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SUBORDINATION FOR BMOA

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1. Introduction.

Let p be arbitrarily fixed with $p>0$ throughout this paper. Let $f(z)$ be a function analytic in the unit disk $U=\{|z|<1\}$. The H_p *norm* $\|f\|_p$ of f is defined by

$$\|f\|_p=\lim_{r\to 1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right\}^{1/p}. \tag{1.1}$$

The family of all f for which $\|f\|_p$ is finite is denoted by $H_p(U)$ and called *Hardy class*. For every $a\in U$, $t_a(z)=(z+a)/(1+\bar{a}z)$ be the conformal map of U onto itself with $t_a(0)=a$, and we set $f_a(z)=f(t_a(z))-f(a)$. The ***BMO*** *norm* $B_p(f)$ of f is defined by

$$B_p(f)=\sup\{\|f_a\|_p : a\in U\}. \tag{1.2}$$

The family of all f for which $B_p(f)$ is finite is denoted by $\boldsymbol{BMOA}(U)$, where ***BMOA*** stands for "*Analytic functions of Bounded Mean Oscillation*". It is known that the family $\boldsymbol{BMOA}(U)$ does not depend on the value of p ([2]). In the following, abbreviating the index p, we simply write as

$$B(f)=B_p(f)^p. \tag{1.3}$$

Let ϕ denote an analytic function in U with $|\phi(z)|<1$ *for* $z\in U$. Applying the *subordination principle*, we easily see that

$$B(f\circ\phi)\leqq B(f) \tag{1.4}$$

holds for every $f\in\boldsymbol{BMOA}(U)$. Here $f\circ\phi$ denotes the composite function of f by ϕ, *i.e.* $(f\circ\phi)(z)\equiv f(\phi(z))$. In this paper, we deal with the equality problem for the inequality (1.4).

A bounded analytic function ϕ in U is called an *inner function*, if its nontangential boundary values are of modulus 1 almost everywhere on the unit circle T. Ryff [5] characterized inner functions by a property of preserving H_p norms of analytic functions:

Ryff's Theorem ([6, Theorem 3, p. 351]). *Let ϕ be an analytic function in*

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U with $|\phi(z)|<1$ for $z\in U$ and $\phi(0)=0$. If ϕ is an inner function, then the equality

$$\|f\circ\phi\|_p=\|f\|_p \tag{1.5}$$

holds for every $f\in H_p(U)$. Conversely, if (1.5) *holds for some nonconstant $f\in H_p(U)$, then ϕ is an inner function.*

It follows from *the Ryff theorem* that if ϕ is an inner function, then equality holds in (1.4) for every $f\in \boldsymbol{BMOA}(U)$ (see Theorem 2.1 in the next section). As for the converse, however, we did not know whether equality occurs or not in (1.4) for some noninner function ϕ. Recently, the author [3] showed that there is a noninner function ϕ for which equality holds in (1.4) for every function f analytic in U and continuous on the closure $\bar{U}$ of U. In the present paper, we generalize this to the desired case for every $f\in \boldsymbol{BMOA}(U)$, that is, we show that there are fairly many noninner functions ϕ such that equality holds in (1.4) for every $f\in \boldsymbol{BMOA}(U)$.

2. Subordination by inner functions.

In this section, to make this paper self-contained, we show that equality holds in (1.4) for every inner function ϕ and for every $f\in \boldsymbol{BMOA}(U)$:

THEOREM 2.1. *If ϕ is an inner function, then the equality*

$$B(f\circ\phi)=B(f) \tag{2.1}$$

holds for every $f\in \boldsymbol{BMOA}(U)$.

Before stating the proof of the theorem, we introduce notation concerning the least harmonic majorants of subharmonic functions, which we use throughout this paper. Let $S^+(U)$ denote the family of all nonnegative subharmonic functions in U. For $s\in S^+(U)$ we denote by $\hat{s}$ the least harmonic majorant of s in U. Here, in case that s admits no harmonic majorants in U, we set $\hat{s}(z)\equiv+\infty$.

The author and Suita [5] generalized *the Ryff theorem* cited above to the case of nonnegative subharmonic functions on Riemann surfaces. We restate it in the case for $S^+(U)$:

LEMMA 2.2. *If ϕ is an inner function, then the equality*

$$(s\circ\phi)\hat{}(z)\equiv(\hat{s}\circ\phi)(z)\,. \tag{2.2}$$

holds for every $s\in S^+(U)$, and conversely if (2.2) *holds for some $s\in S^+(U)$ which is not harmonic in whole U, then ϕ is inner.*

For the proof, see [5, pp. 316-318].

Returning to the case of an analytic function f, we set $s_a(z)=|f(z)-f(a)|^p$ for every $a\in U$. On noting that the equality

$$\|f_a\|_p{}^p=(s_a\circ t_a)\hat{}(0)=(\hat{s}_a\circ t_a)(0)=\hat{s}_a(a)$$

holds for any $f\in \boldsymbol{BMOA}(U)$, where we used (2.2) with $\phi=t_a$, we obtain from (1.2) the following lemma:

LEMMA 2.3. *For every* $f\in \boldsymbol{BMOA}(U)$

$$B_p(f)=\sup\{\hat{s}_a(a):\, a\in U\}. \tag{2.3}$$

Proof of the theorem. By a theorem of Frostman [1], the range $\phi(U)$ of any inner function ϕ covers U except possibly for a set of *(logarithmic) capacity* zero. In particular, we see that $\phi(U)$ is dense in U. Therefore, by using (2.3) and (2.2), we obtain

$$\begin{aligned} B_p(f)&=\sup\{\hat{s}_a(a):\, a\in U\} \\ &=\sup\{\hat{s}_a(a):\, a\in\phi(U)\} \\ &=\sup\{\hat{s}_a(\phi(b)):\, b\in U\} \\ &=\sup\{(s_a\circ\phi)\hat{}(b):\, b\in U\} \\ &=B(f\circ\phi)\,. \end{aligned}$$

Here, we set $a=\phi(b)$.

3. Subordination by noninner functions.

In this section we show that equality holds in (1.4) for every $f\in \boldsymbol{BMOA}(U)$ even for some noninner functions ϕ:

THEOREM 3.1. *There are some noninner functions ϕ for which the equality* (2.1) *holds for every* $f\in \boldsymbol{BMOA}(U)$.

Proof. We construct functions ϕ which satisfy the condition of the theorem. Let ψ be any inner function which has a singularity (a point at which ψ can not be analytically continued) at a point ζ on the unit circle T. We easily see that there are a lot of such inner functions. In fact, we can take as ψ any inner function which has as a factor a *Blaschke product* whose zeros converge to ζ or a *singular inner function* whose associated singular measure has a positive mass at ζ. Let D be a subdomain of U which contains $\{z\in U:\, |z-\zeta|<\delta\}$ for a sufficiently small $\delta>0$. Let λ be a *universal covering map* of D. It is easily seen that λ is inner if and only if $U-D$ is of capacity zero. Now we further assume that $U-D$ is of capacity positive, so that λ is noninner. Finally we set $\phi=\psi\circ\lambda$. Applying the Ryff theorem, we easily see that ϕ is also non-inner.

For $s\in S^{+}(U)$ we denote by $\hat{s}_D(z)$ the least harmonic majorant of s in D. The following lemma is merely a restatement of a wellknown fact that a universal covering map preserves the least harmonic majorant of any subharmonic function. As for the proof, see for example [5, Lemma 1, p. 316]:

LEMMA 3.2. *The equality*

$$(s\circ\lambda)^{\wedge}(z)\equiv(\hat{s}\circ\lambda)(z) \tag{3.1}$$

holds for every $s\in S^{+}(U)$.

We set $s(z)=|g(z)|^{p}$ for $g\in H_p(U)$. Let D_1 be a simply-connected subdomain of D whose boundary contains an open interval on T containing ζ. In fact, for example, we can take as $D_1=\{z\in U: |z-\zeta|<\delta\}$ for a sufficiently small $\delta>0$. It is known that $\hat{s}(z)$ and $\hat{s}_D(z)$ are expressed as integrals by the *harmonic measure* for the point z in D_1 of their boundary values on the boundary of D_1, respectively. Here note that these integrals are expressed via the *Poisson integral formula* by conformally mapping the domain D_1 onto the unit disk U. Since their boundary values coincide almost everywhere on the open interval on T containing ζ, we see that $\hat{s}(z)-\hat{s}_D(z)$ approaches to 0 as z approaches to ζ. Therefore, by noting Lemma 3.2, we obtain the following lemma:

LEMMA 3.3 *Set* $s(z)=|g(z)|^{p}$ *for any* $g\in H_p(U)$. *If* $b_n\in U$ *is a sequence such that* $\lambda(b_n)\to\zeta$ *as* $n\to\infty$ *and that the limit*

$$\lim_{n\to\infty}(\hat{s}_D\lambda)(b_n)$$

exists, then the limit

$$\lim_{n\to\infty}(s\circ\lambda)^{\wedge}(b_n)$$

exists and the two limits coincide:

$$\lim_{n\to\infty}(\hat{s}\circ\lambda)(b_n)=\lim_{n\to\infty}(s\circ\lambda)^{\wedge}(b_n)\,. \tag{3.2}$$

Now we continue the proof of the theorem. Since we assumed ψ to be an inner function with a singularity at ζ, we see by noting a local version of the Frostman theorem [1] cited in the previous section that the range set $R(\psi,\zeta)$ at ζ of ψ convers U except possibly for a set of capacity zero. Here, the *range set* $R(\psi,\zeta)$ is defined by

$$R(\psi,\zeta)=\bigcap_{\delta>0}\psi(\{z\in U: |z-\zeta|<\delta\})\,. \tag{3.3}$$

In particular, $R(\psi,\zeta)$ is dense in U. Let ε be an arbitrarily fixed positive number. On noting Lemma 2.2, we can take a point $a\in R(\psi,\zeta)$ such that

(3.4) $$B(f)-\varepsilon<\hat{s}_a(a)\,.$$

The definition (3.3) means that we can take a sequence $a_n\in D$ such that $\psi(a_n)=a$ for every n and that $a_n\to\zeta$ as $n\to\infty$. Finally we take a sequence $b_n\in U$ such that $\lambda(b_n)=a_n$ for every n. By using (2.2) we see

$$\begin{aligned}\hat{s}_a(a)&=\hat{s}_a(\psi(a_n))\\&=(s_a\circ\psi)\hat{}(a_n)\\&=(s_a\circ\psi)\hat{}(\lambda(b_n))\,.\end{aligned}$$

Therefore by letting $n\to\infty$ in this equality and applying Lemma 3.3 as $g(z)=f(\psi(z))-f(a)$, we see

(3.5) $$\begin{aligned}\hat{s}_a(a)&=\lim_{n\to\infty}(s_a\circ\psi)\hat{}(\lambda(b_n))\\&=\lim_{n\to\infty}(s_a\circ\psi\circ\lambda)\hat{}(b_n)\,.\end{aligned}$$

Since we have set $\phi=\psi\circ\lambda$, writing $h(z)=f\circ\phi$, we easily see $|h(z)-h(b_n)|^p=(s_a\circ\psi\circ\lambda)(z)$ for every n. Therefore, on noting Lemma 2.3, we see that for every n

(3.6) $$(s_a\circ\psi\circ\lambda)\hat{}(b_n)\leqq B(f\circ\phi)\,.$$

Combining (1.4), (3.4), (3.5) and (3.6), we obtain (2.1), as asserted, since ε in (3.4) can be chosen arbitrarily small.

4. Concluding remarks.

We do not know as yet whether there are noninner functions ϕ of different type from ones constructed in the proof of Theorem 3.1 for which the equality (2.1) holds for every $f\in \boldsymbol{BMOA}(U)$. The functions ϕ constructed in the proof of Theorem 3.1 are easily seen to satisfy the condition that for some point $\zeta\in T$ the range set $R(\phi,\zeta)$ covers U except possibly for a set of capacity zero. This condition, however, is not sufficient in order that the equality (2.1) holds for every $f\in \boldsymbol{BMOA}(U)$. In fact, for example, let $\mu(z)$ be the conformal map of U onto $U\cap\{z: Re\, z>-k\}$ with $\mu(0)=0$, where $0<k<1$, and ψ to be an inner function with a singularity at a point $\zeta\in T$. If we set $\phi(z)=\{\mu(\psi(z))\}^2$, we can prove that the strict inequality $B(f\circ\phi)<B(f)$ holds for every function f analytic in U and continuous on $\bar{U}$ (cf. [4, Example 6.3, p. 169]).

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