K. WANG
KODAI MATH. J.
12 (1989), 98—108

ON THE EXISTENCE OF LIMIT CYCLES OF THE EQUATION $x'=h(y)-F(x),\ y'=-g(x)$ *

BY KE WANG

1. Introduction.

Owing to their theoretical and practical importance, the Liénard equations have attracted much attention in recent years, in particular, for theory of periodic solutions, see [1-7].

In this paper, we consider the existence of limit cycles of the system

$$\begin{cases} x'=h(y)-F(x) \\ y'=-g(x), \end{cases} \tag{1}$$

which is little more general than the Liénard equation. We assume that F, G, $h: \boldsymbol{R}\to\boldsymbol{R}$ are continuous functions and satisfy the property of uniqueness for the solutions to the Cauchy problems associated to the system (1), and $xg(x)>0$ for every $x\neq 0$, $yh(y)>0$ for every $y\neq 0$. Without loss of generality, we also assume $F(0)=0$. We obtained some new results. The theorems of this paper generalized some results in [5] and [7].

Let Y^+, Y^-, C^+, C^- denote the sets $\{(x, y): y\geqq 0,\ x=0\}$, $\{(x, y): y\leqq 0,\ x=0\}$, $\{(x, y): h(y)=F(x),\ x>0\}$ and $\{(x, y): h(y)=F(x),\ x<0\}$, respectively.

2. Technical Preliminaries.

LEMMA 1. *If we assume*

(i) $$\overline{\lim_{y\to+\infty}} h(y)=+\infty, \quad \textit{and} \quad \underline{\lim_{y\to-\infty}} h(y)=-\infty,$$

then the sufficient and necessary condition that there exists a point $N\in Y^-$ *such that the negative half-trajectory* L_N^- *passing through point* N *does not intersect* C^+ *is*

(ii)$_a$ *there exists a continuously differentiable function* $k_1(x)$ *defined on* $(0, \infty)$ *with positive derivative such that*

$$F(x)\geqq h(k_1(x))+\frac{g(x)}{k_1'(x)} \qquad \textit{for} \quad x>0.$$

* Supported by the National Natural Science Foundation of China

Received December 18, 1987, Revised August 10, 1988

Proof. If the condition $(\mathrm{ii})_a$ holds, let us consider the system

$$\begin{cases} x'=h(y)-\left(h(k_1(x))+\dfrac{g(x)}{k_1'(x)}\right) \\ y'=-g(x). \end{cases} \tag{2}$$

It is easy to see that $y=k_1(x)$ is a solution of (2). From the comparison theorem, it follows that the solution of (1) passing through the point $N=(0, k_1(0))$ is under the curve $y=k_1(x)$, so it will not cross C^+.

If there exists a point $N\in Y^-$ such that L_N^- does not intersect C^+. We can suppose the equation of L_N^- is $y=k_1(x)$. It is easy to see that $k_1(x)$ is continuously differentiable and its derivative is positive. We have

$$F(x)=h(k_1(x))+\frac{g(x)}{k_1'(x)}.$$

Thus the Lemma is proved.

By Lemma 1, we can give concrete conditions on $h(y)$, $F(x)$ and $g(x)$, so long as a concrete function $k_1(x)$ is given. For example, if we set $k_1(x)=CG(x)-M$, where $G(x)=\int_0^x g(s)ds$, C, $M>0$, $x>0$, then we can prove the following corollary.

COROLLARY 1. *Suppose that there exist constants C, $M>0$ such that*

$$F(x)\geqq h(CG(x)-M)+\frac{1}{C} \qquad for \quad x\geqq 0,$$

then there exists a point $N\in Y^-$ such that L_N^- does not cross C^+.

LEMMA 2. *If condition* (i) *is satisfied, and if we assume*

$(\mathrm{ii})_b$ *$h(y)$ is strictly increasing, and there exists a continuous non-increasing function $k_1(x)$ such that*

$$F(x)>h(-k_1(x))$$

and

$$k_1(x)+\int_0^x \frac{g(s)ds}{F(s)-h(-k_1(s))}\leqq M \qquad for \quad x\geqq 0,$$

where M is a positive constant, then there exists a point $N\in Y^-$ such that L_N^- does not cross C^+.

Proof. Let $y_0=M+k_1(0)$. Since $F(x)>h(-k_1(x))$, so $k_1(0)>0$, and $y_0>0$. Set point $N=(0, -y_0)$. Suppose that the equation of the section of the curve L_N^- under C^+ is $y=y(x)$. If L_N^- intersects C^+, because on $C^+F(x)=h(y)$ and $y(0)=-y_0<-k_1(0)<F(0)=0$, so there must exist $\bar{x}\in(0, \infty)$ such that $y(\bar{x})=-k_1(\bar{x})$, and $y(x)<-k_1(x)$, $0\leqq x\leqq \bar{x}$. This implies that

$$-k_1(\bar{x})=y(\bar{x})=-y_0+\int_0^{\bar{x}}\frac{g(s)ds}{F(s)-h(y(s))}.$$

Thus we have,

$$\begin{aligned}0&=-y_0+k_1(\bar{x})+\int_0^{\bar{x}}\frac{g(s)ds}{F(s)-h(y(s))}\\&\leqq -y_0+k_1(\bar{x})+\int_0^{\bar{x}}\frac{g(s)ds}{F(s)-h(-k_1(s))}\\&\leqq -y_0+M\\&=-k_1(0).\end{aligned}$$

This is a contradiction, and the Lemma is proved.

Remark. If $h(y)$ is strictly increasing, then the condition $(\mathrm{ii})_\mathrm{b}$ in Lemma 2 is necessary as well.

By Lemma 2, we can give concrete conditions on $h(y)$, $F(x)$ and $g(x)$, sc long as a concrete function $k_1(x)$ is given. For example, if we set $k_1(x)=-h^{-1}(-C)$, where C is a positive constant, then we can prove the following corollary.

COROLLARY 2. *If there exist constants M, $C>0$ such that*

$$F(x)+C>0$$

and

$$\int_0^x\frac{g(s)}{F(s)+C}ds\leqq M\qquad for\quad x\geqq 0,$$

then there exists a point $N\in Y^-$ such that L_N^- does not intersect C^+.

Suppose that there exists a strictly increasing function $h_1(y)$ which satisfies the following condition

$$h(y)\geqq h_1(y)\qquad \text{for}\quad y\geqq y_1\geqq 0. \tag{3}$$

Let $e(x)=h_1^{-1}(F(x))$, and

$$e^+(x)=\begin{cases}e(x) & \text{for}\quad e(x)\geqq y_1\\ y_1 & \text{for}\quad e(x)<y_1.\end{cases}$$

Let E^+ denote the set $\{(x, y): x\leqq 0, y>e^+(x)\}$.

LEMMA 3. $C^-\cap E^+=\varnothing$.

Proof. Let point $A(x, y)\in E^+$. If $F(x)\geqq h_1(y_1)$, since $y>e^+(x)=h_1^{-1}(F(x))\geqq y_1$, so $h_1(y)>F(x)$ and hence $h(y)\geqq h_1(y)>F(x)$, thus $A\overline{\in} C^-$.

If $F(x)<h_1(y_1)$, then $h_1^{-1}(F(x))<y_1$ and $e^+(x)=y_1$, since $y>y_1$, so $h(y)\geqq h_1(y)$

$\geqq h_1(y_1) > F(x)$, and $A \overline{\in} C^-$.

LEMMA 4. *If* $\lim_{y\to+\infty} h(y) = +\infty$, $\overline{\lim_{x\to-\infty}} F(x) = -a$, *where* $a>0$ *is a constant, then there exist constants* $b_1, b_2, b_3 > 0$ *such that*

$$E_1\{(x, y): x < -b_1, y > -b_2\} \cap C^- = \emptyset,$$
$$E_2\{(x, y): 0 \geqq x \geqq -b_1, y \geqq b_3\} \cap C^- = \emptyset.$$

Proof. There exist numbers $b_1, b_2 > 0$ such that $F(x) < -a$, for $x < -b_1$ and $h(y) > -a$ for $y > -b_2$. Let a point $A_1(x_1, y_1) \in E_1\{(x, y): x < -b_1, y > -b_2\}$, then $h(y_1) > -a > F(x_1)$, so $A_1(x_1, y_1) \overline{\in} C^-$.

There exists a number $b_3 > 0$ such that

$$h(y) > \max_{0 \geqq x \geqq -b_1} F(x) \qquad \text{for} \quad y > b_3.$$

If the point $A_2(x_2, y_2) \in E_2$, then $h(y_2) > \max_{0\geqq x \geqq -b_1} F(x) \geqq F(x_2)$, so $A_2 \overline{\in} C^-$.

LEMMA 5. *If the point* $N \in E^+$, *then the positive half-trajectory* L_N^+ *passing through* N *must intersect* Y^+.

Proof. From Lemma 3 it is clear that E^+ is above C^- and that E^+ is a connected set. Let the equation of L_N^+ on the left plane is $(x(t), y(t))$. Since $y(t)$ strictly increases at t increases, and $x(t)$ is strictly increases when L_N^+ is in E^+, we assert that L_N^+ will not escape from the set $\{(x, y): x_N \leqq x \leqq 0, y_N \leqq y\}$, and that L_N^+ must reach the set $E^*\{(x, y): x_N \leqq x \leqq 0, y^* \leqq y$, where $y^* = \inf_{x_N \leqq x \leqq 0} e(x)\}$. Onece L_N^+ enters E^*, it will not leave E^* unless it cross Y^+. Because in E^* $x(t)$ and $y(t)$ are strictly increasing, so if L_N^+ does not cross Y^+, we can prove that $x(t) \to a^* \leqq 0$ and $y(t) \to +\infty$ as $t \to +\infty$. But from the given conditions we have

$$\frac{dy}{dx} = \frac{-g(x)}{h(y) - F(x)}.$$

Since $x_N \leqq x \leqq 0$ along L_N^+, we have

$$\lim_{t\to+\infty} \left| \frac{dy}{dx} \right| \leqq \frac{|g(a^*)|}{\lim_{y\to+\infty} h(y)} = 0.$$

This is a contradiction.

LEMMA 6. *Under condition* (i), *if*

(iii)$_a$ *there exists a function* $h_1(y)$ *satisfying* (3) *and there exist a number* $x_0 \leqq 0$ *and a continuously differentiable function* $k_2(x)$ *with negative derivative such that*

$$h(k_2(x)) + \frac{g(x)}{k_2'(x)} \geqq F(x) \qquad \textit{for} \quad x < x_0 \leqq 0,$$

$$\lim_{x\to-\infty} k_2(x)>y_1, \quad \varliminf_{x\to-\infty}(h_1(k_2(x))-F(x))>0, \quad k_2(x_0)<0.$$

Then the positive half-trajectory L_N^+ passing through $N(0, -y_0)$, $0>-y_0\geqq k_2(x_0)$, must cross C^- and next cross Y^+.

Proof. If L_N^+ does not cross the line $x=x_0$, it must cross C^- and Y^+. If L_N^+ cross the line $x=x_0$ at the point P, then $y_P>k_2(x_0)$. It is easy to see that $y=k_2(x)$ is a solution of the system

$$\begin{cases} x'=h(y)-\left(h(k_2(x))+\dfrac{g(x)}{k_2'(x)}\right) \\ y'=-g(x). \end{cases}$$

From the comparison theorem, we can prove that L_N^+ must be located above the curve $y=k_2(x)$ for $x\leqq x_0$. Since

$$\varliminf_{x\to-\infty}(h_1(k_2(x))-F(x))>0,$$

L_N^+ must enter in E^+ when t is sufficiently large. By Lemma 5, we can prove this Lemma.

By Lemma 6, we can give concrete conditions on $h(y)$, $F(x)$ and $g(x)$, so long as concrete functions $k_2(x)$ and $h_1(y)$ are given. For example, if we set $h_1(y)=y/\lambda$ and $k_2(x)=CG(x)-M$ for $x\leqq 0$ with $M>0$ and $C=-1/h(-M)$, we can prove the following corollary.

COROLLARY 3. *If $h(y)\geqq y/\lambda$, $y\geqq 0$, $G(-\infty)=+\infty$ and there exists a number $M>0$ such that*

$$F(x)\leqq h\left(\frac{-G(x)}{h(-M)}-M\right)-h(-M) \qquad \text{for} \quad x\leqq 0,$$

$$\varliminf_{x\to-\infty}\left(\frac{-G(x)}{h(-M)}-\lambda F(x)\right)>M,$$

then the positive half-trajectory L_N^+ passing through $N(0, x_0)$, $0>y_0>-M$, must cross C^- and Y^+.

LEMMA 7. *If*

(iii)$_b$ $\varlimsup_{y\to-\infty} F(x)=-a<0$, *where $a>0$ is a constant and $\lim_{y\to+\infty} h(y)=+\infty$, then the positive half-trajectory L_N^+ passing through the point $N(0, y_0)$, $-b\leqq y_0<0$, must intersect Y^+, where $b=-\sup_{y<0}\{h(y)=-a\}$.*

Proof. From Lemma 4, there exist sets

$$E_1\{(x, y): x<-b_1,\ y>-b_2\},$$

and

$$E_2\{(x, y): -b_1\leqq x<0,\ y>b_3\},$$

such that $(E_1 \cup E_2) \cap C^- = \emptyset$. It is easy to see from the proof of Lemma 4 that $-b_2 > -b$. L_N^+ will not cross the line $y = -b_2$ and $x = -b_1$. If L_N^+ does not enter in E_2 it must cross Y^+. If L_N^+ enters in E_2, it will not escape from E_2 before it cross Y^+, and from Lemma 5, it must cross Y^+.

LEMMA 8. *Suppose that the condition* (i) *holds, and one of the conditions* (ii)$_a$ *and* (ii)$_b$ *holds. If*

(iv) $$\varlimsup_{x\to+\infty} (G(x)+F(x)) = +\infty,$$

then the positive half-trajectory L_N^+ passing through $N \in Y^+$ must cross C^+ and Y^-.

Proof. Suppose condition (i) and (ii)$_a$ hold. It follows from (ii)$_a$ that $F(x) \geqq A$, $A = \inf_{y \geqq k_1(0)} h(y)$. Let point $N_1 = (0, k_1(0))$. Since $L_{N_1}^-$ is located above the line $y = k_1(0)$ and under the curve $y = k_1(x)$ for $x > 0$, so L_N^+ must located under the line $y = y_N$ and above the line $y = k_1(0)$ before it escape from the right half plane.

Let the equation of L_N^+ be $(x(t), y(t))$ with $x(0) = 0$, $y(0) = y_N$. If $\varlimsup_{x\to+\infty} F(x) = +\infty$, then there exists a number $x^* > 0$ such that

$$F(x^*) > \max_{k_1(0) \leqq y \leqq y_N} h(y).$$

Thus, L_N^+ must be located on the left of the line $x = x^*$, because $x'(t) = h(y(t)) - F(x(t))$. If L_N^+ does not leave the region $R: 0 < x < x^*$, $k_1(0) < y < y_N$, there must exist a singular point of (1) and this is impossible, so L_N^+ must cross C^+ and Y^- to leave the region R.

If $\varlimsup_{x\to+\infty} F(x) < +\infty$, then $G(+\infty) = +\infty$. Let us consider the equation

$$\begin{cases} x' = h(y) - A, \\ y' = -g(x). \end{cases} \tag{4}$$

Let L^* denote the positive half-trajectory of (4) passing through the point N, and let $(x^*(t), y^*(t))$ be the solution of L^*. By the comparison theorem, it is easy to see that L_N^+ is located on the left of L^*. If L^* crosses C^+ and Y^-, then so does L_N^+. If L^* does not cross Y^- and $x^*(t)$ is bounded for $x > 0$, then L_N^+ would stay in the region $R^*: 0 \leqq x \leqq x^{**}$, $k_1(0) \leqq y \leqq y_N$, where $x^{**} > 0$ is a upper bound of $x^*(t)$, which implies that (1) has singular points in R^*. This is impossible.

If L^* does not cross Y^- and $x^*(t)$ is unbounded for $x > 0$, then there are points P_k, $k = 1, 2, \cdots$, in L^* such that $x_{P_k} \to +\infty$. Since

$$\frac{dy}{dx} = \frac{-g(x)}{h(y) - A}, \quad x > 0,$$

$$\frac{dy}{dx} \leqq \frac{-g(x)}{\max\limits_{k_1(0)\leq y\leq y_N} h(y)+A} \stackrel{\text{def}}{=\!=} b^*g(x), \quad x>0,$$

and it follows that

$$y_N - y_{P_k} = \int_{x_{P_k}}^{0} \frac{dy}{dx}dx \geqq -b^*G(x_{P_k}) \to +\infty, \qquad \text{as} \quad k\to+\infty.$$

This is a contradiction.

If the condition (i) and $(\mathrm{ii})_b$ hold, the proof is similar.

3. The Main Results.

THEOREM 1. *Suppose that conditions* (i), (iv), *one of* $(\mathrm{ii})_a$, $(\mathrm{ii})_b$ *and one of* $(\mathrm{iii})_a$, $(\mathrm{iii})_b$ *hold. If*

(v) $xF(x)<0$, *for* $0<|x|\ll 1$;

(vi) $k_2(x_0)\leqq k_1(0)$ *when condition* $(\mathrm{iii})_a$ *holds, or* $k_2(x_0)<-b$ *when condition* $(\mathrm{iii})_b$ *holds; then* (1) *has at least one limit cycle.*

Proof. The method of proof is to construct Poincare-Bendixson annular region. Consider $V(x, y)=G(x)+H(y)$, where $H(y)=\int_0^y h(s)ds$. It is obvious that $V(x, y)$ is definite positive in a sufficiently small neighborhood of (0, 0), and we have

$$V'_{(1)}(x, y)=-g(x)F(x)>0.$$

Thus, for sufficiently small c, the trajectory of (1) starting from the point on closed curve S_0: $V(x, y)=c$ go out of the interior region of S_0 at t increases. So we can take S_0 as the interior boundary.

Next, let us construct the exterior boundary. Take point $A=(0, k_2(x_0))$. From Lemma 1, 2, it follows that L_A^- does not cross C^+ and Y^+, and L_A^+ must cross C^-, and then cross Y^+ at point B. It is clear that $y_B>0$. By Lemma 6, L_B^+ must cross C^+, and then cross Y^- at point C. According to the uniqueness of trajectory of (1), we have $y_C>y_A$. We can take the closed curve $\widehat{ABC}\cup\overline{CA}$ as the exterior boundary. The theorem is proved.

THEOREM 2. *If*

1°. $xg(x)>0$, $x\neq 0$, $yh(y)>0$, $y\neq 0$, $xF(x)<0$ *for* $0<|x|\ll 1$;

2°. $F(x)\geqq h(CG(x)-M)+\frac{1}{C}$ *for* $x\leqq x_0<0$ *and* $x\geqq 0$, C, $M>0$;

3°. $\frac{h(y)}{y}\geqq\frac{1}{\lambda}$, $y\neq 0$, $\lambda>0$, $G(-\infty)=+\infty$, $\underline{\lim}_{x\to-\infty}(CG(x)-F(x))>M$;

4°. $\overline{\lim}_{x\to+\infty}(G(x)+F(x))=+\infty$;

then (1) *has at least one limit cycle.*

This theorem follows from Corollary 1, 2 and Theorem 1 immediately.

Suppose the strictly increasing function $h_2(y)$ satisfies the condition

$$h(y) \leqq h_2(y) \qquad \text{for} \quad 0 \leqq y_1 \leqq y. \tag{5}$$

Let the function $y=Q_2(x)$ be the inverse function of

$$x=a^{-1}\int_0^y h_2(s)ds, \quad \text{where } a>0 \text{ is a parameter.}$$

It is easy to see that

$$a^{-1}h_2(Q_2(G(x))=\frac{g(x)}{\frac{d}{dx}Q_2(G(x))}=\frac{1}{Q_2'(G(x))}. \tag{6}$$

LEMMA 9. *If* $h_2(y)$ *satisfies the condition* (5), *and*

$$F(x) \geqq (1+a^{-1})h_2(Q_2(G(x)) \qquad \text{for} \quad x \geqq x_1 \geqq 0, \tag{7}$$

then there exists a point $N \in Y^-$ *such that the negative half-trajectory* L_N^- *of* (1) *passing through* N *does not intersect* C^+.

Proof. From (6) (7) we have

$$\begin{aligned} F(x) &\geqq h_2(Q_2(G(x)))+a^{-1}h_2(Q_2(G(x))) \\ &= h_2(Q_2(G(x)))+\frac{g(x)}{\frac{d}{dx}Q_2(G(x))}. \end{aligned} \tag{8}$$

Since $\overline{\lim_{y\to+\infty}}\, h(y)=+\infty$, so $\lim_{y\to+\infty} h_2(y)=+\infty$. There exists a number $M>0$ such that

$$h_2(y) \geqq \max_{0\leqq y\leqq y_1} h(y) \qquad \text{for} \quad y \geqq M.$$

Now we will prove

$$h_2(Q_2(G(x))) \geqq h(Q_2(G(x))-M). \tag{9}$$

If $Q_2(G(x))-M \leqq 0$, then $h_2(Q_2(G(x)))>0 \geqq h(Q_2(G(x))-M)$. If $Q_2(G(x))-M \geqq y_1$, then

$$h_2(Q_2(G(x))) \geqq h_2(Q_2(G(x))-M) \geqq h(Q_2(G(x))-M).$$

If $0<Q_2(G(x))-M<y_1$, $M<Q_2(G(x))<M+y_1$,

$$h_2(Q_2(G(x)))>h_2(M) \geqq \max_{0\leqq y\leqq y_1} h(y) \geqq h(Q_2(G(x))-M).$$

Thus, (9) holds. From (8) and (9) we have

$$F(x)\geqq h(Q_2(G(x))-M)+\frac{g(x)}{\frac{d}{dx}(Q_2(G(x))-M)}.$$

By Lemma 1 we can compleat the proof.

COROLLARY 4. *If condition* (i) *holds and there exist positive constants* α_1, β_1, x_1, y_1 *such that*

$$h(y)\leqq\alpha_1 y^{\beta_1} \qquad for \quad y\geqq y_1>0,$$

$$F(x)\geqq a_1 G^{\beta_1/(1+\beta_1)}(x) \qquad for \quad x\geqq x_1>0,$$

where

$$a_1=(1+\beta_1)\left(\frac{1+\beta_1}{\beta_1}\right)^{\beta_1/(1+\beta_1)}\alpha_1{}^{1/(1+\beta_1)},$$

then there exists a point $N\in Y^-$ *such that the negative half-trajectory* L_N^- *does not intersect* C^+.

Proof. Taking $h_2(y)=\alpha_1 y^{\beta_1}$, $a=\frac{1}{\beta_1}$, it is easy to verify that

$$Q_2(x)=\left(\frac{1+\beta_1}{\beta_1}\right)^{1/(1+\beta_1)}\alpha_1{}^{-1/(1+\beta_1)}x^{1/(1+\beta_1)},$$

and

$$\begin{aligned}(1+a^{-1})h_2(Q_2(G(x))&=(1+\beta_1)\alpha_1\left(\frac{1+\beta_1}{\beta_1}\right)^{\beta_1/(1+\beta_1)}\alpha_1{}^{-\beta_1/(1+\beta_1)}G^{\beta_1/(1+\beta_1)}(x)\\&=a_1 G^{\beta_1/(1+\beta_1)}(x).\end{aligned}$$

Hence, the corollary is proved from Lemma 9.

Suppose the strictly increasing function $h_1(y)$ satisfies the condition (3) and $h_1(+\infty)=+\infty$. Let the function $y=Q_1(x)$ be the inverse function of $x=a^{-1}\int_0^y h_1(s)ds$. It is easy to see that

$$a^{-1}h_1(Q_1(G(x))=\frac{g(x)}{\frac{d}{dx}Q_1(G(x))}=\frac{1}{Q_1'(G(x))}.$$

LEMMA 10. *If* $h_1(y)$ *satisfies the condition* (3), *and*

$$F(x)\leqq(1+a^{-1})h_1(Q_1(G(x))) \qquad for \quad x<x_0\leqq 0,$$

$$\varlimsup_{x\to-\infty}(-F(x)+G(x))=+\infty,$$

then the positive half-trajectory L_N^+ *with* $N\in Y^-$ *must cross* C^- *and* Y^+.

Proof. If $\overline{\lim_{x\to-\infty}} F(x)=-\infty$, the Lemma can be proved by Lemma 7. If $\overline{\lim_{x\to-\infty}} F(x)=c>-\infty$, then $G(-\infty)=+\infty$ and for any $M>0$ there exists a $\bar{x}_0<x_0$ such that

$$\begin{aligned} F(x) &\leqq h_1(Q_1(G(x))-M)+a^{-1}h_1(Q_1(G(x))) \\ &= h_1(Q_1(G(x))-M)+\frac{g(x)}{\frac{d}{dx}(Q_1(G(x)))} \\ &\leqq h(Q_1(G(x))-M)+\frac{g(x)}{\frac{d}{dx}(Q_1(G(x))-M)} \qquad \text{for} \quad x<\bar{x}_0<x_0\leqq 0. \end{aligned}$$

Taking $k_2(x)=Q_1(G(x))-M$, we have

$$\lim_{x\to-\infty}(h_1(k_2(x))-F(x))>0,$$

so from Lemma 6, this Lemma is proved.

COROLLARY 5. *Under condition* (i), *if* $\overline{\lim_{x\to-\infty}}(-F(x)+G(x))=+\infty$ *and there exist constants* α_2, β_2, y_1, $x_2>0$ *such that*

$$h(y)\geqq\alpha_2 y^{\beta_2} \qquad \textit{for} \quad y\geqq y_1>0,$$

$$F(x)\leqq b_1 G^{\beta_2/(1+\beta_2)}(x) \qquad \textit{for} \quad x\leqq -x_2\leqq 0,$$

where

$$b_1=(1+\beta_2)\left(\frac{1+\beta_2}{\beta_2}\right)^{\beta_2/(1+\beta_2)}\alpha_2^{1/(1+\beta_2)};$$

then the positive half-trajectory L_N^+ *passing through any point* $N\in Y^-$ *must cross* C^- *and* Y^+.

The following Theorem follows from Lemma 8, 9 and 10.

THEOREM 3. *If*

1°. $xg(x)>0$, $x\neq 0$, $yh(y)>0$, $y\neq 0$, $xF(x)<0$ *for* $0<|x|\ll 1$;

2°. *there exist strictly increasing function* $h_1(y)$, $h_2(y)$ *such that*

$$h_2(y)\geqq h(y)\geqq h_1(y) \qquad \textit{for} \quad 0\leqq y_1\leqq y;$$

3°. $F(x)\leqq(1+a_1^{-1})h_1(Q_1(G(x))) \qquad \textit{for} \quad x\leqq x_1\leqq 0,$

$F(x)\geqq(1+a_2^{-1})h_2(Q_2(G(x))) \qquad \textit{for} \quad x\geqq x_2\geqq 0;$

4°. $\overline{\lim_{x\to\infty}}(F(x)\operatorname{sgn}x+G(x))=+\infty$;

then (1) *has at least one limit cycle.*

COROLLARY 6. *If*

1°. $xg(x)>0,\ x\neq 0,\ yh(y)>0,\ y\neq 0,\ xF(x)<0$ *for* $0<|x|\ll 1$;

2°. $\overline{\lim\limits_{x\to\infty}}\,(F(x)\operatorname{sgn} x+G(x))=+\infty$;

3°. *there exist positive constants* $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ y_1,\ x_1,\ x_2$ *such that*

$$\alpha_1 y^{\beta_1}\geqq h(y)\geqq \alpha_2 y^{\beta_2} \qquad \text{for} \quad y\geqq y_1\geqq 0,$$

$$F(x)\geqq a_1 G^{\beta_1/(1+\beta_1)}(x) \qquad \text{for} \quad 0\leqq x_1\leqq x,$$

$$F(x)\leqq a_2 G^{\beta_2/(1+\beta_2)}(x) \qquad \text{for} \quad 0\geqq -x_2\geqq x,$$

where

$$a_i=(1+\beta_i)\Big(\frac{1+\beta_i}{\beta_i}\Big)^{\beta_i/(1+\beta_i)}\alpha_i^{1/(1+\beta_i)}, \qquad i=1,2;$$

then (1) *has at least one limit cycle.*

REFERENCES

[1] J.R. GRAEF, On the Generalized Liénard Equation with Negative Damping, J. Differential Equations **12** (1972), 34-62.

[2] G. VILLARI, On the existence of periodic solutions for Liénard's equations, Nonl. Anal., Th. Meth. Appl. **7** (1983), 71-78.

[3] G. VILLARI, On the Qualitative Behaviour of Solutions of Liénard Equations, J. Diff. Eqs. **67** (1987), 269-277.

[4] G. VILLARI, Periodic Solutions of the Liénard Equations with One Sided Growth Restrictions, J. Diff. Eqs. **67** (1987), 278-293.

[5] ZHOU YURONG, Existence and Uniqueness of Limiting Cycle of Equations $x'=\varphi(y)-F(x)$, $y'=-g(x)$, and its Existence of Two and only Two Limiting Cycle, Chin. Ann. of Math. **3**(1) 1982, 89-102.

[6] WANG XIAN, The Existence of Limit Cycles for a Class of Liénard Equations, Chin. Ann. of Math. **4A**(6) 1983, 724-730.

[7] ZHU DEMING, Existence of the Limit Cycle and Global Behavior of the Solutions of Liénard Equation, Chin. Ann. of Math. **7A**(5) 1986, 536-546.

MATHEMATICS DEPARTMENT
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN, JILIN
PEOPLE'S REPUBLIC OF CHINA