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ON Z_p-EXTENSIONS OF REAL ABELIAN FIELDS

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In [1], Fukuda and Komatsu gave a sufficient condition for the λ invariants of real quadratic fields to vanish. In this note, we shall show that the results obtained in [1] can be extended in a real abelian case.

Let k be a finite real abelian extension of Q and p an odd prime number which splits completely in k. Let $\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_r$ be the prime factorization of (p) in k, where $r=[k:Q]$. A Z_p-extension

$$k=k_0\subset k_1\subset\cdots\subset k_n\subset\cdots\subset K$$

is the cyclotomic Z_p-extension, since k is a totally real field and Leopoldt's conjecture is valid for k. Let C_n be the ideal class group of k_n, A_n the p-primary part of C_n, σ a topological generator of $\mathrm{Gal}(K/k)$ and we denote also by σ the restriction of σ to k_n. Let

$$C_n'=\{c\in C_n\,;\,c^\sigma=c\} \quad\text{and}\quad B_n=C_n'\cap A_n\,.$$

We have the following result by Greenberg [2], since every $\mathcal{P}_i$ is totally ramified in K.

LEMMA 1. *There exists an integer* n_0 *such that* $|B_0|\leqq|B_1|\leqq\cdots\leqq|B_{n_0}|=|B_{n_0+1}|=\cdots=|B_n|=\cdots$.

Let E be the unit group of k. Then

$$E=\{1,\,-1\}\oplus E'\,,$$

where E' is a free abelian group of rank $r-1$. We denote by $N_{m,n}$ the norm mapping from k_m to k_n $(m\geqq n)$. Let $H=N_{n,0}(k_n)\cap E$, then there exists a base $\{\eta_1,\eta_2,\cdots,\eta_{r-1}\}$ of E' and positive integers $c_1, c_2,\cdots,c_{r-1}$ such that

$$H=\{1,\,-1\}\oplus[\eta_1^{c_1},\,\eta_2^{c_2},\,\cdots,\,\eta_{r-1}^{c_{r-1}}] \tag{1}$$

and $c_i\,|\,c_{i+1}$ for $i=1,\cdots,r-2$. Since $[k_n:k]=p^n$,

$$H\supset\{1,\,-1\}\oplus[\eta_1^{p^n},\,\eta_2^{p^n},\,\cdots,\,\eta_{r-1}^{p^n}]\,.$$

Then we have

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$$c_1=p^{n-s_1},\ c_2=p^{n-s_2},\ \cdots,\ c_{r-1}=p^{n-s_{r-1}},$$

where $s_1, s_2, \cdots, s_{r-1}$ are integers such that $n\geqq s_1\geqq s_2\cdots\geqq s_{r-1}\geqq 0$. Let $l_n=s_1+s_2+\cdots+s_{r-1}$, then $[E:H]=p^{n(r-1)-l_n}$.

We denote by $e(\mathfrak{P})$ the ramification index of a place $\mathfrak{P}$ of k with respect to k_n/k. Since k_n is a finite cyclic extension of k, by Yokoi [3] we have

$$\begin{aligned}|C_n'|&=|C_0|\times\frac{\prod_{\mathfrak{P}}e(\mathfrak{P})}{[k_n:k][E:H]}\\&=\frac{|C_0|\times\overbrace{p^n\times\cdots\times p^n}^{r}}{p^n\times p^{n(r-1)-l_n}}\\&=|C_0|p^{l_n}.\end{aligned}$$

Hence $|B_n|=p^{c+l_n}$, where c is the integer such that p^c divides $|C_0|$ exactly. Then, by lemma 1

$$l_0\leqq l_1\leqq\cdots\leqq l_{n_0}=l_{n_0+1}=\cdots=l.$$

Since $k_{\mathfrak{P}_i}\cong Q_p(1\leqq i\leqq r)$, if ε is any element of E' which is not equal to 1, then there exists the positive integer $m_{\varepsilon,i}$ for $\mathfrak{P}_i$ such that $\varepsilon^{p-1}\in 1+p^{m_{\varepsilon,i}}Z_p$ and $\varepsilon^{p-1}\notin 1+p^{m_{\varepsilon,i}+1}Z_p$. Let $m_\varepsilon=\mathrm{Min}\{m_{\varepsilon,i};\ 1\leqq i\leqq r\}$.

LEMMA 2. *If $n\geqq m_\varepsilon$, then $\varepsilon'=\varepsilon^{(p-1)p^{n-m_\varepsilon+1}}$ is contained in H.*

Proof. $\varepsilon^{(p-1)p^{n-m_\varepsilon+1}}\in 1+p^{n+1+(m_{\varepsilon,i}-m_\varepsilon)}Z_p\subset 1+p^{n+1}Z_p$. Hence, ε' is a $\mathfrak{P}_i$-adic norm for k_n/k at every $\mathfrak{P}_i$. By local class field theory, ε' is also a $\mathfrak{P}$-adic norm at every prime $\mathfrak{P}$ of k_n prime to p. Then, by Hasse's norm theorem, ε' is a global norm for k_n/k. This implies $\varepsilon'\in E\cap N_{n,0}(k_n)=H$.

Remark. If $m_\varepsilon=m_{\varepsilon,j}$, then $\varepsilon^{(p-1)p^{n-m_\varepsilon+1}}\in 1+p^{n+1}Z_p$ and $\notin 1+p^{n+2}Z_p$ in $k_{\mathfrak{P}_j}$.

Let $\{\zeta_1, \zeta_2, \cdots, \zeta_{r-1}\}$ be any base of E' and $H'=[\zeta_1^{(p-1)p^{n-m_{\zeta_1}+1}}, \zeta_2^{(p-1)p^{n-m_{\zeta_2}+1}}, \cdots, \zeta_{r-1}^{(p-1)p^{n-m_{\zeta_{r-1}}+1}}]$ for $n\geqq\mathrm{Max}\{m_{\zeta_i};\ 1\leqq i\leqq r-1\}$. Then $H\supset\{1,-1\}\oplus H'$, since $\zeta_i^{(p-1)p^{n-m_{\zeta_i}+1}}\in H$. Let $m(\zeta)=m_{\zeta_1}+m_{\zeta_2}+\cdots+m_{\zeta_{r-1}}$.

PROPOSITION 1. $m(\zeta)\leqq l_n+(r-1)$.

Proof. $[E:H]=p^{n(r-1)-l_n}$ and $[E:\{1,-1\}\oplus H']=(p-1)^{r-1}p^{n(r-1)-m(\zeta)+(r-1)}$. Here, $p^{n(r-1)-m(\zeta)+(r-1)}\geqq p^{n(r-1)-l_n}$, since $[E:H]$ divides $[E:\{1,-1\}\oplus H']$ and $(p-1)^{r-1}$ is prime to p. Then $m(\zeta)\leqq l_n+(r-1)$.

COROLLARY. *$m(\zeta)$ remains bounded for all bases of E'.*

Proof. $m(\zeta)\leqq l_n+(r-1)\leqq l+(r-1)$.

Let $m=\mathrm{Max}\{m(\zeta);$ any base $\{\zeta_1,\cdots,\zeta_{r-1}\}$ of $E'\}$ and, for $n\geqq m$, we take a base $\{\eta_1, \eta_2, \cdots, \eta_{r-1}\}$ of E' as in (1). Then $m(\eta)\leqq m$, but we can show that $m(\eta)=m$.

PROPOSITION 2. $|B_n|=p^{c+m-(r-1)}$ *for* $n\geqq m$.

Proof. We have $\eta_i^{(p-1)p^{n-m_{\eta_i}+1}}\in H(1\leqq i\leqq r-1)$. Then there exists an integer d_i such that $\eta_i^{(p-1)p^{n-m_{\eta_i}+1}}=\eta_i^{d_i p^{n-s_i}}$, so $d_i=(p-1)p^{d'_i}$ for a non negative integer d'_i. On the other hand, $\eta_i^{(p-1)p^{n-s_i}}\in 1+p^{n+1}Z_p$ and $\eta_i^{(p-1)p^{n-m_{\eta_i}+1}}$ is a topological generator of $1+p^{n+1}Z_p$ in $k_{\mathcal{P}_j}$, where $m_{\eta_i}=m_{\eta_{i,j}}$. Hence, $d'_i=0$, so $n-m_{\eta_i}+1=n-s_i$. Then we have $[E:H]=p^{n(r-1)-m(\eta)+(r-1)}$ and $l_n=m(\eta)-(r-1)$. By proposition 1, $m\leqq l_n+(r-1)=m(\eta)$. Hence, $m=m(\eta)$ and $|B_n|=p^{c+l_n}=p^{c+m-(r-1)}$.

This proposition was proved by Fukuda and Komatsu [1] in a real qudratic case. We can also prove the following result by the same way as in the theorem of Fukuda and Komatsu [1].

THEOREM. *If* $N_{m-1,0}\ (E_{m-1})=E$ *and the class number of* k *is prime to* p, *then* $\lambda_p(k)=0$, *where* E_n *is the unit group of* k_n *and* $\lambda_p(k)$ *is the Iwasawa invariant of* k *for the* p.

Let $\{\zeta_1, \zeta_2, \cdots, \zeta_{r-1}\}$ be any base of E' and $n_{\zeta,i}=\mathrm{Max}\{m_{\zeta_j,i}\,;\,1\leqq j\leqq r-1\}$ for $\mathcal{P}_i$. Then $n_{\zeta,i}$ does not remain bounded for all bases of E'.

Example. Let k be a real cubic field and $\{\varepsilon_1, \varepsilon_2\}$ a base of E'. Assume that

$$\varepsilon_1^{p-1}=1+xp^{a_1}+\cdots$$

and

$$\varepsilon_2^{p-1}=1+yp^{a_2}+\cdots$$

in $k_{\mathcal{P}_i}$, where $1\leqq x,\ y\leqq p-1$ and $a_2\geqq a_1$. We take a non negative integer z such that

$$zx\equiv -y(\text{mod. } p) \quad \text{and} \quad 1\leqq z\leqq p-1\,.$$

Let $\varepsilon'_2=\varepsilon_1^{zp^{a_2-a_1}}\varepsilon_2$. Then $\{\varepsilon_1, \varepsilon'_2\}$ is a base of E' and $m_{\varepsilon'_2,i}>a_2=m_{\varepsilon_2,i}\geqq a_1=m_{\varepsilon_1,i}$.

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