) denotes the irreducible Gaussian diffusion process on $\boldsymbol{R}^d$ associated with $A \in M(\boldsymbol{R}^d)$ and $B \in M_+(\boldsymbol{R}^d)$. For the proof of Theorem 4.1 we prepare several lemmas.

LEMMA 4.2. *Suppose that the irreducible Gaussian diffusion process (X_t, P_x) satisfies*

(4.2) $$R_0(0, U) = \infty \quad \text{for every open set } U \text{ containing } 0.$$

Then the process (X_t, P_x) is recurrent.

Proof. Clearly (X_t, P_x) satisfies the strong Feller property, hence the proof is immediate from Remark 1 after Theorem 2.2 and Theorem 2.3. Q. E. D.

LEMMA 4.3. *Let $V(t)$ be the covariance matrix of the transition probability, whose explicit form is given in Lemma 3.1.*

(i) *If*

$$\int_1^\infty (det\ V(t))^{-1/2} dt < \infty, \tag{4.3}$$

then the process (X_t, P_x) ist ransient. In this case, it holds that

$$P_x(\lim_{t\to\infty} |X_t| = \infty) = 1 \qquad \text{for every } x \in \boldsymbol{R}^d.$$

(ii) *If*

$$\int_1^\infty (det\ V(t))^{-1/2} dt = \infty, \tag{4.4}$$

then the process (X_t, P_x) is recurrent.

Proof. For $B \in \mathcal{B}(\boldsymbol{R}^d)$, we denote by $\lambda(B)$ the Lebesgue measure of B.

(i) For every compact subset K of $\boldsymbol{R}^d$ and every $x \in \boldsymbol{R}^d$

$$P_t(x, K) \leq (2\pi)^{-d/2} \lambda(K) (det\ V(t))^{-1/2},$$

hence by (4.3), $R_0(x, K) < \infty$, so that Theorem 2.1 is applicable and the process (X_t, P_x) is transient. The latter statement can be easily shown by applying the arguements in the proof of Theorem 2.1.

(ii) By Lemma 4.2, it suffices to show that for every open ball U containing 0,

$$R_0(0, U) = \infty. \tag{4.2}$$

The explicit form of $V(t)$ in Lemma 3.1 implies that $V(t)$ is increasing in $t \geq 0$, so its inverse $V(t)^{-1}$ is decreasing in $t \geq 0$, hence for every $t \geq 1$,

$$C(t) \equiv \sup_{y \in U} \langle V(t)^{-1} y, y \rangle \leq C(1).$$

Using this we get for $t \geq 1$,

$$P_t(0, U) = (2\pi)^{-d/2} (det\ V(t))^{-1/2} \int_U e^{-1/2\langle V(t)^{-1} y, y\rangle} dy$$

$$\geq e^{-C(1)/2} (2\pi)^{-d/2} \lambda(U) (det\ V(t))^{-1/2},$$

from which and the assumption of (ii), (4.2) follows, and the process (X_t, P_x) is recurrent. Q. E. D.

LEMMA 4.4. *Let (Y_t, Q_x) be the Gaussian diffusion process on $\boldsymbol{R}^d$ associated with the same A as of (X_t, P_x) and $I_{d\times d}$ (identity) in place of B. If (Y_t, Q_x) is reccurrent, then (X_t, P_x) is also recurrent.*

Proof. By Example 1 of section 3, (Y_t, Q_x) is irreducible. Denote by $V_1(t)$

and $V_2(t)$ the covariance matrices of the transition probabilities $P_t(x, dy)$ and $Q_t(x, dy)$ of (X_t, P_x) and (Y_t, Q_x) respectively. Since (Y_t, Q_x) is recurrent, by Lemma 4.3

$$\int_1^\infty (det\ V_2(t))^{-1/2} dt = \infty .$$

Note that by Lemma 3.1

$$V_1(t) = \int_0^t e^{sA} B e^{sA^*} ds$$

and

$$V_2(t) = \int_0^t e^{sA} e^{sA^*} ds ,$$

which implies $V_1(t) \leq \|B\| V_2(t)$. In particular we get

$$det\ V_1(t) \leq \|B\|^d\ det\ V_2(t) .$$

Accordingly we have

$$\int_1^\infty (det\ V_1(t))^{-1/2} dt = \infty .$$

Therefore using Lemma 4.3 again we see that (X_t, P_x) is recurrent. Q. E. D.

LEMMA 4.5. *Let $V(t)$ be the covariance matrix of its transition probability $P_t(x, dy)$. For some $1 \leq m \leq d$, let $V_m(t)$ be the principal minor of $V(t)$, that is,*

$$V_m(t) = \begin{pmatrix} V_{11} & \cdots & V_{1m} \\ V_{21} & \cdots & V_{2m} \\ \vdots & & \\ V_{m1} & \cdots & V_{mm} \end{pmatrix} .$$

Then there is a constant $C>0$ such that for every $t \geq 1$,

$$det\ V(t) \geq C\ det\ V_m(t) . \tag{4.5}$$

Furthermore, if

$$\int_1^\infty (det\ V_m(t))^{-1/2} dt < \infty , \tag{4.6}$$

the process (X_t, P_x) is transiedt.

Proof. By virtue of Lemma 4.3, we have only to prove (4.5). Let $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_d(t)$ be eigen-values of $V(t)$. We know for every $t > t'$ and $x \in \boldsymbol{R}^d$

$$\langle V(t)x, x \rangle \geq \langle V(t')x, x \rangle .$$

This implies, by the mini-max theorem, that

$$\lambda_j(t) \geq \lambda_j(t') , \qquad j = 1, 2, \cdots, d ,$$

if $t \geq t'$. In particular, if $t \geq 1$, then

$$\lambda_j(t) \geq \lambda_0, \qquad j=1, 2, \cdots, d,$$

where λ_0 denotes $\lambda_d(1)$. Therefore,

$$\begin{aligned} \det V(t) &= \lambda_1(t)\lambda_2(t) \cdots \lambda_d(t) \\ &\geq \lambda_1(t)\lambda_2(t) \cdots \lambda_m(t)\lambda_0^{d-m}, \end{aligned} \tag{4.7}$$

if $t \geq 1$. Applying the mini-max theorem again we see

$$\lambda_1(t)\lambda_2(t) \cdots \lambda_m(t) \geq \det V_m(t), \tag{4.8}$$

which proves (4.5). Q. E. D.

We are now in position to prove Theorem 4.1. Without loss of generality we may assume that A is of the real Jordan canonical form, because by a suitable regular transformation of the state space $\boldsymbol{R}^d$, the process (X_t, P_x) is transformed to another irreducible Gaussian diffusion process having a matrix of the real Jordan canonical form in place of A and with its recurrence properties unchanged.

Proof of the part (I).

It is easy to see that for every $x \in \boldsymbol{R}^d$, its transition probability $P_t(x, \cdot)$ converges to a Gaussian distribution on $\boldsymbol{R}^d$ with the mean vector 0 and the covariance matrix

$$\int_0^\infty e^{sA}Be^{sA^*}ds. \tag{4.9}$$

It is easy to see that the integral of (4.9) is convergent if and only if every eigen-value of A has negative real part (see the computation for the case 2 below). This implies that there is a unique stationary distribution if and only if every eigen-value of A has negative real part.

Proof of the part (II)-(a).

By the assumption of (II)-(a),

$$A = \begin{pmatrix} 0 & -b & \boldsymbol{0} \\ b & 0 & \\ \boldsymbol{0} & & \tilde{A} \end{pmatrix} \qquad (b \neq 0),$$

where $\tilde{A} \in M(\boldsymbol{R}^{d-2})$ and all the eigen-values of $\tilde{A}$ have negative real parts. By Lemma 4.4 we may assume $B=I$. By (3.1), we get

$$V(t(= \begin{pmatrix} tI_{2\times 2} & 0 \\ 0 & \int_0^t e^{s\tilde{A}}e^{s\tilde{A}^*}ds \end{pmatrix}.$$

and see

$$\det V(t) \sim Const.\, t^2 \qquad as \quad t \to \infty.$$

Accordingly, by Lemma 4.3 (ii), the process (X_t, P_x) is recurrent.

Proof of the part (II)-(b).

The proof is the same as part (II)-(a).

Proof of the part (III).

Recall that we are assuming that A itself is of a real Jordan canonical form. Under the assumption of irreducibility, all remaining cases in part (III) of Theorem 4.1 can be devided into the following six cases (1)~(6); The cases 1 and 2 correspond to the case when $Re\,\lambda>0$ for some eigen-value λ of A, and the cases 3~6 correspond to the cases when $Re\,\lambda\leq 0$ for all eigen-values λ of A except the cases in part (II).

Case 1.

$$A=\begin{pmatrix} a & 0 \\ * & \tilde{A} \end{pmatrix},$$

where $a>0$ and $\tilde{A}\in M(R^{d-1})$.

Since $V(t)$ is positive definite and by (3.1), we have for $x=(1, 0, \cdots, 0)$,

$$\begin{aligned} 0&< V_{11}(t) \\ &=\langle V(t)x, x\rangle \\ &=\int_0^t \langle e^{sA}Be^{sA^*}x, x\rangle ds \\ &=\int_0^t e^{sa}b_{11}e^{sa}ds \\ &=\frac{b_{11}}{2a}(e^{2ta}-1). \end{aligned}$$

We can apply Lemma 4.5 with $m=1$ and conclude that the process is transient.

Case 2.

$$A=\begin{pmatrix} a & -b & 0 \\ b & a & \\ & * & \tilde{A} \end{pmatrix},$$

where $a>0$ and $\tilde{A}\in M(\boldsymbol{R}^{d-2})$.

Since

$$e^{sA}=\begin{pmatrix} e^{sa}\cos bs & -e^{sa}\sin bs & 0 \\ e^{sa}\sin bs & e^{sa}\cos bs & \\ * & * & * \end{pmatrix},$$

using (3.1) and the fact that

$$\forall a>0,\ \exists t_0(a);\ \forall s>t_0,\ e^{sa}>s^2,$$

we have

$$V_{11}(t)=\int_0^t e^{sa}[b_{11}\cos^2 bs+b_{22}\sin^2 bs-(b_{12}+b_{21})\cos bs\sin bs]ds$$

$$\geqq\int_{t_0}^t s^2[b_{11}\cos^2 bs+b_{22}\sin^2 bs-(b_{12}+b_{21})\cos bs\sin bs]ds,$$

where we have also employed

$$(*)\qquad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\in M_+(\boldsymbol{R}^2).$$

Thus we get

$$V_{11}(t)\geq\frac{1}{6}(b_{11}+b_{22})t^3+O(t^2).$$

It therefore suffices to see

$$(4.10)\qquad b_{11}+b_{22}>0.$$

But if (4.10) fails, then by $(*)$ $b_{12}=b_{21}=0$, so that $V_{11}(t)=0$, the contradiction.

Case 3.

$$A=\begin{pmatrix} 0 & 0 & \mathbf{0} \\ 1 & 0 & \\ & * & \tilde{A} \end{pmatrix}.$$

Using (3.1) again we can calculate

$$V_{11}(t)=b_{11}t$$

$$V_{22}(t)=\frac{1}{3}b_{11}t^3+O(t^2).$$

Applying Lemma 4.5 with $m=1$ to $V_{22}(t)$, we conclude that the process is transient.

Case 4.

$$A=\begin{pmatrix} 0 & & \mathbf{0} \\ & D & \\ \mathbf{0} & & \tilde{A} \end{pmatrix},$$

where $\tilde{A}\in M(\boldsymbol{R}^{d-3})$ and $D=\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$. For convenience we here include the case $b=0$.

We have by (3.1)

$$V_{11}(t)=b_{11}t$$

$$V_{22}(t)=V_{33}(t)$$

$$=\frac{1}{2}(b_{22}+b_{33})t+O(1)$$

$$
\begin{aligned}
V_{12}(t) &= V_{13} \\
&= V_{23} \\
&= O(1).
\end{aligned}
$$

Since $V_{11}(t)>0$ for every $t>0$, $b_{11}>0$. And $b_{22}+b_{33}>0$ follows from the fact that

$$0<V_{22}(t)=\int_0^t (b_{22}\cos^2 bs+b_{33}\sin^2 bs)ds.$$

Applying Lemma 4.5 with $m=3$, we conclude that the process is transient.

Case 5.

$$A=\begin{pmatrix} D & & \mathbf{0} \\ I & D & \\ & * & \tilde{A} \end{pmatrix},$$

where $\tilde{A}\in M(\boldsymbol{R}^{d-4})$, $D=\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ and $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have by (3.1)

$$
\begin{aligned}
V_{11}(t) &= V_{22}(t) \\
&= \frac{1}{2}(b_{11}+b_{22})t+O(1) \\
V_{33}(t) &= V_{44}(t) \\
&= \frac{1}{6}(b_{11}+b_{22})t^3+O(t^2) \\
V_{12}(t) &= V_{14} \\
&= V_{24} \\
&= O(t) \\
V_{13}(t) &= V_{23}(t) \\
&= \frac{1}{4}(b_{11}+b_{22})t^2+O(t) \\
V_{34}(t) &= O(t^2).
\end{aligned}
$$

Since $V_{11}(t)>0$, we can prove that $b_{11}+b_{22}>0$ by the same discussion as in the proof of (4.10) for the case 2. We apply Lemma 4.5 with $m=1$ to $V_{33}(t)$ and conclude that the process is transient.

Case 6.

$$A=\begin{pmatrix} D & & \mathbf{0} \\ & D & \\ \mathbf{0} & & \tilde{A} \end{pmatrix},$$

where $\tilde{A}\in M(\boldsymbol{R}^{d-4})$ and $D=\begin{pmatrix}0 & -b\\ b & 0\end{pmatrix}$ $(b\neq 0)$.

We have by (3.1)

$$\begin{aligned} V_{11}(t)&=V_{22}(t)\\ &=\frac{1}{2}(b_{11}+b_{22})t+O(1)\\ V_{33}(t)&=V_{44}(t)\\ &=\frac{1}{2}(b_{33}+b_{44})t+O(1)\\ V_{12}(t)&=V_{34}\\ &=O(1)\\ V_{13}(t)&=V_{24}(t)\\ &=\frac{1}{2}(b_{13}+b_{24})t+O(1)\\ V_{14}(t)&=\frac{1}{2}(b_{14}-b_{23})t+O(1)\\ V_{23}(t)&=\frac{1}{2}(b_{23}-b_{14})t+O(1) \end{aligned}$$

We want to apply Lemma 4.5 with $m=4$. Let

$$V_4^{\infty}=\lim_{t\to\infty}\frac{1}{t}V_4(t).$$

Then

$$det\ V_4(t)=t^4(det\ V_4^{\infty})+O(t^3).$$

By virtue of Lemma (4.5), we have only to prove

$$det\ V_4^{\infty}>0. \tag{4.11}$$

We have by (3.1)

$$V_{ij}=\int_0^t\sum_{k,l=1}^4 a_{ik}b_{kl}a_{jl}\qquad (1\leq i,\ j\leq 4),$$

where

$$(a_{ij})_{4\times 4}=\begin{pmatrix}\cos bs & -\sin bs & & \mathbf{0}\\ \sin bs & \cos bs & & \\ & & \cos bs & -\sin bs\\ \mathbf{0} & & \sin bs & \cos bs\end{pmatrix}.$$

Since $V_{ij}(t)$ is an integral of a periodic function of s with the period $2\pi/|b|$, we get

$$V_4^\infty = \frac{b}{2\pi} V_4(2\pi/|b|),$$

hence (4.11).

Accordingly, Theorem 4.1 has completely been Proved.

Q. E. D.

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