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TOPOLOGICAL TYPES OF COMPLEX ISOLATED HYPERSURFACE SINGULARITIES

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§1. Introduction.

Let $f: \boldsymbol{C}^n, 0 \to \boldsymbol{C}, 0$ be a holomorphic function germ which has an isolated critical point at the origin ($n \geqq 2$). Then there are several ways to define a topological type of f. Let g be another holomorphic function germ with an isolated critical point at the origin.

DEFINITION 1. f and g are *topologically right equivalent* if there is a homeomorphism germ $\varphi: \boldsymbol{C}^n, 0 \to \boldsymbol{C}^n, 0$ satisfying $f = g \circ \varphi$.

DEFINITION 2. f and g are *topologically right-left equivalent* if there are homeomorphism germs $\varphi: \boldsymbol{C}^n, 0 \to \boldsymbol{C}^n, 0$ and $\phi: \boldsymbol{C}, 0 \to \boldsymbol{C}, 0$ satisfying $f = \phi \circ g \circ \varphi$.

Set $V_f = f^{-1}(0)$ and $V_g = g^{-1}(0)$, which are germs of complex analytic varieties at the origin.

DEFINITION 3. f and g are *topologically V-equivalent* if there is a homeomorphism germ $\varphi: \boldsymbol{C}^n, 0 \to \boldsymbol{C}^n, 0$ satisfying $\varphi(V_f) = V_g$.

Set $D_\varepsilon^{2n} = \{z \in \boldsymbol{C}^n ; \|z\| \leqq \varepsilon\}$ and $S_\varepsilon^{2n-1} = \partial D_\varepsilon^{2n}$ for $\varepsilon > 0$. Then by Milnor [9] $S_\varepsilon^{2n-1} \cap V_f$ is a smooth $(2n-3)$-dimensional manifold for $\varepsilon > 0$ sufficiently small. The pair $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is called the *link* of the singularity of f.

DEFINITION 4. f and g are *link equivalent* if $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is homeomorphic to $(S_{\varepsilon'}^{2n-1}, S_{\varepsilon'}^{2n-1} \cap V_g)$ for all sufficiently small ε and ε'.

Our problem is whether these definitions are equivalent or not. By the definitions, the right equivalence implies the right-left equivalence, which in turn implies the V-equivalence. Furthermore, since $(D_\varepsilon^{2n}, D_\varepsilon^{2n} \cap V_f)$ is homeomorphic to the cone over the link $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ ([9]), the link equivalence obviously implies the V-equivalence. The first non-trivial result was obtained by King [5]: the right-left equivalence is equivalent to the link equivalence for $n \neq 3$. After that Perron [12] showed that it also holds for $n = 3$. Our main result of this paper is the following.

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THEOREM 1. *If two holomorphic function germs with isolated critical points at the origin are topologically V-equivalent, then they are link equivalent.*

Combining this with the results of King and Perron, we obtain the following immediately.

COROLLARY 2. *Let f and g be holomorphic function germs with isolated critical points at the origin. Then the following three are equivalent.*

(a) *f and g are topologically right-left equivalent.*
(b) *f and g are topologically V-equivalent.*
(c) *f and g are link equivalent.*

Remark. The real analogue of Corollary 2 does not hold in general. In fact, for all $n \geqq 7$ King [4] gives an example of two real polynomial functions $f, g: \boldsymbol{R}^n, 0 \to \boldsymbol{R}, 0$ which are topologically V-equivalent but not topologically right-left equivalent, nor link equivalent. (Note that even in the real case the link equivalence implies the right-left equivalence [4].) Furthermore it is easy to show that the right-left equivalence does not imply the right equivalence in the real case. For example, consider x^2+y^2 and $-(x^2+y^2)$.

Our proof of Theorem 1 is divided into three cases. When $n=2$, we use results of Burau [1] and Zariski [15] concerning links of singularities in $\boldsymbol{C}^2$. When $n \geqq 4$, we use the topological h-cobordism theorem together with the existence and uniqueness theorem of topological normal disk bundles for codimension two locally flat embeddings [6]. When $n=3$, we need an additional lemma due to Perron [12].

In § 3 we show that the homeomorphism which gives the V-equivalence can actually be made to be a diffeomorphism except possibly at the origin. Some relations between the right equivalence and the right-left equivalence are also discussed in § 3.

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§ 2. Proof of theorem 1.

Let f and g be as in § 1, and suppose they are topologically V-equivalent. We shall show $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is homeomorphic to $(S_{\varepsilon'}^{2n-1}, S_{\varepsilon'}^{2n-1} \cap V_g)$.

Case 1. $n=2$.

Let $\Delta_f(t)$ (resp. $\Delta_g(t)$) be the characteristic polynomial for the Milnor fibration of f (resp. g) [9]. Then Lê [8] shows that if f and g are V-equivalent, $\Delta_f(t)=\Delta_g(t)$. Furthermore, Burau [1] shows that if f and g are irreducible and $\Delta_f(t)=\Delta_g(t)$, $(S_\varepsilon^3, S_\varepsilon^3 \cap V_f)$ is homeomorphic to $(S_{\varepsilon'}^3, S_{\varepsilon'}^3 \cap V_g)$.

Now let $f=f_1f_2\cdots f_r$ and $g=g_1g_2\cdots g_s$ be irreducible factorizations. Since f and g are V-equivalent, we have $r=s$ and may assume f_i and g_i are V-equivalent. Set $K_i=S_\varepsilon^3\cap V_{f_i}$ and $K_i'=S_{\varepsilon'}^3\cap V_{g_i}$, then (S_ε^3, K_i) is homeomorphic to $(S_{\varepsilon'}^3, K_i')$ by the above facts. (Note that $S_\varepsilon^3\cap V_f=\cup K_i$ and $S_{\varepsilon'}^3\cap V_g=\cup K_i'$.) Then Zariski [15] shows that $(S_\varepsilon^3, S_\varepsilon^3\cap V_f)$ is homeomorphic to $(S_{\varepsilon'}^3, S_{\varepsilon'}^3\cap V_g)$ if and only if the linking number of K_i and K_j is the same as that of K_i' and K_j' $(1\leqq i,\ j\leqq r;\ i\neq j)$. The linking number of K_i and K_j in $S_\varepsilon^3=\partial D_\varepsilon^4$ is defined to be the algebraic intersection number of C_i and C_j in D_ε^4, where C_i (resp. C_j) is a 2-chain in D_ε^4 with $\partial C_i=K_i$ (resp. $\partial C_j=K_j$). Thus V-equivalence obviously implies that the corresponding linking numbers are the same. Hence the links $(S_\varepsilon^3, S_\varepsilon^3\cap V_f)$ and $(S_{\varepsilon'}^3, S_{\varepsilon'}^3\cap V_g)$ are homeomorphic.

Case 2. $n\geqq 4$.

Since f and g are topologically V-equivalent, there is a homeomorphism germ $\varphi: \boldsymbol{C}^n, 0\to \boldsymbol{C}^n, 0$ satisfying $\varphi(V_f)=V_g$. Take sufficiently small ε, $\varepsilon'>0$ such that $\varphi(D_\varepsilon^{2n})\subset \operatorname{Int} D_{\varepsilon'}^{2n}$. Set $K_f=S_\varepsilon^{2n-1}\cap V_f$, $K_g=S_{\varepsilon'}^{2n-1}\cap V_g$ and $W=V_g\cap(D_{\varepsilon'}^{2n}-\operatorname{Int}\varphi(D_\varepsilon^{2n}))$ (Fig. 1). Using the cone structures of V_f and V_g at the origin ([9]), we see easily that W is an invertible cobordism between $\varphi(K_f)$ and K_g; hence, it is an h-cobordism (e.g. see [5]). Since K_f and K_g are simply connected ([9]), W is homeomorphic to the product $K_g\times I$ by the topological h-cobordism theorem ([7]).

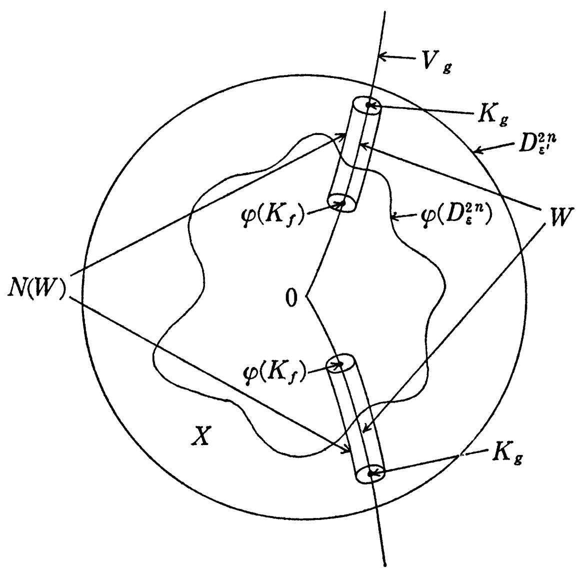

Figure 1.

Let $N(K_f)$ (resp. $N(K_g)$) be the tubular disk neighborhood of K_f (resp. K_g) in S_ε^{2n-1} (resp. $S_{\varepsilon'}^{2n-1}$). By the existence theorem of tubular disk neighborhoods for codimension two locally flat embeddings [6] we can extend $\varphi(N(K_f))$ and $N(K_g)$ to a tubular disk neighborhood $N(W)$ of W in $D_{\varepsilon'}^{2n}-\mathrm{Int}\,\varphi(D_{\varepsilon}^{2n})$. Since W is homeomorphic to the product $K_g\times I$, $N(W)$ is also homeomorphic to the product $N(K_g)\times I$. Set $E_g=S_{\varepsilon'}^{2n-1}-\mathrm{Int}\,N(K_g)$, $E_f=S_{\varepsilon}^{2n-1}-\mathrm{Int}\,N(K_f)$ and $X=Cl(D_{\varepsilon'}^{2n}-(\varphi(D_{\varepsilon}^{2n})\cup N(W)))$, where Cl denotes the closure in $\boldsymbol{C}^n$. Then X is a cobordism relative to the boundary between E_g and $\varphi(E_f)$. Using the uniqueness of tubular disk neighborhoods in codimension two [6], we see easily that X is an invertible cobordism; hence, it is an h-cobordism. Since the fundamental groups $\pi_1(E_g)$ and $\pi_1(E_f)$ are isomorphic to the infinite cyclic group $\boldsymbol{Z}$ ([9]) and the Whitehead group of $\boldsymbol{Z}$ vanishes [3], the h-cobordism X is in fact an s-cobordism. Hence by the topological s-cobordism theorem [7], there is a homeomorphism between X and $E_g\times I$ which extends the product structure on the boundary. Thus there is a homeomorphism η from $\varphi(E_f)\cup\varphi(N(K_f))=\varphi(S_{\varepsilon}^{2n-1})$ to $E_g\cup N(K_g)=S_{\varepsilon'}^{2n-1}$ such that $\eta(\varphi(K_f))=K_g$. Hence $(S_{\varepsilon}^{2n-1}, K_f)$ is homeomorphic to $(S_{\varepsilon'}^{2n-1}, K_g)$.

Case 3. $n=3$.

Since we do not know whether the invertible cobordism W above is homeomorphic to the product, the above argument does not apply to this case. Instead, we use the following lemma due to Perron.

LEMMA 3. ([12]) *Let $f, g:\boldsymbol{C}^3, 0\to\boldsymbol{C}, 0$ be holomorphic function germs with isolated critical points at the origin. Let F_f (resp. F_g) be the closure in S_{ε}^5 (resp. $S_{\varepsilon'}^5$) of a fiber of the Milnor fibration of f (resp. g). Suppose there is a topological embedding $\xi:F_f\to F_g$ with the following properties.*

(1) *$\xi(F_f)\subset\mathrm{Int}\,F_g$ and $F_g-\mathrm{Int}\,\xi(F_f)$ is an h-cobordism between ∂F_g and $\xi(\partial F_f)$.*
(2) *The induced homomorphism $\xi_*:H_2(F_f;\boldsymbol{Z})\to H_2(F_g;\boldsymbol{Z})$ is an isomorphism which preserves the Seifert forms of f and g.*

Then the links $(S_{\varepsilon}^5, S_{\varepsilon}^5\cap V_f)$ and $(S_{\varepsilon'}^5, S_{\varepsilon'}^5\cap V_g)$ are homeomorphic. (In fact, they are diffeomorphic.)

(For the definition of Seifert forms, see [12].)

In the following, we use the notations used when $n\geqq 4$. Set $M=\partial(\varphi(D_{\varepsilon}^6)\cup N(W))$. Then using an s-cobordism argument similar to that above, we see that there is a homeomorphism $\eta:(M, K_g)\to(S_{\varepsilon'}^5, K_g)$. Let $p_0:S_{\varepsilon}^5-\mathrm{Int}\,N(K_f)\to S^1$ be the restriction of the Milnor fibration of f. Using the uniqueness of the tubular disk neighborhood, we see easily that the tubular disk neighborhood $N(W)$ is trivial; i.e. $N(W)$ is homeomorphic to the product $W\times D^2$. Thus the fibration $p_1=p_0\circ\varphi^{-1}:\varphi(S_{\varepsilon}^5-\mathrm{Int}\,N(K_f))\to S^1$ extends to a fibration $p_2:M-\mathrm{Int}\,N(K_g)\to S^1$. Let p_3 be the fibration given by $p_2\circ\eta^{-1}:S_{\varepsilon'}^5-\mathrm{Int}\,N(K_g)\to S^1$ and let $p_4:S_{\varepsilon'}^5-\mathrm{Int}\,N(K_g)\to S^1$ be the restriction of the Milnor fibration of g. Note that p_4 is a smooth fibration, while p_3 is only a topological fibration. However, using the method of Perron [12], we can show, changing the orientation of the base space S^1 of p_4 if necessary, that there is a homeomorphism $\eta' : (S^5_{\varepsilon'}, K_g) \to (S^5_{\varepsilon'}, K_g)$ such that $\eta'(N(K_g)) = N(K_g)$ and $p_4 \circ \eta' = p_3$ (Diagram 2). (Perron uses Freedman's result: every compact 1-connected 5-dimensional *smooth* h-cobordism is homeomorphic to the product. However, by Quinn [13] the smoothness of the h-cobordism is not necessary.)

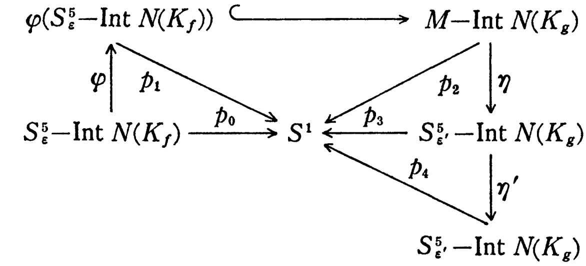

Diagram 2.

Let F_j $(j=0, 1, \cdots, 4)$ be the fiber of the fibration p_j over $1 \in S^1$. Then $F_2 - \mathrm{Int}\, F_1 \cong W$ is an h-cobordism and the inclusion map $i : F_1 \to F_2$ induces an isomorphism on homology, which obviously preserves the Seifert forms of p_1 and p_2. (See the proof of Lemma 12 of [12].) Thus the topological embedding $\xi = \eta' \circ \eta \circ i \circ \varphi : F_0 \to F_4$ satisfies the conditions in Lemma 3. Thus the links (S^5_ε, K_f) and $(S^5_{\varepsilon'}, K_g)$ are homeomorphic. This completes the proof of Theorem 1.

§ 3. Further results.

First we show the following.

PROPOSITION 4. *Let $f, g : \boldsymbol{C}^n, 0 \to \boldsymbol{C}, 0$ be holomorphic function germs with isolated critical points at the origin. If f and g are topologically V-equivalent, then there exists a homeomorphism germ $\varphi : \boldsymbol{C}^n, 0 \to \boldsymbol{C}^n, 0$ which is a diffeomorphism except possibly at the origin such that $\varphi(V_f) = V_g$.*

Proposition 4 follows from Theorem 1, Theorem 2.10 of [9], and the following lemma.

LEMMA 5. *Let f and g be as in Proposition 4. If $(S^{2n-1}_\varepsilon, S^{2n-1}_\varepsilon \cap V_f)$ is homeomorphic to $(S^{2n-1}_{\varepsilon'}, S^{2n-1}_{\varepsilon'} \cap V_g)$, then they are diffeomorphic.*

Remark. In fact, Lemma 5 holds for fibered knots ([2]) in general.

Proof of Lemma 5. For $n=2$, this is a well-known fact. Thus we assume $n \geqq 3$. Let F_f (resp. F_g) be the closure in S^{2n-1}_ε (resp. $S^{2n-1}_{\varepsilon'}$) of the Milnor fiber of f (resp. g). Using the infinite cyclic covering method and the topological h-cobordism theorem ([7], [13]), we see that F_f is homeomorphic to F_g.

Furthermore the Wall construction ([14, p. 140]) shows that the geometric monodromies of the Milnor fibrations of f and g are topologically pseudo-isotopic relative to the boundary. (For details see [12, § 3].) This topological pseudo-isotopy gives a homeomorphism between $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1}\cap V_f)$ and $(S_{\varepsilon'}^{2n-1}, S_{\varepsilon'}^{2n-1}\cap V_g)$ which preserves the Milnor fibers in a neighborhood of a fiber. Therefore, changing the orientations if necessary, we see that the Seifert forms of f and g are isomorphic; hence, $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1}\cap V_f)$ is smoothly isotopic to $(S_{\varepsilon'}^{2n-1}, S_{\varepsilon'}^{2n-1}\cap V_g)$ by [2] ($n\neq 3$). For $n=3$, we can use Lemma 3 and get the same result. This completes the proof.

Next we consider the right equivalence and the right-left equivalence. If f and g are right equivalent, then by the definitions they are right-left equivalent. Conversely, King [5] shows that if f and g are right-left equivalent, either f or $\bar{f}$ is right equivalent to g, where $\bar{f}$ denotes the germ of the conjugate of f. Thus if we can show that f is right equivalent to $\bar{f}$ for every holomorphic function germ f with an isolated critical point, then the right-left equivalence implies the right equivalence. Along these lines, Nishimura shows the following.

PROPOSITION 6. ([10]) *Let $f: \boldsymbol{C}^n, 0\to \boldsymbol{C}, 0$ be a holomorphic function germ with an isolated critical point at the origin. If* (a) $n=2$ *or* (b) *f has a non-degenerate Newton principal part, then f is topologically right equivalent to $\bar{f}$.*

(For the definition of the non-degeneracy of the Newton principal part, see [11] for example.)

Nishimura proves part (a) of Proposition 6 using the fact that the links of singularities in $\boldsymbol{C}^2$ are invertible. Part (b) is proved by the fact that if f has a non-degenerate Newton principal part, then it can be deformed through a topologically trivial family into a polynomial function germ whose coefficients are real ([11]). Note that if f is a holomorphic function germ whose power series expansion has real coefficients, then $f(\bar{z})=\bar{f}(z)$; hence, f is right equivalent to $\bar{f}$.

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