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ON THE ZERO-ONE-POLE SET OF A MEROMORPHIC FUNCTION, II

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0. Let $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit points. If it is possible to construct a meromorphic function f in the plane C whose zeros, d-points and poles are exactly $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively, where their multiplicities are taken into consideration, then the given triad $(\{a_n\}, \{b_n\}, \{p_n\})$ is called a *zero-d-pole set*. Here of course d is a nonzero complex number. Further if there exists only one meromorphic function f whose zero-d-pole set is just the given triad, then the triad is called *unique*. It is well known that unicity in this sense does not hold in general.

In Sections 1 and 2, the letter E will denote sets of finite linear measure which will not necessarily be the same at each occurrence.

1. Let f and g be meromorphic functions in the plane C. If f and g assume the value $a \in C \cup \{\infty\}$ at the same points with the same multiplicities, we denote this by $f=a \Leftrightarrow g=a$. With this notation, our first result of this note is stated as follows.

THEOREM 1. *Let f and g be nonconstant meromorphic functions satisfying $f=0 \Leftrightarrow g=0$, $f=1 \Leftrightarrow g=1$ and $f=\infty \Leftrightarrow g=\infty$. If*

$$\bar{K}(f)=\limsup_{r\to\infty}\{\bar{N}(r, 0, f)+\bar{N}(r, \infty, f)\}/T(r, f)<1/2, \tag{1.1}$$

then $f \equiv g$ or $fg \equiv 1$.

From this we immediately deduce the following

COROLLARY 1. *Let f be a nonconstant meromorphic function satisfying $n(r, 0, f)+n(r, \infty, f) \not\equiv 0$ and $\bar{K}(f)<1/2$. Then the zero-one-pole set of f is unique.*

The estimate (1.1) is sharp. For example, let us consider $f=e^{\alpha}(1-e^{\alpha})$ and $g=e^{-\alpha}(1-e^{-\alpha})$ with a nonconstant entire function α. Then we easily see that $f=0 \Leftrightarrow g=0$, $f=1 \Leftrightarrow g=1$ and $f=\infty \Leftrightarrow g=\infty$. Also, $f \not\equiv g$ and $fg \not\equiv 1$ are evident.

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To determine the value $\bar{K}(f)$, we may note that $T(r, f) \sim T(r, f-1/4) = 2m(r, e^{\alpha}-1/2) \sim 2m(r, e^{\alpha})$ $(r \to \infty)$, $\bar{N}(r, 0, f) = \bar{N}(r, 1, e^{\alpha}) \leqq m(r, e^{\alpha}) + O(1)$ $(r \to \infty)$, and $\bar{N}(r, 1, e^{\alpha}) \sim m(r, e^{\alpha})$ $(r \notin E, r \to \infty)$. The combination of these estimates gives $\bar{K}(f) = 1/2$, and thus (1.1) is actually sharp. Also we remark that Theorem 1 is an improvement of [4, Theorem 2].

Before proceeding to the proof of Theorem 1, we state two lemmas.

LEMMA 1. *If α is a nonconstant entire function, then*

$$m(r, \alpha') = o\{m(r, e^{\alpha})\} \qquad (r \notin E, r \to \infty), \tag{1.2}$$

and for any nonzero constant c

$$\bar{N}(r, c, e^{\alpha}) \sim N(r, c, e^{\alpha}) \sim m(r, e^{\alpha}) \qquad (r \notin E, r \to \infty). \tag{1.3}$$

(1.2) is immediately deduced from the fact that $\alpha' = (e^{\alpha})'/e^{\alpha}$ and [1, Theorem 2.3.]. (1.3) is easily obtained from the first and the second fundamental theorems.

LEMMA 2. *If f is meromorphic and not constant in the plane C, then we have*

$$N\{r, \infty, 2(f'/f)^2 - f''/f\} \leqq 2\bar{N}(r, 0, f) + \bar{N}(r, \infty, f) + \bar{N}_1(r, \infty, f). \tag{1.4}$$

This estimate is easily verified from the computation which was done in [4, p. 28].

Proof of Theorem 1. We make use of notations and argument in the proof of [4, Theorem 2]. Our assumptions of this theorem imply

$$f = e^{\alpha} g, \qquad f-1 = e^{\beta}(g-1) \tag{1.5}$$

with two entire functions α and β. If e^{β} or $e^{\beta-\alpha}$ is identically equal to one, we deduce $f \equiv g$ from (1.5) at once. We divide our argument into the following three cases.

Case 1. e^{β} is a constant $c(\neq 0, 1)$,

Case 2. $e^{\beta-\alpha}$ is a constant $c(\neq 0, 1)$,

Case 3. neither e^{β} nor $e^{\beta-\alpha}$ are constants.

In *Cases* 1, 2, and 3 with $\Delta \equiv 0$ and $C \neq 0$, the argument in [4, pp. 29-30] and (1.3) are combined to show that $\bar{K}(f) = 1$. This is inconsistent with (1.1). In *Case* 3 with $\Delta \equiv 0$ and $C = 0$, the argument in [4, p. 30] gives $fg \equiv 1$. Consider *Case* 3 with $\Delta \not\equiv 0$. The argument in [4, p. 30] yields

$$m(r, f) \leqq O\{\log T(r, f) + \log r\} + N(r, \infty, \Delta) \qquad (r \notin E, r \to \infty), \tag{1.6}$$

and

$$\left.\begin{array}{l} m(r, e^{\alpha}) \\ m(r, e^{\beta}) \end{array}\right\} \leqq \{4 + o(1)\} T(r, f) \qquad (r \notin E, r \to \infty). \tag{1.7}$$

Since $f=(1-e^{\beta})/(1-e^{\beta-\alpha})$, we readily obtain from (1.2) and (1.7)

$$\bar{N}_1(r, \infty, f) \leqq N_1(r, \infty, f) \leqq N(r, 0, \beta'-\alpha') \leqq m(r, \beta'-\alpha')+O(1) \tag{1.8}$$
$$\leqq O\{\log T(r, f)+\log r\} \qquad (r \notin E, r \to \infty).$$

By the definition of Δ and (1.4)

$$N(r, \infty, \Delta) \leqq 2\bar{N}(r, 0, f)+\bar{N}(r, \infty, f)+\bar{N}_1(r, \infty, f). \tag{1.9}$$

Combining (1.6), (1.8) and (1.9), we have

$$T(r, f)=m(r, f)+\bar{N}(r, \infty, f)+N_1(r, \infty, f) \tag{1.10}$$
$$\leqq 2\{\bar{N}(r, 0, f)+\bar{N}(r, \infty, f)\}+O\{\log T(r, f)+\log r\}$$
$$(r \notin E, r \to \infty).$$

The nonconstancy of β and (1.7) imply that f is transcendental, and thus (1.10) gives $\bar{K}(f) \geqq 1/2$. This is also inconsistent with (1.1). This completes the proof of Theorem 1.

2. Suppose that f is a nonconstant meromorphic function in the plane C. We denote the zero-one-pole set of f by $(\{a_n\}, \{b_n\}, \{p_n\})$. Let $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ be subsequences of $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively such that $\{c_n\} \cup \{d_n\} \cup \{q_n\} \neq \emptyset$. Then for shortening we write $\{a_n\} \cup \{p_n\} = \{s_n\}$ and $\{c_n\} \cup \{q_n\} = \{t_n\}$. Further we define a subsequence $\{u_n\}$ of $\{s_n\}$ as follows: $u_n \in \{u_n\}$ if and only if u_n occurs in $\{s_n\}$ only once but never in $\{t_n\}$.

In this section we prove

THEOREM 2. *Let f, $(\{a_n\}, \{b_n\}, \{p_n\})$, $\{c_n\}$, $\{d_n\}$, $\{q_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{u_n\}$ be given as above. If $\{s_n\} \neq \emptyset$ and*

$$\limsup_{r \to \infty} \frac{4\bar{N}(r, \{s_n\} \cup \{d_n\})+N(r, \{s_n\})+\bar{N}(r, \{t_n\})-\bar{N}(r, \{u_n\})}{T(r, f)} < 2, \tag{2.1}$$

then $(\{a_n\} \setminus \{c_n\}, \{b_n\} \setminus \{d_n\}, \{p_n\} \setminus \{q_n\})$ is not a zero-one-pole set of any nonconstant meromorphic function.

In view of Corollary 1 the zero-one-pole set of f in Theorem 2 is unique. Also, the estimate (2.1) is sharp. In fact, let P and Q be canonical products with no common zeros, let L be a transcendental entire function, and consider $g=(P/Q)e^L$ and $f=g^2$. Then simple computations show that the left hand side of (2.1)=2. Further we notice that Theorem 2 improves [5, Theorem 4] in some sense.

The proof of Theorem 2 needs the following estimate of Weierstrass products.

LEMMA 3. ([3, Lemma 4]) *Let $\{a_{\nu\mu}\}$ be n sequences $(1 \leqq \mu \leqq n)$ of complex* *numbers satisfying* $|a_{1\mu}|\leqq|a_{2\mu}|\leqq\cdots$, $\lim_{\nu\to\infty}|a_{\nu\mu}|=+\infty$ *for each* μ. *Then we can construct the Weierstrass products* P_μ *of* $\{a_{\nu\mu}\}$ $(1\leqq\mu\leqq n)$ *with the property that*

$$\frac{\sum_{\mu=1}^{n}\log m(r, P_\mu)}{\sum_{\mu=1}^{n}N(r, 0, P_\mu)}\longrightarrow 0$$

holds as $r\to\infty$ *inside a certain set* Ω *of infinite linear measure.*

Proof of Theorem 2. We shall seek a contradiction from the assumption that $(\{a_n\}\setminus\{c_n\}, \{b_n\}\setminus\{d_n\}, \{p_n\}\setminus\{q_n\})$ is the zero-one-pole set of a nonconstant meromorphic function g. First, we construct entire functions P, R and Q whose zeros are $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ respectively as follows:

(i) If $\{c_n\}=\emptyset$, then $P\equiv 1$. It is the same with R and Q.

(ii) If $1\leqq\#\{c_n\}<+\infty$, then P is the polynomial $\prod_n(z-c_n)$. It is the same with R and Q.

$$\text{(iii)}\qquad \frac{\log m(r, P)+\log m(r, R)+\log m(r, Q)}{N(r, 0, P)+N(r, 0, R)+N(r, 0, Q)}\longrightarrow 0$$

holds as $r\to\infty$ inside a suitable set Ω of infinite linear measure.

The condition (iii) is possible by means of (ii) and Lemma 3. From the first fundamental theorem it follows that $N(r, 0, P)+N(r, 0, R)+N(r, 0, Q)\leqq N(r, 0, f)+N(r, 1, f)+N(r, \infty, f)\leqq 3T(r, f)+O(1)$, and hence by (iii)

$$\frac{\log m(r, P)+\log m(r, R)+\log m(r, Q)}{T(r, f)}\longrightarrow 0\qquad (r\in\Omega,\ r\to\infty). \tag{2.2}$$

Now, under our assumptions there are two entire functions α and β such that

$$gP/Q=fe^{\alpha}, \qquad (g-1)R/Q=(f-1)e^{\beta}. \tag{2.3}$$

Eliminating g from (2.3), we have

$$f-fSe^{\gamma}+Te^{-\beta}=1, \quad\text{or}\quad 1/f-Te^{-\beta}/f+Se^{\gamma}=1, \tag{2.4}$$

where $S=R/P$, $T=R/Q$ and $\gamma=\alpha-\beta$. For simplicity's sake, we write $\phi_1=f$, $\phi_2=-fSe^{\gamma}$, $\phi_3=Te^{-\beta}$, $\psi_1=1/f$, $\psi_2=-Te^{-\beta}/f$ and $\psi_3=Se^{\gamma}$. With these ϕ_j $(j=1, 2, 3)$ define Δ and Δ' by

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ \phi_1'/\phi_1 & \phi_2'/\phi_2 & \phi_3'/\phi_3\\ \phi_1''/\phi_1 & \phi_2''/\phi_2 & \phi_3''/\phi_3\end{vmatrix}, \qquad \Delta'=\begin{vmatrix}\phi_2'/\phi_2 & \phi_3'/\phi_3\\ \phi_2''/\phi_2 & \phi_3''/\phi_3\end{vmatrix},$$

and further with ϕ_j replaced by ψ_j we define Δ_1 and Δ_1' similarly.

(A) First we consider the case $\Delta\equiv 0$. By (2.4)

$$-fSe^{\gamma}=\phi_2=C\phi_1+D=Cf+D\,, \tag{2.5}$$

$$Te^{-\beta}=\phi_3=1-\phi_1-\phi_2=1-D-(C+1)f \tag{2.6}$$

with two constants C and D. Eliminating f from (2.5) and (2.6), we get

$$R\{CPe^{-\beta}+Re^{\gamma-\beta}+(D-1)Qe^{\gamma}\}=(C+D)PQ\,. \tag{2.7}$$

(A_1) If $\{d_n\}\neq\emptyset$, then (2.7) implies $C+D=0$ and $(C+1)Qe^{\beta+\gamma}-Re^{\gamma}=CP$. It is easily verified that $C\neq 0,\ -1$. Hence from (2.5) and (2.6) we deduce that $\{c_n\}=\{a_n\}$, $\{d_n\}=\{b_n\}$ and $\{q_n\}=\{p_n\}$. This is contradictory to the assumption that g is nonconstant.

(A_2) Now, we proceed to the case $\{d_n\}=\emptyset$, i.e. $R\equiv 1$. If $D=0$, (2.5) implies $P\equiv 1$, so that Q has at least one zero. Hence by (2.6) $C\neq -1$ and $f=(1-1/Qe^{\beta})/(C+1)$, from which we have $\bar{K}(f)\geqq 1$. (Here we remark that (2.1) implies $\bar{K}(f)<1/2$. This is an immediate consequence of the fact that $\{u_n\}$ is a subsequence of $\{s_n\}$.) If $D=1$, (2.7) implies $P\equiv 1$ and $C\neq 0$. Hence in view of (2.5) $f=-(C+e^{\gamma})^{-1}$, from which we have $\bar{K}(f)=1$. It remains to consider the case $D\neq 0,\ 1$. If $C=-1$, (2.6) implies $e^{\beta}\equiv(1-D)^{-1}$. From this and (2.5) it follows that $f=DP/\{P+(D-1)e^{\alpha}\}$, so that $\bar{K}(f)\geqq 1$. If $C\neq -1$, (2.6) gives $f=(1-D-1/Qe^{\beta})/(C+1)$, which also yields $\bar{K}(f)\geqq 1$.

(B) The case $\Delta_1\equiv 0$ can be handled in all the same way as the case $\Delta\equiv 0$, and after all $\Delta_1\equiv 0$ leads us to imcompatible results with our assumptions.

(C) Next we suppose that neither Δ nor Δ_1 are identically zero. From (2.4) it follows that $f=\Delta'/\Delta$. Using the same reasoning as in *Case* 3 in the proof of Theorem 1, we obtain the following estimates:

$$m(r,\ f)\leqq m(r,\ \Delta')+m(r,\ \Delta)+N(r,\ \infty,\ \Delta)+O(1)\,, \tag{2.8}$$

$$\begin{aligned} m(r,\ \Delta')+m(r,\ \Delta)=O\{&\log T(r,\ f)+\log r+\log m(r,\ P)\\ &+\log m(r,\ R)+\log m(r,\ Q)\}\quad (r\notin E,\ r\to\infty)\,, \end{aligned} \tag{2.9}$$

$$\begin{aligned} N(r,\ \infty,\ \Delta)\leqq\ & 2\bar{N}(r,\ 0,\ f)+\bar{N}(r,\ \infty,\ f)+\bar{N}(r,\ 0,\ P)+\bar{N}(r,\ 0,\ Q)+2\bar{N}(r,\ 0,\ R)\\ &+\bar{N}_1(r,\ \infty,\ fQ)-\bar{N}(r,\ \{Q=0\}\cap\{\textit{ multiple poles of } fQ\})\,, \end{aligned} \tag{2.10}$$

$$N(r,\infty,f)=\bar{N}(r,\infty,f)+N_1(r,0,Q)+N_1(r,\infty,fQ)+\bar{N}(r,\{Q=0\}\cap\{fQ=\infty\}). \tag{2.11}$$

In particular, if f is a rational function, (2.9) can be replaced by

$$m(r,\ \Delta')+m(r,\ \Delta)=O(1)\,. \tag{2.9$'$}$$

Indeed, we may use the first and the second fundamental theorems to find that g is rational, and next note from (ii) that all of P, R and Q are polynomials, so that e^{α} and e^{β} are constants. Hence ϕ_1, ϕ_2 and ϕ_3 are all rational functions, and thus (2.9)$'$ holds.

After (2.2), (2.9) ((2.9)' in case that f is a rational function) and (2.10) are taken into account, (2.8) and (2.11) yield

$$\begin{aligned}(2.12)\qquad &\{1-o(1)\}T(r, f)\leqq 2\{\bar{N}(r, 0, f)+\bar{N}(r, \infty, f)+\bar{N}(r, 0, R)\}\\ &+\bar{N}(r, 0, P)+N(r, 0, Q)+(N_1+\bar{N}_1)(r, \infty, fQ)\\ &+\bar{N}(r, \{Q=0\}\cap\{\textit{simple poles of } fQ\}) \quad (r\in\Omega\setminus E,\ r\to\infty).\end{aligned}$$

In the same way, starting from $1/f=\Delta_1'/\Delta_1$ we deduce

$$\begin{aligned}(2.13)\qquad &\{1-o(1)\}T(r, 1/f)\leqq 2\{\bar{N}(r, \infty, f)+\bar{N}(r, 0, f)+\bar{N}(r, 0, R)\}\\ &+\bar{N}(r, 0, Q)+N(r, 0, P)+(N_1+\bar{N}_1)(r, 0, f/P)\\ &+\bar{N}(r, \{P=0\}\cap\{\textit{simple zeros of } f/P\}) \quad (r\in\Omega\setminus E,\ r\to\infty),\end{aligned}$$

where Ω and E are the same as in (2.12). Summing up (2.12) and (2.13), we have

$$\begin{aligned}&\{2-o(1)\}T(r, f)\leqq 4\{\bar{N}(r, 0, f)+\bar{N}(r, \infty, f)+\bar{N}(r, 0, R)\}+(N+\bar{N})(r, 0, P)\\ &+(N+\bar{N})(r, 0, Q)+(N_1+\bar{N}_1)(r, 0, f/P)+(N_1+\bar{N}_1)(r, \infty, fQ)\\ &+\bar{N}(r, \{P=0\}\cap\{\textit{simple zeros of } f/P\})\\ &+\bar{N}(r, \{Q=0\}\cap\{\textit{simple poles of } fQ\}) \quad (r\in\Omega\setminus E,\ r\to\infty),\end{aligned}$$

which is also inconsistent with (2.1).

Thus $(\{a_n\}\setminus\{c_n\}, \{b_n\}\setminus\{d_n\}, \{p_n\}\setminus\{q_n\})$ is not a zero-one-pole set of any nonconstant meromorphic function.

3. Suppose that f is a nonconstant meromorphic function in the plane C whose zero-d-pole set is not unique, where $d(\neq 0, 1)$ is a constant. Let $(\{a_n\}, \{b_n\}, \{p_n\})$ be the zero-one-pole set of f, and let $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ be subsequences of $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively such that $\{c_n\}\cup\{d_n\}\cup\{q_n\}\neq\emptyset$ and such that

$$(3.1)\qquad \sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}+\sum_{q_n\neq 0}|q_n|^{-1}<+\infty .$$

Under these assumptions we prove the following result.

THEOREM 3. *Let f, d, $(\{a_n\}, \{b_n\}, \{p_n\})$, $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ be given as above. Then $(\{a_n\}\setminus\{c_n\}, \{b_n\}\setminus\{d_n\}, \{p_n\}\setminus\{q_n\})$ is not a zero-one-pole set of any nonconstant meromorphic function.*

We have already showed the corresponding result for the case $d=1$ in [5, Theorem 1]. Also, Ozawa [2, Section 4] has proved this result for $\{p_n\}=\{q_n\}=\{c_n\}=\emptyset$ and $1\leqq\#\{d_n\}<+\infty$. The assumption (3.1) cannot be omitted. For example, let us consider $f=d(e^z-1)/(e^z-d)$, $N=df$ and $g=e^z/(e^z-d)$ with a constant $d(\neq 0, 1)$. Then we easily see that f and N have the same zero-d-pole set, say $(\{a_n\}, \phi, \{p_n\})$, and therefore the zero-d-pole set of f is not unique. On the other hand, the zero-one-pole sets of f and g are $(\{a_n\}, \phi, \{p_n\})$ and $(\phi, \phi, \{p_n\})$ respectively, and $\sum_{a_n\neq 0}|a_n|^{-1}=\pi^{-1}\sum_{k=1}^{\infty}k^{-1}=+\infty$. Further we remark that this result does not hold in general in the case that the zero-d-pole set of f is unique for any $d\neq 0$. In fact, let g be a nonconstant meromorphic function of order less than one, and consider $f=g^2$. See [1, p. 25, Lemma 1.4.].

In the proof of Theorem 3, we frequently use the following form of the impossibility of Borel's identity.

LEMMA 4. (cf. [5]) *Let $P_0, P_1, \cdots, P_n$ ($P_j\not\equiv 0$, $0\leqq j\leqq n$, $n\geqq 1$) be entire functions satisfying $m(r, P_j)=o(r)$ $(r\to\infty)$, and let $g_1, g_2, \cdots, g_n$ be nonconstant entire functions. Then an identity of the following form is impossible:* $\sum_{j=1}^{n}P_je^{g_j}=P_0$.

Proof of Theorem 3. We suppose that $(\{a_n\}\setminus\{c_n\}, \{b_n\}\setminus\{d_n\}, \{p_n\}\setminus\{q_n\})$ is the zero-one-pole set of a nonconstant meromorphic function g, To begin with, we construct entire functions P, R and Q whose zeros are $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ respectively in the following manner.

(i) If $\{c_n\}$ is empty, then $P\equiv 1$. It is the same with R and Q.

(ii) All of P, R and Q have genus zero, so that $m(r, P)+m(r, R)+m(r, Q)=o(r)$ $(r\to\infty)$.

The condition (ii) is possible from (3.1). Let $N(\not\equiv f)$ be the meromorphic function whose zero-d-pole set is the same as the one of f.

According to our assumptions, there are four entire functions α, β, γ and δ such that

$$N=fe^{\alpha},\quad N-d=(f-d)e^{\beta},\quad gP/Q=fe^{\gamma},\quad (g-1)R/Q=(f-1)e^{\delta}. \tag{3.2}$$

We note that each of e^{α}, e^{β} and $e^{\alpha-\beta}$ is not identically equal to one, otherwise we immediately deduce from (3.2) $f\equiv N$. The elimination of N, g and f from (3.2) gives

$$PRe^{\alpha}-PRe^{\beta}-dQRe^{\gamma}+dQRe^{\beta+\gamma}+dPQe^{\delta}-PQe^{\alpha+\delta}+(1-d)PQe^{\beta+\delta}=0. \tag{3.3}$$

Suppose that e^{α} is a constant $c(\neq 0, 1)$. Then e^{β} is not a constant because of the nonconstancy of f, and by (3.3)

$$PRe^{\beta}+dQRe^{\gamma}+(c-d)PQe^{\delta}-dQRe^{\beta+\gamma}+(d-1)PQe^{\beta+\delta}=cPR. \tag{3.4}$$

We first consider the case $c=d$. Recall that P, R and Q satisfy the condition (ii). Then applying Lemma 4 to (3.4), we find that at least one of e^{γ}, $e^{\beta+\gamma}$ and $e^{\beta+\delta}$ is a constant, say x. If $e^{\gamma}\equiv x$, (3.4) becomes $(P-dxQ)Re^{\beta}+(d-1)PQe^{\beta+\delta}$

$=d(P-xQ)R$. It is easily seen that $R\equiv 1$, $P-dxQ\not\equiv 0$ and $P-xQ\not\equiv 0$, so that $e^{\beta+\delta}$ is a constant, say y, and hence $(P-dxQ)e^{\beta}=d(P-xQ)-y(d-1)PQ$. This is impossible. If $e^{\beta+\gamma}\equiv x$, then $PRe^{\beta}+dxQRe^{-\beta}+(d-1)PQe^{\beta+\delta}=d(P+xQ)R$, which implies that $R\equiv 1$ and $P+xQ\not\equiv 0$. Hence $e^{\beta+\delta}$ must be a constant, say y, and thus $Pe^{\beta}+dxQe^{-\beta}=d(P+xQ)-y(d-1)PQ\not\equiv 0$, which is absurd. If $e^{\beta+\delta}\equiv x$ but neither e^{γ} nor $e^{\beta+\gamma}$ are constants, then $PRe^{\beta}+dQRe^{\gamma}-dQRe^{\beta+\gamma}=\{dR-x(d-1)Q\}P$ by (3.4), so that $dR-x(d-1)Q\equiv 0$. Hence $R\equiv Q\equiv 1$ and $de^{\beta}-Pe^{\beta-\gamma}=d$. This is also untenable. We can discuss the case $c\neq d$ in much the same way as the case $c=d$, and in each subcase we make an appeal to Lemma 4 to obtain an absurd result. Thus we see that e^{α} is not a constant. Similarly, we can make sure that e^{β}, e^{γ} and e^{δ} are not constants.

Suppose next that $e^{\beta-\alpha}$ is a constant $c(\neq 0, 1)$. From (3.3) it follows that

$$cdQRe^{\gamma}+\{c(1-d)-1\}PQe^{\delta}-dQRe^{\gamma-\alpha}+dPQe^{\delta-\alpha}=(c-1)PR\,, \tag{3.5}$$

which implies that at least one of $e^{\gamma-\alpha}$ and $e^{\delta-\alpha}$ is a constant, say x. First assume that $e^{\gamma-\alpha}\equiv x$. In view of (3.5)

$$cdxQRe^{\alpha}+\{c(1-d)-1\}PQe^{\delta}+dPQe^{\delta-\alpha}=\{(c-1)P+dxQ\}R\,. \tag{3.6}$$

If $(c-1)P+dxQ\equiv 0$, then $P\equiv Q\equiv 1$ and $c-1+dx=0$. Substituting these into (3.6), we have $cxRe^{\alpha-\delta}+e^{-\alpha}=c+x$. Since R has at least one zero, $c+x\neq 0$, and so $e^{\alpha-\delta}$ must be a constant, say y. Thus $e^{-\alpha}=c+x-cxyR$. This is untenable. If $(c-1)P+dxQ\not\equiv 0$, then (3.6) yields that $e^{\delta-\alpha}$ is a constant, say y, and that $[cdxR+y\{c(1-d)-1\}P]Qe^{\alpha}=\{(c-1)P+dxQ\}R-dyPQ\not\equiv 0$. This is also impossible. Next assume that $e^{\delta-\alpha}\equiv x$. By means of (3.5) $cdQRe^{\gamma}+\{c(1-d)-1\}PQe^{\delta}-dQRe^{\gamma-\alpha}=\{(c-1)R-dxQ\}P$, from which we have $(c-1)R-dxQ\equiv 0$. Hence $Q\equiv R\equiv 1$, $c-1=dx$, and so $(c-x)Pe^{\delta-\gamma}+e^{-\alpha}=c$. This is absurd. Thus we may assume that $e^{\beta-\alpha}$ is not a constant. In the similar manner, we can ascertain the fact that $e^{\gamma-\alpha}$, $e^{\beta+\gamma-\alpha}$, $e^{\delta-\alpha}$ and $e^{\beta+\delta-\alpha}$ are not constants.

It remains to consider the case that none of e^{α}, e^{β}, e^{γ}, e^{δ}, $e^{\beta-\alpha}$, $e^{\gamma-\alpha}$, $e^{\beta+\gamma-\alpha}$, $e^{\delta-\alpha}$ and $e^{\beta+\delta-\alpha}$ are constants. Using (3.3) once more, we have $PQe^{\delta}+PRe^{\beta-\alpha}+dQRe^{\gamma-\alpha}-dQRe^{\beta+\gamma-\alpha}-dPQe^{\delta-\alpha}+(d-1)PQe^{\beta+\delta-\alpha}=PR$. This is also impossible because of Lemma 4.

All the above arguments are combined to show that $(\{a_n\}\setminus\{c_n\}, \{b_n\}\setminus\{d_n\}, \{p_n\}\setminus\{q_n\})$ is not a zero-one-pole set of any nonconstant meromorphic function.

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