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ON THE THEOREM OF TUMURA-CLUNIE

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1. Introduction and Main Results.

Let f be a nonconstant meromorphic function in the complex plane. It is assumed that the reader is familiar with the notations of Nevanlinna theory (see, for example [3]). We denote by $S(r, f)$ any function satisfying $S(r, f)= o(T(r, f))$ as $r\to+\infty$, possibly outside a set E of finite linear measure. Throughout this paper we denote by $a_j(z)$ meromorphic functions which satisfying $T(r, a_j)=S(r, f)$ $(j=0, 1, \cdots, n)$. If $a_n\not\equiv 0$, we call

$$P[f]=a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0$$

a polynomial in f with degree n. If $n_0, n_1, \cdots, n_k$ are nonnegative integers, we call

$$M[f]=f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k} \tag{1}$$

a differential monomial in f of degree $\gamma_M=n_0+n_1+\cdots+n_k$ and of weight $\Gamma_M =n_0+2n_1+\cdots+(k+1)n_k$. If $M_1, \cdots, M_n$ are differential monomials in f, we call

$$Q[f]=\sum_{j=1}^{n}a_j(z)M_j[f] \tag{2}$$

a differential polynomial in f, and define the degree γ_Q and the weight Γ_Q by $\gamma_Q=\max_{j=1}^{n}\gamma_{M_j}$ and $\Gamma_Q=\max_{j=1}^{n}\Gamma_{M_j}$ respectively. If Q is a differential polynomial, then Q' denotes the differential polynomial which satisfies $Q'[f(z)]=\frac{d}{dz}Q[f(z)]$ for any meromorphic function f. (See, for example, Mues and Steinmetz [4, P 115]).

The following theorem was first stated by Tumura [6] and proved completely by Clunie [1]:

THEOREM A. *Let f and g be entire functions, and*

$$F=a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0 \qquad (a_n\not\equiv 0). \tag{3}$$

If $F=be^g$, where $b(z)$ is a meromorphic function satisfying $T(r, b)=S(r, f)$, then

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$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

Hayman proved the following theorem:

THEOREM B (see [3, P 69-70]). *Suppose that f is meromorphic and not constant in the plane, that*

$$F=a_nf^n+a_{n-1}f^{n-1}+Q[f] \tag{4}$$

where $Q[f]$ is a differential polynomial of degree at most $n-2$ in f. If $N(r, f)+N\left(r, \frac{1}{F}\right)=S(r, f)$, then

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

Mues and Steinmetz have given the following theorem:

THEOREM C (see [4]). *Let f be a nonconstant meromorphic function. Suppose that F is given by* (3). *If $\overline{N}(r, f)+\overline{N}\left(r, \frac{1}{F}\right)=S(r, f)$, then*

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

Toda proved the following theorem:

THEOREM D (see [5]). *Let f be a nonconstant meromorphic function. Suppose that F is given by* (3). *If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}}\frac{\overline{N}\left(r, \frac{1}{F}\right)+2\overline{N}(r, f)}{T(r, f)}<\frac{1}{2},$$

then

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

Recently Weissenborn has given the following theorem:

THEOREM E (see [7]). *Let f be a nonconstant meromorphic function. Suppose that F is given by* (3). *Then either*

$$T(r, f)\leqq\overline{N}\left(r, \frac{1}{F}\right)+\overline{N}(r, f)+S(r, f),$$

or

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

In this paper we improve the above results and obtain the following:

THEOREM. *Suppose that f is a nonconstant meromorphic function, that F is given by* (4). *If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}\left(r, \frac{1}{F}\right)+\alpha\overline{N}(r, f)+\overline{N}\left(r, \left(f+\frac{a_{n-1}}{na_n}\right)^{-1}\right)}{T(r, f)}<2,$$

where $\alpha=\max\{1, \Gamma_Q+3-n\}$, Γ_Q *is the weight of* $Q[f]$, *then*

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

The proof of the Theorem is left to §4. In the special case that F is given by (3), our result is

COROLLARY. *Suppose that f is a nonconstant meromorphic function and that F is given by* (3). *If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}\left(r, \frac{1}{F}\right)+\overline{N}(r, f)+\overline{N}\left(r, \left(f+\frac{a_{n-1}}{na_n}\right)^{-1}\right)}{T(r, f)}<2$$

then

$$F=a_n\left(f+\frac{a_{n-1}}{na_n}\right)^n.$$

The above Corollary improves Theorems A, C, D and E. To illustrate our results we give an example.

Let $f(z)=e^z$, $F=f^n+f^{n-2}(n\geqq 2)$, we can easily verify

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}\left(r, \frac{1}{F}\right)+\alpha\overline{N}(r, f)+\overline{N}\left(r, \left(f+\frac{a_{n-1}}{na_n}\right)^{-1}\right)}{T(r, f)}=2.$$

This example shows that our results are sharp.

2. Some Lemmas.

The following four lemmas will be needed in the proof of our Theorem.

LEMMA 1 (see [2]). *Let f be a meromorphic function, and $Q[f]$ be a differential polynomial in f of degree γ_Q. Then*

$$m(r, Q[f])\leqq\gamma_Q m(r, f)+S(r, f).$$

LEMMA 2. *Suppose that $M[f]$ is given by* (1). *If f has a pole at $z=z_0$ of order p, then z_0 is a pole of $M[f]$ of order* $(p-1)\gamma_M+\Gamma_M$.

Proof. Obviously, the order of $M[f]$ at the pole z_0 is

$$pn_0+(p+1)n_1+\cdots+(p+k)n_k=(p-1)\gamma_M+\Gamma_M.$$

LEMMA 3. *Suppose that $Q[f]$ is given by* (2). *Let z_0 be a pole of f of order p, and not a zero nor a pole of coefficients of $Q[f]$. Then z_0 is a pole of $Q[f]$ of order at most $p\gamma_Q+(\Gamma_Q-\gamma_Q)$.*

Proof. By Lemma 2, z_0 is a pole of $M_j[f]$ of order $(p-1)\ \gamma_{M_j}+\Gamma_{M_j}$ $(j=1, 2, \cdots, n)$. Therefore, z_0 is a pole of $Q[f]$ of order at most

$$\max\{(p-1)\gamma_{M_j}+\Gamma_{M_j}\}\leqq(p-1)\gamma_Q+\Gamma_Q=p\gamma_Q+(\Gamma_Q-\gamma_Q),$$

which proves Lemma 3.

LEMMA 4. *Let f be a nonconstant meromorphic function, and $F=f^n+Q[f]$, where $Q[f]$ is a differential polynomial in f of degree γ_Q and of weight Γ_Q. If $Q[f]\not\equiv 0$, then*

$$(n-\gamma_Q)T(r, f)\leqq\overline{N}\Big(r, \frac{1}{F}\Big)+\overline{N}\Big(r, \frac{1}{f}\Big)+(\Gamma_Q-\gamma_Q+1)\overline{N}(r, f)+S(r, f).$$

The proof of Lemma 4 is given in § 3.

3. Proof of Lemma 4.

If $n\leqq\gamma_Q$, the conclusion of Lemma 4 holds obviously. In the following we suppose $n>\gamma_Q$. By $F=f^n+Q[f]$, we have

$$F'=\frac{F'}{F}f^n+\frac{F'}{F}Q[f], \qquad F'=nf^{n-1}f'+Q'[f],$$

and hence

$$f^n\Big(\frac{F'}{F}-\frac{nf'}{f}\Big)=Q[f]\Big(\frac{Q'[f]}{Q[f]}-\frac{F'}{F}\Big).$$

Let

$$\Omega_1[f]=\frac{F'}{F}-\frac{nf'}{f}, \qquad \Omega_2[f]=Q[f]\Big(\frac{Q'[f]}{Q[f]}-\frac{F'}{F}\Big).$$

Then

$$f^n\Omega_1[f]=\Omega_2[f]. \tag{5}$$

If $\Omega_1[f]\equiv 0$, then $\Omega_2[f]\equiv 0$. By integration we get

$$f^n=cQ[f] \qquad (c\neq 0),$$

and hence

$$T(r, f^n)=T(r, Q[f])+O(1),$$

that is

$$nT(r, f)=m(r, Q[f])+N(r, Q[f])+O(1).$$

Using Lemma 1 and Lemma 3, we have

$$m(r, Q[f])\leqq\gamma_Q m(r, f)+S(r, f),$$
$$N(r, Q[f])\leqq\gamma_Q N(r, f)+(\Gamma_Q-\gamma_Q)\bar{N}(r, f)+S(r, f).$$

From the above we get

$$(n-\gamma_Q)T(r, f)\leqq(\Gamma_Q-\gamma_Q)\bar{N}(r, f)+S(r, f),$$

the conclusion of Lemma 4 holds. In the following we suppose that $\Omega_1[f]\not\equiv 0$. Noting $\Omega_1[f]=\frac{F'}{F}-\frac{nf'}{f}$, we have $m(r, \Omega_1[f])=S(r, f)$. From

$$\Omega_2[f]=Q[f]\left(\frac{Q'[f]}{Q[f]}-\frac{F'}{F}\right),$$

we have

$$m(r, \Omega_2[f])\leqq m(r, Q[f])+S(r, f)$$
$$\leqq\gamma_Q m(r, f)+S(r, f),$$

using Lemma 1. By (5) we have $f^n=\frac{\Omega_2[f]}{\Omega_1[f]}$, and hence

$$m(r, f^n)\leqq m(r, \Omega_2[f])+m\left(r, \frac{1}{\Omega_1[f]}\right),$$

that is

$$nm(r, f)\leqq\gamma_Q m(r, f)+m\left(r, \frac{1}{\Omega_1[f]}\right)+S(r, f).$$

Again by the first fundamental theorem (see [3]), we get

$$m\left(r, \frac{1}{\Omega_1[f]}\right)=N(r, \Omega_1[f])-N\left(r, \frac{1}{\Omega_1[f]}\right)+S(r, f).$$

Obviously, a pole of $\Omega_1[f]$ occurs at one of the zeros of F and f, poles of f, zeros and poles of coefficients of $Q[f]$. Let z_0 be a pole of f of order p, and not a zero nor a pole of coefficients of $Q[f]$. Then z_0 is a pole of f^n of order pn. From Lemma 3 we know that z_0 is a pole of $\Omega_2[f]$ of order at most $p\gamma_Q+(\Gamma_Q-\gamma_Q)+1$. If z_0 is a pole of $\Omega_1[f]$, since $\Omega_1[f]=\frac{\Omega_2[f]}{f^n}$, z_0 is the pole of $\Omega_1[f]$ of order at most

$$p\gamma_Q+(\Gamma_Q-\gamma_Q+1)-pn=(\Gamma_Q-\gamma_Q+1)-p(n-\gamma_Q).$$

If z_0 is not a pole of $\Omega_1[f]$, since

$$\frac{1}{\Omega_1[f]}=\frac{f^n}{\Omega_2[f]}$$

z_0 is a zero of $\Omega_1[f]$ of order at least

$$pn-\{p\gamma_Q+(\Gamma_Q-\gamma_Q)+1\}=p(n-\gamma_Q)-(\Gamma_Q-\gamma_Q+1).$$

Hence we have

$$N(r, \Omega_1[f])-N\left(r, \frac{1}{\Omega_1[f]}\right)\leqq \bar{N}\left(r, \frac{1}{F}\right)+\bar{N}\left(r, \frac{1}{f}\right)$$
$$+(\Gamma_Q-\gamma_Q+1)\bar{N}(r, f)-(n-\gamma_Q)N(r, f)+S(r, f).$$

From the above we get

$$(n-\gamma_Q)T(r, f)\leqq \bar{N}\left(r, \frac{1}{F}\right)+\bar{N}\left(r, \frac{1}{f}\right)+(\Gamma_Q-\gamma_Q+1)\bar{N}(r, f)+S(r, f).$$

This completes the proof of Lemma 4.

4. Proof of the Theorem.

Let $g=f+\frac{a_{n-1}}{na_n}$, and $G=\frac{F}{a_n}$; then

$$G=g^n+Q^*[g],$$

where $Q^*[g]$ is a differential polynomial in g of degree γ_{Q^*} and of weight Γ_{Q^*}. Obviously,

$$\gamma_{Q^*}\leqq n-2$$
$$\Gamma_{Q^*}\leqq \max\{n-2, \Gamma_Q\}.$$

If $\Gamma_Q>n-2$, then $\Gamma_{Q^*}=\Gamma_Q$, $\alpha=\Gamma_Q+3-n$. If $\Gamma_Q\leqq n-2$, then $\Gamma_{Q^*}\leqq n-2$, $\alpha=1$. Therefore, $\Gamma_{Q^*}-\gamma_{Q^*}+1\leqq \alpha+(n-2-\gamma_{Q^*})$.

Suppose $Q^*[g]\not\equiv 0$. From Lemma 4, we have

$$\begin{aligned}(n-\gamma_{Q^*})T(r, g)&\leqq \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}\left(r, \frac{1}{g}\right)+(\Gamma_{Q^*}-\gamma_{Q^*}+1)\bar{N}(r, g)+S(r, g)\\ &\leqq \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}\left(r, \frac{1}{g}\right)+\alpha\bar{N}(r, g)\\ &\quad +(n-2-\gamma_{Q^*})\bar{N}(r, g)+S(r, g)\\ &\leqq \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}\left(r, \frac{1}{g}\right)+\alpha\bar{N}(r, g)\\ &\quad +(n-2-\gamma_{Q^*})T(r, g)+S(r, g).\end{aligned}$$

Thus, we have

$$2T(r, g)\leqq \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}\left(r, \frac{1}{g}\right)+\alpha\bar{N}(r, g)+S(r, g).$$

Noting

$$T(r, g)=T(r, f)+S(r, f),$$

$$\bar{N}\left(r, \frac{1}{G}\right)=\bar{N}\left(r, \frac{1}{F}\right)+S(r, f),$$

$$\bar{N}\left(r, \frac{1}{g}\right)=\bar{N}\left(r, \left(f+\frac{a_{n-1}}{n a_n}\right)^{-1}\right),$$

$$\bar{N}(r, g)=\bar{N}(r, f)+S(r, f),$$

we get

$$2T(r, f) \leqq \bar{N}\left(r, \frac{1}{F}\right)+\alpha \bar{N}(r, f)+\bar{N}\left(r, \left(f+\frac{a_{n-1}}{n a_n}\right)^{-1}\right)+S(r, f).$$

So

$$\limsup_{\substack{r \to \infty \\ r \notin E}} \frac{\bar{N}\left(r, \frac{1}{F}\right)+\alpha \bar{N}(r, f)+\bar{N}\left(r, \left(f+\frac{a_{n-1}}{n a_n}\right)^{-1}\right)}{T(r, f)} \geqq 2,$$

which is a contradiction. This shows that $Q^*[g] \equiv 0$, that is

$$F=a_n\left(f+\frac{a_{n-1}}{n a_n}\right)^n.$$

The Theorem is thus proved.

References

[1] J. Clunie, On integral and meromorphic functions, J. London Math. Soc., **37** (1962), 17-27.

[2] W. Doeringer, Exceptional values of differential polynomials, Pacific J. Math. **98** (1982), 55-62.

[3] W.K. Hayman, Meromorphic Functions, Oxford, Clarendon Press, 1964.

[4] E. Mues and N. Steinmetz, The theorem of Tumura-Clunie for meromorphic functions, J. London Math. Soc. **23** (1981), 113-122.

[5] N. Toda, On an extension of the Theorem of Tumura-Clunie, Contemporary Mathematics, **25** (1983), 215-219.

[6] Y. Tumura, On the extensions of Borel's theorem and Saxer-Csillag's theorem, Proc. Phys. Math. Soc. Japan **19** (1937), 29-35.

[7] G. Weissenborn, On the Theorem of Tumura and Clunie, Bull. London Math. Soc. **18** (1986), 371-373.

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