K. ATSUYAMA
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THE CONNECTION BETWEEN THE SYMMETRIC SPACE $E_8/Ss(16)$ AND PROJECTIVE PLANES

BY KENJI ATSUYAMA

§ 0. Introduction.

Simple Lie groups are already classified, and they have four kinds of infinite series of classical types and have five exceptional types. H. Freudenthal wrote many papers to obtain the geometrical and intuitive image of the exceptional Lie groups (cf. [5]). We have now the same aim as his. Our methods to solve the problem were first devised by B. A. Rozenfeld [7], but he didn't succeed completely in explaining the all cases which contain the exceptional Lie groups. For lack of the associativity in Cayley algebras, his explanations were incomplete (cf. [5]). To justify his assertions, we gave first a unified construction of real simple Lie algebras which were easy to handle directly [1]. Namely we made representative spaces for the exceptional Lie groups. Three symmetric spaces with the types $E\mathrm{III}$, $E\mathrm{VI}$ and $E\mathrm{VIII}$ in the E. Cartan's sense were next constructed explicitly as orbits of some projections in the sets of endomorphisms of the Lie algebras. We asked whether several similar properties to projective planes hold in the symmetric spaces by regarding the antipodal sets as lines [2], [3]. In this paper we continue to study the type $E_8/Ss(16)$, where $Ss(16)=\mathrm{Spin}(16)/\boldsymbol{Z}_2$, and we assert that this space is also a projective plane in the wider sense of Theorem 4.16.

§ 1. A construction of real simple Lie algebras.

The coefficient field is the field $\boldsymbol{R}$ of real numbers. Composition algebras are classified and have the seven following types:

	real	complex	quaternion	Cayley
division	$\boldsymbol{R}$	$\boldsymbol{C}$	$\boldsymbol{Q}$	$\mathfrak{C}$
split		$\boldsymbol{C}_s$	$\boldsymbol{Q}_s$	$\mathfrak{C}_s$

Let M^n be the $n\times n$ matrix algebra with coefficients in $\boldsymbol{R}$. Set $\mathrm{tr}(X)=(x_{11}+\cdots+x_{nn})/n$ for $X=(x_{ij})\in M^n$ and let $T: X\to X^T$ be the transposed operator. E is the unit matrix of M^n. If $\mathfrak{A}$ is a composition algebra, it has the

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usual conjugation $-: a \to \bar{a}$ and has an inner product $(a, b)=(ab+\overline{ab})/2$ for $a, b \in \mathfrak{A}$. Let $\mathfrak{A}^{(1)} \otimes M^n \otimes \mathfrak{A}^{(2)}$ denote the tensor product over $\boldsymbol{R}$ of these algebras. If the confusion does not occur, we write aXs simply instead of $a \otimes X \otimes s$, where $a \in \mathfrak{A}^{(1)}$, $s \in \mathfrak{A}^{(2)}$ and $X \in M^n$. $\mathfrak{A}^{(1)} \otimes M^n \otimes \mathfrak{A}^{(2)}$ has the following operations except for the addition.

$$\begin{aligned} &\text{the product} &&: (aXs)(bYt)=abXYst, \\ &\text{the involution} &&: aXs \longrightarrow \bar{a}X^T\bar{s}, \\ &\text{the trace} &&: \mathrm{Tr}(aXs)=a\,\mathrm{tr}(X)Es. \end{aligned}$$

Let $\mathfrak{M}$ be the linear subspace of $\mathfrak{A}^{(1)} \otimes M^n \otimes \mathfrak{A}^{(2)}$ such that any element in $\mathfrak{M}$ has the value 0 for the trace Tr and also has the skew-symmetric form for the above involution. We denote by $\mathrm{Der}\,\mathfrak{A}^{(i)}$ the Lie algebra of inner derivations $D_{a,b}$ of $\mathfrak{A}^{(i)}$, where $D_{a,b}(c)=[[a, b], c]-3(a, b, c)$ for $a, b, c \in \mathfrak{A}^{(i)}$ if we put $[a, b]=ab-ba$ and $(a, b, c)=(ab)c-a(bc)$.

Let $L(\mathfrak{A}^{(1)}, M^n, \mathfrak{A}^{(2)})$ be the vector space $\mathrm{Der}\,\mathfrak{A}^{(1)} \oplus \mathfrak{M} \oplus \mathrm{Der}\,\mathfrak{A}^{(2)}$ (direct sum). This becomes a Lie algebra by the following anti-commutative product [1]:

(1) $$[D^{(i)}, D^{(j)}]=\begin{cases} \text{the Lie product of } \mathrm{Der}\,\mathfrak{A}^{(i)} & (i=j), \\ 0 & (i \neq j), \end{cases}$$

(2) $$[D^{(1)}+D^{(2)}, aXs]=(D^{(1)}a)Xs+aX(D^{(2)}s),$$

(3) For $x=aXs$ and $y=bYt$ in $\mathfrak{M}$,

$$[x, y]=(X, Y)(s, t)D_{a,b}+(xy-yx-\mathrm{Tr}(xy-yx))+(X, Y)(a, b)D_{s,t},$$

where $D^{(i)} \in \mathrm{Der}\,\mathfrak{A}^{(i)}$ and $(X, Y)=\mathrm{tr}(XY)$.

If we restrict the composition algebras $\mathfrak{A}^{(i)}$ to $\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{Q}$ or $\mathfrak{C}$, then the Lie algebra $L(\mathfrak{A}^{(1)}, M^n, \mathfrak{A}^{(2)})$ becomes a compact real Lie algebra. It is generally simple. For instance, $E_8=L(\mathfrak{C}, M^3, \mathfrak{C})$ holds:

	$\boldsymbol{R}$	$\boldsymbol{C}$	$\boldsymbol{Q}$	$\mathfrak{C}$
$\boldsymbol{R}$	B_1	A_2	C_3	F_4
$\boldsymbol{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\boldsymbol{Q}$	C_3	A_5	D_6	E_7
$\mathfrak{C}$	F_4	E_6	E_7	E_8

$(n=3)$

	$\boldsymbol{R}$	$\boldsymbol{C}$	$\boldsymbol{Q}$
$\boldsymbol{R}$	B or D	A_{n-1}	C_n
$\boldsymbol{C}$	A_{n-1}	$A_{n-1} \oplus A_{n-1}$	A_{2n-1}
$\boldsymbol{Q}$	C_n	A_{2n-1}	D_{2n}

$(n \geqq 2)$

The Killing form B of $L(\mathfrak{A}^{(1)}, M^n, \mathfrak{A}^{(2)})$ can be given by

$$\begin{aligned} &B(D^{(1)}+aXs+D^{(2)}, D^{(1)}+aXs+D^{(2)}) \\ &\qquad =c_1B^{(1)}(D^{(1)}, D^{(1)})+c_0(a, a)(X, X)(s, s)+c_2B^{(2)}(D^{(2)}, D^{(2)}), \end{aligned}$$

and

$$c_0=n((n-2)d_1d_2+4(d_1+d_2-2)),$$

$$c_i=c_0/48 \qquad (i=1, 2),$$

where $D^{(1)}+aXs+D^{(2)}\in L(\mathfrak{A}^{(1)}, M^n, \mathfrak{A}^{(2)})$ and $d_i=\dim\mathfrak{A}^{(i)}$. There are two remarks for the coefficients c_0, c_i. (i) Since the inner product (X, X) contains $1/n$ in its definition, the factor $(n-2)d_1d_2+4(d_1+d_2-2)$ is essential in c_0. (ii) $B^{(i)}$ denotes the Killing form of $\mathrm{Der}\,\mathfrak{C}$. In the case of $\mathfrak{A}^{(i)}=\boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{Q}$, we also use $B^{(i)}$ instead of the Killing form of $\mathrm{Der}\,\mathfrak{A}^{(i)}$ because $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{Q}$ can be realized as subalgebras of $\mathfrak{C}$ naturally.

A basis of $\mathfrak{C}$ which we use usually is given explicitly.

a basis: $e_0, e_1, \cdots, e_7$,

rules of product:

$$e_1e_2=e_3,\ e_1e_4=e_5,\ e_6e_7=e_1,\ e_2e_5=e_7,\ e_3e_4=e_7,\ e_3e_5=e_6,\ e_6e_4=e_2,$$

$$e_ie_j=-e_je_i\ (i,\ j\geqq 1 \text{ and } i\neq j),\ e_ie_i=-e_0\ (i\geqq 1),$$

e_0 is the unit element,

the conjugation $-$: $\quad e_0\longrightarrow e_0,\ e_i\longrightarrow -e_i\ (1\leqq i\leqq 7)$.

Then $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{Q}$ can be generated by the bases $\{e_0\}$, $\{e_0, e_1\}$ or $\{e_0, e_1, e_2, e_3\}$ respectively.

§2. A construction of symmetric spaces Π.

We construct some symmetric spaces on which the exceptional Lie groups act, by making use of the compact exceptional Lie algebras $L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$.

Set $\mathfrak{G}=L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$ simply. Let $\mathfrak{X}$ be the subset of $\mathfrak{G}$ such that any element x in $\mathfrak{X}$ satisfies the identity

$$(\mathrm{ad}\,x)((\mathrm{ad}\,x)^2+1)((\mathrm{ad}\,x)^2+4)=0,$$

where $\mathrm{ad}\,x$ is the adjoint representation of $\mathfrak{G}$ and 1 means the identity transformation of $\mathfrak{G}$. The eigenspaces of $\mathrm{ad}\,x$, for each $x\in\mathfrak{X}$, can be given by

$$\mathfrak{G}_0(x)=\{z\in\mathfrak{G}\,|\,(\mathrm{ad}\,x)z=0\},$$

$$\mathfrak{G}_i(x)=\{z\in\mathfrak{G}\,|\,(\mathrm{ad}\,x)^2z=-i^2z\},\quad (i=1, 2).$$

For $x\in\mathfrak{X}$ we define three transformations $P_i(x)$ of $\mathfrak{G}$ by

$$P_0(x)=1+5/4(\mathrm{ad}\,x)^2+1/4(\mathrm{ad}\,x)^4,$$

$$P_1(x)=-4/3(\mathrm{ad}\,x)^2-1/3(\mathrm{ad}\,x)^4,$$

$$P_2(x)=1/12(\mathrm{ad}\,x)^2+1/12(\mathrm{ad}\,x)^4.$$

These satisfy $P_i(x)P_i(x)=P_i(x)$, $P_i(x)P_j(x)=0$ $(i\neq j)$ and $P_0(x)+P_1(x)+P_2(x)=1$. Namely each $P_i(x)$ is the projection of $\mathfrak{G}$ onto $\mathfrak{G}_i(x)$. Therefore $\mathfrak{G}$ has a direct sum decomposition $\mathfrak{G}=\mathfrak{G}_0(x)\oplus\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$. Then $(\mathfrak{G}_0(x)\oplus\mathfrak{G}_2(x))\oplus\mathfrak{G}_1(x)$ is a Cartan decomposition with respect to the involutive automorphism $1-2P_1(x)$ $(=\exp\pi(\mathrm{ad}\,x))$.

Example. Let $\mathfrak{G}=L(\mathfrak{C}, M^3, \mathfrak{C})$ and take $K_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$ in $\mathfrak{M}\cap\mathfrak{X}$. Then the eigenspaces $\{\mathfrak{G}_i(x)\}$ can be given by the following.

		dimension
$\mathfrak{G}_0(K_1)$:	$\mathrm{Der}\,\mathfrak{C}\oplus\begin{pmatrix}2a&0&0\\0&-a&b\\0&-b&-a\end{pmatrix}\oplus\mathrm{Der}\,\mathfrak{C}$	$14+64+14=92$,
$\mathfrak{G}_1(K_1)$:	$\begin{pmatrix}0&b_1&b_2\\-b_1&0&0\\-b_2&0&0\end{pmatrix}\oplus\begin{pmatrix}0&a_1&a_2\\a_1&0&0\\a_2&0&0\end{pmatrix}$	$100+28=128$,
$\mathfrak{G}_2(K_1)$:	$\begin{pmatrix}0&0&0\\0&0&a\\0&a&0\end{pmatrix}\oplus\begin{pmatrix}0&0&0\\0&a&0\\0&0&-a\end{pmatrix}$	$14+14=28$,

where a, a_1, a_2 (resp. b, b_1, b_2) are linear combinations of $e_0\otimes e_i$ and $e_j\otimes e_0$ (resp. $e_0\otimes e_0$ and $e_i\otimes e_j$), $i, j=1, 2, \cdots, 7$.

We now construct a symmetric space Π in the set $\mathrm{End}\,\mathfrak{G}$ of endomorphisms of $\mathfrak{G}$. The action of the adjoint group G of $\mathfrak{G}$ on $\mathrm{End}\,\mathfrak{G}$ is defined by $g\cdot h=ghg^{-1}$ for $g\in G$ and $h\in\mathrm{End}\,\mathfrak{G}$. Let Π be the orbit of the projection $P_1(K_1)$ by G under this action, i.e., $\Pi=\{g\cdot P_1(K_1)\,|\,g\in G\}$. Note that $g\cdot P_1(K_1)=P_1(gK_1)$. Then the eigenspace $\mathfrak{G}_1(gK_1)$ can be regarded as the tangent space at $P_1(gK_1)$ of Π, and the eigenspace $\mathfrak{G}_0(gK_1)\oplus\mathfrak{G}_2(gK_1)$ can also be regarded as the Lie algebra of the isotropy group at $P_1(gK_1)$ (cf. [2], Proposition 2.4). When we introduce a G-invariant Riemannian structure into Π by restricting the Killing form B of $\mathfrak{G}$ to each tangent space $\mathfrak{G}_1(gK_1)$, the adjoint group G equals to the identity component of the isometry group of Π. If $\mathfrak{G}$ is a compact exceptional Lie algebra, Π is simply connected from [4], p. 411.

PROPOSITION 2.1. *Π is a compact symmetric space in which each point $P_1(gK_1)$ has the geodesic symmetry $1-2P_1(gK_1)$. The type of Π can be given by the table:*

	$\boldsymbol{R}$	$\boldsymbol{C}$	$\boldsymbol{Q}$	$\mathfrak{C}$
$\boldsymbol{R}$	$\boldsymbol{R}P_2$	$\boldsymbol{C}P_2$	$\boldsymbol{Q}P_2$	$\mathfrak{C}P_2$
$\boldsymbol{C}$	$\boldsymbol{C}P_2$	$\boldsymbol{C}P_2\times\boldsymbol{C}P_2$	$G^C(4,2)$	$E\,\mathrm{III}$
$\boldsymbol{Q}$	$\boldsymbol{Q}P_2$	$G^C(4,2)$	$G(8,4)$	$E\,\mathrm{VI}$
$\mathfrak{C}$	$\mathfrak{C}P_2$	$E\,\mathrm{III}$	$E\,\mathrm{VI}$	$E\,\mathrm{VIII}$

where $G^C(4,2)=SU(6)/S(U(4)\times U(2))$, $G(8,4)=SO(12)/SO(8)\times SO(4)$ *and the first column contains four planes from the real projective plane to the Cayley plane.*

Proof. We can obtain the table by the direct calculation.

One has an involutive automorphism

$$\beta:\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\longrightarrow\begin{pmatrix}a_{11} & -a_{12} & -a_{13}\\ -a_{21} & a_{22} & a_{23}\\ -a_{31} & a_{32} & a_{33}\end{pmatrix}$$

in the matrix algebra M^3. This can be extended easily to the Lie algebra $\mathfrak{G}=L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$ by

$$\beta: D^{(1)}+a\otimes X\otimes s+D^{(2)}\longrightarrow D^{(1)}+a\otimes\beta X\otimes s+D^{(2)}.$$

Denote this extended map by β again. Then $\beta=\exp\pi(\mathrm{ad}\,K_1)$ and $P_1(K_1)=(1-\beta)/2$ hold. Hence the orbit of β by G is the same as Π essentially. We notice moreover that all the symmetric spaces in Proposition 2.1 are constructed by a single transformation β of M^3 and the spaces in the first column have the structure of projective planes. Therefore one can expect that the remaining symmetric spaces may have the similar structure. For each point P in Π we will regard later the antipodal set $L(P)$ of P as a line and investigate Π from the viewpoint of projective planes. All lines are transitive one another (see Proposition 3.8 in the case of $\Pi=E\,\mathrm{VIII}$).

PROPOSITION 2.2. *$L(P)$ is a compact connected symmetric space. The following table gives the type of $L(P)$ in each case: S^n is an n-dimensional sphere.*

	$\boldsymbol{R}$	$\boldsymbol{C}$	$\boldsymbol{Q}$	$\mathfrak{C}$
$\boldsymbol{R}$	S^1	S^2	S^4	S^8
$\boldsymbol{C}$	S^2	$S^2\times S^2$	$G^C(2,2)$	$G(8,2)$
$\boldsymbol{Q}$	S^4	$G^C(2,2)$	$G(4,4)$	$G(8,4)$
$\mathfrak{C}$	S^8	$G(8,2)$	$G(8,4)$	$G(8,8)$

§3. A maximal flat torus in Π.

In the remaining sections, let $\mathfrak{G}=L(\mathfrak{C}, M^3, \mathfrak{C})$ and $\Pi=E_8/Ss(16)$ $(=E\text{VIII})$. For simplicity we write $P(x)$ instead of $P_1(x)$. We analyze here the structure of the integer lattice of a maximal flat torus and study the subset of Π which consists of all points commuting with $P(K_1)$.

Define three elements $\{K_\iota\}$ in $\mathfrak{X}\subset\mathfrak{G}$ by

$$K_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},\quad K_2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad K_3=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

where the unit element e_0 and the notation $\otimes$ are omitted. Let $\mathfrak{T}_0$ be the 8-dimensional abelian subspace of $\mathfrak{M}$ spanned by $\{K_2, e_1K_2e_1, \cdots, e_7K_2e_7\}$, and set $T_0=\{\exp(\operatorname{ad} z)\cdot P(K_1)\,|\,z\in\mathfrak{T}_0\}$. T_0 is a maximal flat torus in Π passing through $P(K_1)$. Then, with respect to $\mathfrak{T}_0$, $\mathfrak{G}$ has a root space decomposition

$$\mathfrak{G}=\mathfrak{T}_0\oplus\sum\mathfrak{G}_\lambda \quad (\text{over } \boldsymbol{C}).$$

The 240 roots are given, with respect to the operation $\operatorname{ad}(\sum a_i e_i K_2 e_\iota)$, $a_i\in\boldsymbol{R}$, by

$$\pm 2(a_\iota \pm a_j)\boldsymbol{i} \qquad (0\leqq i<j\leqq 7),$$

$$\pm(a_0+\varepsilon_1 a_1+\varepsilon_2 a_2+\cdots+\varepsilon_7 a_7)\boldsymbol{i} \qquad (\varepsilon_\iota=\pm 1 \text{ and the product } \varepsilon_1\varepsilon_2\cdots\varepsilon_7=1),$$

where $\boldsymbol{i}=\sqrt{-1}$. A fundamental root system consists of

$$\lambda_1=-2(a_1-a_2)\boldsymbol{i}, \qquad \lambda_2=-2(a_2-a_3)\boldsymbol{i},$$

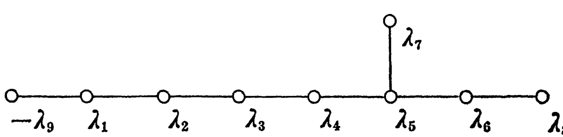
$$\lambda_3=-2(a_3-a_4)\boldsymbol{i}, \qquad \lambda_4=-2(a_4-a_5)\boldsymbol{i},$$

$$\lambda_5=-2(a_5-a_6)\boldsymbol{i}, \qquad \lambda_6=-2(a_6-a_7)\boldsymbol{i},$$

$$\lambda_7=-2(a_6+a_7),\boldsymbol{i}$$

$$\lambda_8=-(a_0-a_1-a_2-a_3-a_4-a_5-a_6+a_7)\boldsymbol{i}.$$

The highest root is $\lambda_9=-2(a_0+a_1)\boldsymbol{i}$. Then $\lambda_9=2\lambda_1+3\lambda_2+4\lambda_3+5\lambda_4+6\lambda_5+4\lambda_6+3\lambda_7+2\lambda_8$ holds and the extended Dynkin diagram becomes



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Define an 8-dimensional simplex in $\mathfrak{T}_0$ by

$$\mathfrak{S}(\Pi)=\{x\in\mathfrak{T}_0\,|\,\lambda_1(x\boldsymbol{i})\geqq 0, \cdots, \lambda_8(x\boldsymbol{i})\geqq 0, \lambda_9(x\boldsymbol{i})\leqq\pi\}.$$

This contains the origin of $\mathfrak{T}_0$, and any point in Π is conjugate to some point in $\exp(\mathrm{ad}\,\mathfrak{S}(\Pi))\cdot P(K_1)$. Let x_i denote the vertex of $\mathfrak{S}(\Pi)$ which corresponds to the root λ_i, then the coefficients of the vectors $\{x_i/\pi \mid i=1, 2, \cdots, 8\}$, with respect to the basis $\{e_i K_2 e_i \mid i=0, 1, \cdots, 7\}$, can be given by the table:

x_i/π	0	1	2	3	4	5	6	7
1	1/4	1/4	0	0	0	0	0	0
2	1/3	1/6	1/6	0	0	0	0	0
3	3/8	1/8	1/8	1/8	0	0	0	0
4	2/5	1/10	1/10	1/10	1/10	0	0	0
5	5/12	1/12	1/12	1/12	1/12	1/12	0	0
6	7/16	1/16	1/16	1/16	1/16	1/16	1/16	−1/16
7	5/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
8	1/2	0	0	0	0	0	0	0

Remark. The Lie algebras of the isotropy group at the points $\exp(\mathrm{ad}\,x_i)\cdot P(K_1)$ in Π have respectively the types (1) $A_7 \oplus I$, (2) $C_4 \oplus B_1$, (3) $2B_2 \oplus D_2$, (4) $2B_2$, (5) $I \oplus B_1 \oplus D_3$, (6) $D_4 \oplus I$, (7) B_4 and (8) $2D_4$, where I is a one-dimensional center. If x_9=the origin 0, one has (9) D_8.

LEMMA 3.1. *Let $z \in \mathfrak{T}_0$. Then $\exp(\mathrm{ad}\,z)$ is the identity transformation of $\mathfrak{G}$ if and only if each fundamental root λ_i satisfies $\lambda_i(z) \in 2\pi \boldsymbol{Z} \boldsymbol{i}$, where $\boldsymbol{Z}$ is the integer ring.*

Proof. Assume that $\exp(\mathrm{ad}\,z)=1$ and take a non-zero element $g_\lambda \in \mathfrak{G}_\lambda$ for each root λ. Since $g_\lambda = \exp(\mathrm{ad}\,z) g_\lambda = (\exp \lambda(z)) g_\lambda$ holds, we see $\exp \lambda(z)=1$. This means $\lambda(z) \in 2\pi \boldsymbol{Z}\boldsymbol{i}$. We next show the converse. Since any root λ can be written as $\lambda = \sum n_i \lambda_i$ ($n_i \in \boldsymbol{Z}$), we have $\exp(\mathrm{ad}\,z) g_\lambda = (\exp \lambda(z)) g_\lambda = (\exp \sum n_i \lambda_i(z)) g_\lambda = g_\lambda$.

For the highest root λ_9, set $\lambda_9 = \sum m_i \lambda_i$. Then we see that $m_1=2$, $m_2=3, \cdots, m_8=2$ by the above. The integer lattice $\mathfrak{L}$ in $\mathfrak{T}_0$ is defined by $\mathfrak{L} = \{z \in \mathfrak{T}_0 \mid \exp(\mathrm{ad}\,z)=1\}$.

PROPOSITION 3.2. $\mathfrak{L} = \{2n_1 m_1 x_1 + \cdots + 2n_8 m_8 x_8 \mid n_i \in \boldsymbol{Z}\}$ *holds, where $\{x_i\}$ are the vertexes of $\mathfrak{S}(\Pi)$.*

Proof. Assume that $z \in \mathfrak{T}_0$ satisfies $\exp(\mathrm{ad}\,z)=1$ and set $z = \sum \xi_i x_i$ ($\xi_i \in \boldsymbol{R}$). From Lemma 3.1, there exists an integer $n_i \in \boldsymbol{Z}$ for each i such that $\lambda_i(z\boldsymbol{i}) = 2\pi n_i$. Since $\lambda_i(x_j \boldsymbol{i}) = 0$ $(i \neq j)$ and $\lambda_i(x_i \boldsymbol{i}) = \pi/m_i$ hold, we obtain $\xi_i = 2n_i m_i$. Conversely, let $z = \sum 2n_i m_i x_i$ ($n_i \in \boldsymbol{Z}$). Then $\lambda_j(z\boldsymbol{i}) = \sum 2n_i m_i \lambda_j(x_i \boldsymbol{i}) = 2n_j \pi$ holds. This shows $\exp(\mathrm{ad}\,z)=1$ from Lemma 3.1.

We rewrite Proposition 3.2 by making use of the basis $\{e_i K_2 e_i\}$ of $\mathfrak{T}_0$.

Then we obtain the following where $z=\sum t_i e_i K_2 e_i$ $(t_i \in \boldsymbol{R})$.

PROPOSITION 3.3. *Let* $z \in \mathfrak{T}_0$. *Then* $z \in \mathfrak{L}$ *holds if and only if* z *satisfies* (i) *or* (ii):

(i) $t_i \in \pi \boldsymbol{Z}$, *for each* i, *and* $\sum t_i \in 2\pi \boldsymbol{Z}$,

(ii) $t_i - \pi/2 \in \pi \boldsymbol{Z}$, *for each* i, *and* $\sum t_i \in 2\pi \boldsymbol{Z}$.

$\mathfrak{G}_1(x)$ is the tangent space at $P(x)$ of Π.

LEMMA 3.4. *Let* $z \in \mathfrak{G}_1(x)$. *Then* $\exp(\operatorname{ad} z)$ *leaves* $P(x)$ *fixed if and only if* $\exp(\operatorname{ad} 2z)=1$ *holds in* $\mathfrak{G}$.

Proof. Put $e(z)=\exp(\operatorname{ad} z)$ simply. Assume that $e(z)\cdot P(x)=P(x)$. Then we obtain $e(z)P(x)=P(x)e(z)$. Hence it holds that $e(-z)=e((1-2P(x))z)=(1-2P(x))e(z)(1-2P(x))=e(z)$ because $1-2P(x)$ is the geodesic symmetry at $P(x)$ in Π. This identity implies $e(2z)=1$. We next show the converse. Since $e(2z)=1$ gives $e(z)=e(-z)$, we have $e(z)=e(-z)=e((1-2P(x))z)=(1-2P(x))e(z)(1-2P(x))$. Therefore $e(z)P(x)=P(x)e(z)$ holds.

PROPOSITION 3.5. *Let* $z \in \mathfrak{T}_0$. *Then* $\exp(\operatorname{ad} z)\cdot P(K_1)=P(K_1)$ *holds if and only if* z *satisfies* (i) *or* (ii) *where* $\pi/2\boldsymbol{Z}=\{n\pi/2 \mid n \in \boldsymbol{Z}\}$:

(i) $t_i \in \pi/2\boldsymbol{Z}$, *for each* i, *and* $\sum t_i \in \pi \boldsymbol{Z}$,

(ii) $t_i - \pi/4 \in \pi/2\boldsymbol{Z}$, *for each* i, *and* $\sum t_i \in \pi \boldsymbol{Z}$.

Proof. One obtains this from Proposition 3.3 and Lemma 3.4 because $\mathfrak{T}_0 \subset \mathfrak{G}_1(K_1)$.

LEMMA 3.6. *Let* $z \in \mathfrak{G}_1(K_1)$. *The point* $\exp(\operatorname{ad} z)\cdot P(K_1)$ *commutes with* $P(K_1)$ *if and only if* $\exp(\operatorname{ad} 2z)\cdot P(K_1)=P(K_1)$ *holds.*

Proof. This is the same as Lemma 3.4 [3] essentially.

PROPOSITION 3.7. *Let* $z \in \mathfrak{T}_0$. *The point* $\exp(\operatorname{ad} z)\cdot P(K_1)$ *commutes with* $P(K_1)$ *if and only if* z *satisfies* (i) *or* (ii):

(i) $t_i \in \pi/4\boldsymbol{Z}$, *for each* i, *and* $\sum t_i \in \pi/2\boldsymbol{Z}$,

(ii) $t_i - \pi/8 \in \pi/4\boldsymbol{Z}$, *for each* i, *and* $\sum t_i \in \pi/2\boldsymbol{Z}$.

Proof. This can be derived easily from Proposition 3.5 and Lemma 3.6.

Let $L(P(K_1))$ denote in Π the antipodal set associated to $P(K_1)$, and Ω be the submanifold of Π of which points are the middle points of the shortest closed geodesics with the initial point $P(K_1)$.

PROPOSITION 3.8. *The subset of all points in* Π *commuting with* $P(K_1)$ *becomes two connected submanifolds* $L(P(K_1))$ *and* Ω. *They are totally geodesic* *and compact.*

Proof. Take $z\in\mathfrak{S}(\Pi)$. If $\exp(\mathrm{ad}\,z)\cdot P(K_1)$ commutes with $P(K_1)$, $z=x_1$ or x_8 holds from Proposition 3.7, where x_1 and x_8 are the vertexes of $\mathfrak{S}(\Pi)$. Hence, the points commuting with $P(K_1)$ form two orbits Ω and $L(P(K_1))$ since they are transitive to $\exp(\mathrm{ad}\,x_1)\cdot P(K_1)$ or $\exp(\mathrm{ad}\,x_8)\cdot P(K_1)$. The types as symmetric spaces are $SO(16)/U(8)$ and $G(8,8)$ respectively. Note that $P(K_3)=\exp(\mathrm{ad}\,x_8)\cdot P(K_1)$ holds. The distance between $P(K_1)$ and $L(P(K_1))$ can be given by

$$\int_0^1(-B(\dot\gamma(t),\,\dot\gamma(t)))^{1/2}dt=\int_0^1(-B(x_8,\,x_8))^{1/2}dt=2\sqrt{15}\pi\,,$$

where B is the Killing form of $\mathfrak{G}$ and $\gamma(t)=\exp t(\mathrm{ad}\,x_8)\cdot P(K_1)$ is the shortest geodesic from $P(K_1)$ to $P(K_3)$.

Remark. If Π is another exceptional type in Proposition 2.1, we have the following table where the values mean the distance from $P(K_1)$:

	$\mathfrak{C}P_2$	$E\,\mathrm{III}$	$E\,\mathrm{VI}$	$E\,\mathrm{VIII}$
Ω		$SO(10)/U(5)$	$(SO(12)/U(6))\times S^2$	$SO(16)/U(8)$
		$2\sqrt{3}\,\pi$	$3\sqrt{2}\,\pi$	$\sqrt{30}\pi$
$L(P(K_1))$	S^8	$G(8,2)$	$G(8,4)$	$G(8,8)$
	$3\sqrt{2}\,\pi$	$2\sqrt{6}\,\pi$	6π	$2\sqrt{15}\pi$

§ 4. Π as a projective plane in a wider sense.

We introduce two geometrical objects, points and lines, into Π in order to study Π from the viewpoint of projective planes. Let $P\in\Pi$. Then we call the antipodal set $L(P)$ *a line* (associated with P) and call P *a point* again in the sense of the projective geometry. The incidence structure is defined by the inclusion relation.

Let Π^L denote the set of all lines of Π. First we show that the correspondence L from Π to Π^L defined by $L: P\to L(P)$ gives the notion of the polarity to Π (Proposition 4.3) and next analyze the structure of the intersection $L(P)\cap L(Q)$. Finally we assert that Π is a projective plane in the wider sense of Theorem 4.16.

(NOTATIONS). T_0 is the maximal flat torus defined in § 3 and contains $P(K_1)$. $\mathfrak{T}_0$ is its Lie algebra at $P(K_1)$ spanned by the basis $\{e_iK_2e_i\}$, and $\dim\mathfrak{T}_0=8$ holds. $\mathfrak{G}_1(K_1)$ is the tangent space at $P(K_1)$ of Π and contains $\mathfrak{T}_0$. The subset $\mathfrak{X}$ of $\mathfrak{G}$ is defined in § 2. G is the identity component of the isometry group of Π. Let $\mathrm{Iso}(P)$ denote the isotropy group at P in Π with respect to G. Set $U(Q)=\mathrm{Iso}(P(K_1))\cap\mathrm{Iso}(Q)$ for $Q\in\Pi$. $\mathfrak{U}(Q)$ is the Lie algebra of $U(Q)$. The distance between P and Q is denoted by $d(P, Q)$. The notation $\Leftrightarrow$ means the equivalency.

LEMMA 4.1. *$L(P)=\{Q\in\Pi \mid PQ=QP$ and $d(P, Q)=2\sqrt{15}\pi\}$ holds.*

Proof. We can prove this by Proposition 3.8 and by the transitivity of points in Π.

LEMMA 4.2. *The correspondence $L:\Pi\rightarrow\Pi^L$ is a bijective map.*

Proof. The definition of L gives the surjectivity. So we show that L is injective. From the transitivity of points in Π, it is sufficient to see that $L(P(K_1))=L(Q)$, for $Q\in\Pi$, implies $P(K_1)=Q$. Assume $L(P(K_1))=L(Q)$. Then there exists an element $\alpha\in\mathrm{Iso}(P(K_1))$ such that $\alpha\cdot Q\in T_0$. Hence one has $L(P(K_1))=L(\alpha\cdot Q)$. This means $d(\alpha\cdot Q, L(P(K_1)))=2\sqrt{15}\pi$, but such a point in T_0 is $P(K_1)$ only. In fact this holds by the following assertion and then one obtains $\alpha\cdot Q=P(K_1)$, i.e., $Q=P(K_1)$.

The points in $T_0\cap L(P(K_1))$ have the forms $\exp(\mathrm{ad}\,z)\cdot P(K_1)$ such that $z=\sum t_i\pi e_iK_2e_i$ ($t_i\in\boldsymbol{R}$). We can write down then their coordinates (t_i) by Lemma 4.1 and Proposition 3.7. Namely they are the permutations of the (t_i) as below. The number of the points is 135 from Proposition 3.5, and $P(K_1)$ is a single point in T_0 which commutes with all the points and also has the distance $2\sqrt{15}\pi$ from them.

(t_0 ,	t_1 ,	t_2 ,	t_3 ,	t_4 ,	t_5 ,	t_6 ,	t_7)	permutations
(1/2,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0)	1
(5/8,	1/8,	1/8,	1/8,	1/8,	1/8,	1/8,	1/8)	1
(7/8,	3/8,	1/8,	1/8,	1/8,	1/8,	1/8,	1/8)	28
(7/8,	3/8,	3/8,	3/8,	1/8,	1/8,	1/8,	1/8)	35
(1/4,	1/4,	1/4,	1/4,	0 ,	0 ,	0 ,	0)	35
(3/4,	1/4,	1/4,	1/4,	0 ,	0 ,	0 ,	0)	35

Since the correspondence L is bijective, we can introduce the structure of the symmetric space Π into Π^L. If we also use L instead of L^{-1}, then L^2 is the identity map of $\Pi\cup\Pi^L$.

PROPOSITION 4.3. *The correspondence L gives the polarity of Π, i.e., L satisfies* (i) *and* (ii):

(i) $L^2=1$ *on* $\Pi\cup\Pi^L$,

(ii) $P\in L(Q)\Leftrightarrow Q\in L(P)$.

Proof. The result (ii) is easy from Lemma 4.1.

We are going to prepare some facts from Lemma 4.4 to Corollary 4.15 in order to analyze the structure of the intersection $L(P)\cap L(Q)$. The goal is

Corollary 4.15. The explicit classification by this is listed up in § 5.

LEMMA 4.4. *Let $Q\in\Pi$ commute with all points in T_0. Then all isometries* $\exp(\operatorname{ad} z)$, $z\in\mathfrak{T}_0$, *leave Q fixed.*

Proof. Any point in T_0 has the form $\alpha\cdot P(K_1)$ where $\alpha=\exp(\operatorname{ad}\sum t_i e_i K_2 e_i)$, $t_i\in\boldsymbol{R}$. Put $\gamma=1-2P(K_1)$. If Q commutes with T_0, one has $\gamma\cdot(\alpha^{-1}\cdot Q)=\alpha^{-1}\cdot Q$ because $Q(\alpha\cdot P(K_1))=(\alpha\cdot P(K_1))Q$. On the other hand, we have $\gamma\cdot Q=Q$ from $P(K_1)\in T_0$ and also have $\gamma(e_i K_2 e_i)=-e_i K_2 e_i$ since γ is the geodesic symmetry at $P(K_1)$. Hence it holds that $\gamma\cdot(\alpha^{-1}\cdot Q)=(\gamma\alpha^{-1}\gamma^{-1})\cdot(\gamma\cdot Q)=\exp(\operatorname{ad}\gamma\sum(-t_i)e_i K_2 e_i)\cdot Q=\alpha\cdot Q$. We obtain then $\alpha^{-1}\cdot Q=\gamma\cdot(\alpha^{-1}\cdot Q)=\alpha\cdot Q$. This implies $\alpha^2\cdot Q=Q$. Since t_i is arbitrary, $\alpha\cdot Q=Q$ holds.

LEMMA 4.5. *If $Q\in\Pi$ commutes with all points in T_0, there exists an element x in $\mathfrak{T}_0\cap\mathfrak{X}$ such that $Q=P(x)$.*

Proof. Let Q commute with T_0 and let $Q=g\cdot P(K_1)$, $g\in G$. Put $y=gK_1$, then we have $Q=P(y)$. The set $\{ge_i K_1 e_i \mid i=0, 1, \cdots, 7\}$ generates a Cartan subalgebra $\mathfrak{T}_1$ of $\mathfrak{G}$ such that $y\in\mathfrak{T}_1\subset\mathfrak{G}_0(y)$. On the other hand, since all isometries $\exp t(\operatorname{ad} e_i K_2 e_i)$, $t\in\boldsymbol{R}$, leave Q fixed by Lemma 4.4, we obtain $e_i K_2 e_i\in\mathfrak{G}_0(y)\oplus\mathfrak{G}_2(y)$ for each i. Hence the Cartan subalgebra $\mathfrak{T}_0$ spanned by $\{e_i K_2 e_i\}$ is contained in $\mathfrak{G}_0(y)\oplus\mathfrak{G}_2(y)$. Since $\mathfrak{G}_0(y)\oplus\mathfrak{G}_2(y)$ is a compact simple Lie algebra $so(16)$ with the rank 8, both $\mathfrak{T}_0$ and $\mathfrak{T}_1$ are Cartan subalgebras also in $\mathfrak{G}_0(y)\oplus\mathfrak{G}_2(y)$. Therefore there exists an element h in the Lie subgroup $\exp(\operatorname{ad}(\mathfrak{G}_0(y)\oplus\mathfrak{G}_2(y)))$ of G such that $h\mathfrak{T}_1=\mathfrak{T}_0$. Then, for $x=hy\in\mathfrak{T}_0\cap\mathfrak{X}$, we obtain $P(x)=P(hy)=h\cdot P(y)=P(y)=Q$.

LEMMA 4.6. *If $Q\in\Pi$ commutes with all points in T_0 and also has the distance $2\sqrt{15}\pi$ from $P(K_1)$, there exists an element $k\in\operatorname{Iso}(P(K_1))$ such that $k\mathfrak{T}_0=\mathfrak{T}_0$ and $Q=P(kK_2)$.*

Proof. Let Q satisfy the above assumption. Then the line $L(P(K_1))$ contains Q and also does $P(K_2)$ because $P(K_2)=\exp(\pi/2\operatorname{ad} K_3)\cdot P(K_1)$. Hence there exists an element $g\in\operatorname{Iso}(P(K_1))$ by Proposition 3.8 such that $Q=g\cdot P(K_2)$. Since the Lie algebra of the isotropy group at $P(K_2)$ is $\mathfrak{G}_0(K_2)\oplus\mathfrak{G}_2(K_2)$, one has $g\mathfrak{G}_0(K_2)\oplus g\mathfrak{G}_2(K_2)$ (denoted by $\mathfrak{J}(Q)$) as the isotropy algebra at Q. We show next $\mathfrak{G}_1(K_1)\cap\mathfrak{J}(Q)\supset g\mathfrak{T}_0\cup\mathfrak{T}_0$. $\mathfrak{J}(Q)\supset\mathfrak{T}_0$ holds by the same reason as the proof of Lemma 4.5. Since $g\cdot P(K_1)=P(K_1)$ is equivalent to $g\gamma=\gamma g$ where $\gamma=1-2P(K_1)$, one has $\gamma(gx)=g\gamma x=-gx$ for $x\in\mathfrak{T}_0$. This gives $\mathfrak{G}_1(K_1)\supset g\mathfrak{T}_0$. Let $U(Q)_0$ denote the identity component of the subgroup $U(Q)$ of G. This group is compact. Take an element $x\in g\mathfrak{T}_0$ (resp. $y\in\mathfrak{T}_0$) such that its centralizer in $\mathfrak{G}$ is equal to $g\mathfrak{T}_0$ (resp. $\mathfrak{T}_0$). Define a differentiable function F on $U(Q)_0$ by $F(h)=B(x, hy)$ where B is the Killing form of $\mathfrak{G}$. If F has an extremal value at $h_0\in U(Q)_0$, one obtains for any $z\in\mathfrak{U}(Q)$ (where $\mathfrak{U}(Q)$ is the Lie algebra of $U(Q)_0$)

$$0=\left\{\frac{d}{dt}B(x,\ (\exp t(\mathrm{ad}\,z))h_0y)\right\}_{t=0}=B(x,\ [z,\ h_0y])$$

$$=-B([x,\ h_0y],\ z).$$

Hence we have $[x, h_0y]=0$ because $[x, h_0y]\in\mathfrak{U}(Q)$ and B is non-degenerate in $\mathfrak{U}(Q)$. The first property can be derived from x, $h_0y\in\mathfrak{G}_1(K_1)\cap\mathfrak{Z}(Q)$ and so by $[x, h_0y]\in(\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1))\cap\mathfrak{Z}(Q)$ $(=\mathfrak{U}(Q))$. Now $[x, h_0y]=0$ gives $h_0\mathfrak{T}_0=g\mathfrak{T}_0$. Put $k=h_0^{-1}g$, then one has $k\mathfrak{T}_0=\mathfrak{T}_0$ and $k\in\mathrm{Iso}(P(K_1))$. Finally it holds that $P(kK_2)=h_0^{-1}\cdot P(gK_2)=h_0^{-1}\cdot Q=Q$.

We define two sets S and S_0 by

$$S=\{Q\in\Pi\mid QP=PQ \text{ and } d(P,\ Q)=2\sqrt{15}\pi \quad \text{for all } P\in T_0\},$$

$$S_0=\{Q\in T_0\mid QP(K_1)=P(K_1)Q \text{ and } d(P(K_1),\ Q)=2\sqrt{15}\pi\}.$$

Since $Q\in S$ has the form $Q=P(x)$ for some $x\in\mathfrak{T}_0$ from Lemma 4.5, we define the map $f: S\to T_0$ by

$$f(P(x))=\exp(\pi/2\mathrm{ad}\,x)\cdot P(K_1).$$

One can see from Lemma 4.7 that f is well-defined and injective. Furthermore Lemma 4.8 asserts that $f(S)=S_0$ holds.

LEMMA 4.7. *For x, $y\in\mathfrak{T}_0$, $P(x)=P(y)$ holds if and only if $f(P(x))=f(P(y))$ holds.*

Proof. Let $e(x)=\exp(\mathrm{ad}\,x)$. Then, $P(x)=P(y)\Leftrightarrow e(\pi x)=e(\pi y)$ (because $e(\pi x)=1-2P(x))\Leftrightarrow e(\pi x/2)e(-\gamma\pi x/2)=e(\pi y/2)e(-\gamma\pi y/2)$ (because x, $y\in\mathfrak{T}_0\subset\mathfrak{G}_1(K_1)$ and $\gamma=1-2P(K_1))\Leftrightarrow e(\pi x/2)\gamma e(-\pi x/2)=e(\pi y/2)\gamma e(-\pi y/2)\Leftrightarrow e(\pi x/2)\cdot P(K_1)=e(\pi y/2)\cdot P(K_1)\Leftrightarrow f(P(x))=f(P(y))$ hold.

PROPOSITION 4.8. *$f(S)=S_0$ holds.*

Proof. We show first $f(S)\supset S_0$. S_0 contains $P(K_3)$ because $P(K_3)=\exp\pi/2(\mathrm{ad}\,K_2)\cdot P(K_1)$. Take any point Q in S_0. Since any element in $\mathfrak{T}_0$ is transitive to a point in $\mathfrak{S}(\Pi)$ by the affine Weyl group of $\mathfrak{T}_0$, there exists an element $g\in\mathrm{Iso}(P(K_1))$ such that $g\cdot T_0=T_0$ and $g\cdot P(K_3)=Q$. This g also satisfies $P(gK_2)\in S$. In fact, for $P\in T_0$ one has $PP(gK_2)=g(g^{-1}\cdot P)P(K_2)g^{-1}=gP(K_2)(g^{-1}\cdot P)g^{-1}$ (because $g^{-1}\cdot P\in T_0$ and $P(K_2)\in S)=P(gK_2)P$. Moreover the distance is given by $d(P(gK_2),\ P))=d(P(K_2),\ g^{-1}\cdot P)=2\sqrt{15}\pi$. Hence we obtain $f(P(gK_2))=\exp\pi/2(\mathrm{ad}\,gK_2)\cdot P(K_1)=g\exp\pi/2(\mathrm{ad}\,K_2)\cdot P(K_1)$ (because $g^{-1}\in\mathrm{Iso}(P(K_1)))=g\cdot P(K_3)=Q$. This means $f(S)\supset S_0$.

Next the converse $f(S)\subset S_0$ is shown. For any $Q\in S$, there exists an element $k\in\mathrm{Iso}(P(K_1))$ by Lemma 4.6 such that $k\mathfrak{T}_0=\mathfrak{T}_0$ and $Q=P(kK_2)$. One has then $f(Q)=\exp\pi/2(\mathrm{ad}\,kK_2)\cdot P(K_1)\in T_0$ and also has $f(Q)=k\cdot P(K_3)$ because $k^{-1}\in\mathrm{Iso}(P(K_1))$. This gives the commutativity of $P(K_1)$ and $f(Q)$, from

$P(K_1)P(K_3)=P(K_3)P(K_1)$. The distance is: $d(P(K_1), f(Q))=d(P(K_1), k\cdot P(K_3))=d(P(K_1), P(K_3))=2\sqrt{15}\pi$. Hence we obtain $f(Q)\subset S_0$.

LEMMA 4.9. *For $Q\in S$, two identities hold:*

$$f(Q)=P(K_1)+(1-2P(K_1))Q\,,$$

and

$$Q=P(K_1)+(1-2P(K_1))f(Q)\,.$$

Proof. The Lie algebra $\mathfrak{G}$ has three involutive automorphisms $\{1-2P(K_i)\}$, $i=1, 2, 3$. They are commutative with one another and satisfy the identity

$$(1-2P(K_1))(1-2P(K_2))(1-2P(K_3))=1\,.$$

For any $Q\in S$ we can take an element $k\in \mathrm{Iso}(P(K_1))$ by Lemma 4.6 such that $k\mathfrak{T}_0=\mathfrak{T}_0$ and $Q=P(kK_2)$. Since $f(Q)=P(kK_3)$ also holds, one obtains

$$(1-2P(K_1))(1-2Q)(1-2f(Q))=1\,.$$

This gives the above identities.

Two following lemmas can be proved easily by Lemma 4.9 and by $U(Q)=\mathrm{Iso}(P(K_1))\cap \mathrm{Iso}(Q)$.

LEMMA 4.10. *For $Q\in S$, $U(Q)=U(f(Q))$ holds.*

LEMMA 4.11. *Let $g\in \mathrm{Iso}(P(K_1))$ and $P, Q\in S$. Then $g\cdot P=Q$ holds if and only if $g\cdot f(P)=f(Q)$ holds.*

The study of the intersection $L(P)\cap L(Q)$ is equivalent, by Proposition 4.3, to that of all lines passing through P and Q. By the transitivity of points and lines in Π, we may take then $P(K_1)$ and $P\in T_0$ as such two points. Hence we define, for each $P\in T_0$, the subset $N(P)$ of Π by

$$N(P)=\{Q\in \Pi \mid P(K_1)\in L(Q) \quad \text{and} \quad P\in L(Q)\}\,.$$

PROPOSITION 4.12. *Let V_1 and V_2 be maximal flat tori of Π and let both pass through $P(K_1)$ and P. Then there exists an element $z\in\mathfrak{U}(P)$ such that* $\exp(\mathrm{ad}\,z)\cdot V_1=V_2$.

Proof. This is the same proof as Lemma 5.9 [2] essentially.

LEMMA 4.13. *For each $P\in T_0$, $N(P)=\{g\cdot Q \mid Q\in S$ and $g\in U(P)_0\}$ holds where $U(P)_0$ is the identity component of $U(P)$.*

Proof. Take any $Q\in N(P)$. The line $L(Q)$ has the rank 8 as a symmetric space because its type is $G(8,8)$. Hence there exists a maximal flat torus V of Π such that $P(K_1)$, $P\in V$ and $V\subset L(Q)$. Since we can find an element $g\in U(P)_0$ by Proposition 4.12 such that $g\cdot V=T_0$, we obtain $g\cdot Q\in S$. This gives $Q=g^{-1}\cdot$

$(g\cdot Q)\in g^{-1}\cdot S$. Conversely let $Q\in S$ and $g\in U(P)_0$. Since $P(K_1)$, $P\in L(Q)$ holds from the definition of S, one has $P(K_1)$, $P\in L(g\cdot Q)$.

For each $P\in T_0$ we define the subset $N_0(P)$ of Π by

$$N_0(P)=\{g\cdot Q \mid Q\in S_0 \quad \text{and} \quad g\in U(P)_0\},$$

and we also define the map $\bar{f}\colon N(P)\to N_0(P)$ by $\bar{f}(g\cdot Q)=g\cdot f(Q)$ where $N(P)$ is in Lemma 4.13 and f appears in Lemma 4.7.

LEMMA 4.14. *$\bar{f}$ is a diffeomorphism for each $P\in T_0$.*

Proof. First the bijectivity of $\bar{f}$ is shown. Let $Q_1, Q_2\in S$ and $g_1, g_2\in U(P)_0$. Then it holds that: $\bar{f}(g_1\cdot Q_1)=\bar{f}(g_2\cdot Q_2)\Leftrightarrow g_2^{-1}g_1\cdot f(Q_1)=f(Q_2)\Leftrightarrow g_2^{-1}g_1\cdot Q_1=Q_2$ (by Lemma 4.11)$\Leftrightarrow g_1\cdot Q_1=g_2\cdot Q_2$. Next let C be any connected component of $N(P)$ and then there exists a point Q in $C\cap S$ by Lemma 4.13. Since $U(Q)_0\cap U(P)_0=U(f(Q))_0\cap U(P)_0$ holds from Lemma 4.10, the components C and $f(C)$ are homogeneous spaces with the same type $U(P)_0/(U(f(Q))_0\cap U(P)_0)$. Furthermore they have the same differentiable structure induced from Π because the identity $g\cdot f(Q)=P(K_1)+(1-2P(K_1))(g\cdot Q)$ holds for $g\in U(P)_0$.

COROLLARY 4.15. *The analysis of the set of all lines passing through $P(K_1)$ and $P\in T_0$ can be reduced to the classification of the orbit $N_0(P)$ of S_0 by $U(P)_0$.*

Let $\mathfrak{U}$ be the Lie algebra of the isotropy group $\mathrm{Iso}(P(K_1))$ at $P(K_1)$. In the following definition we use the roots $\lambda\in\Delta$ of the symmetric space Π (=the restricted roots of $\mathfrak{G}$ to $\mathfrak{T}_0\cap\mathfrak{G}_1(P(K_1))$).

$$\mathfrak{U}_\lambda=\{z\in\mathfrak{U} \mid [x,[x,z]]=\lambda(z)^2 z \quad \text{for} \quad x\in\mathfrak{T}_0\},$$

$$S_\lambda=\{Q\in T_0 \mid Q=\exp(\mathrm{ad}\,x)\cdot P(K_1) \quad \text{with} \quad \lambda(x)\in\pi \boldsymbol{Z}\boldsymbol{i}\}.$$

$$\mathfrak{U}(\mathfrak{T}_0)=\{z\in\mathfrak{U} \mid [z,\mathfrak{T}_0]=\{0\}\}.$$

Note that $0\notin\Delta$. In the case of $\Pi=E_8/Ss(16)$, $\mathfrak{U}(\mathfrak{T}_0)=\{0\}$ holds and Δ is the same as the roots of $\mathfrak{G}$. For $P\in T_0$, the Lie algebra of $U(P)_0$ is given generally by

$$\mathfrak{U}(P)=\mathfrak{U}(\mathfrak{T}_0)\oplus\sum\mathfrak{U}_\lambda,$$

where the index λ runs over the positive roots λ such that $P\in S_\lambda$ (cf. [6], p. 64). We denote the set of such roots λ by $\Lambda(P)$.

Set $\Delta_0=\{\lambda\in\Delta \mid S_0\subset S_\lambda\}$. If P satisfies $\Lambda(P)\subset\Delta_0$, P is said to be in the general position with respect to $P(K_1)$ (in the sense of the projective plane). If P satisfies $\Lambda(P)\cap(\Delta-\Delta_0)\neq\emptyset$, P is said to be in the singular position. Since one has here $\Delta_0=\emptyset$ and $\mathfrak{U}(\mathfrak{T}_0)=\{0\}$, that P is in the singular position is equivalent to that P is a singular point with respect to $P(K_1)$.

Remark. If $\Pi=\mathfrak{C}P_2$, it holds that $S_0=\{P(K_3)\}$ and $\Delta_0=\Delta$. Hence all points except $P(K_1)$ itself are in the general position with respect to $P(K_1)$. This asserts that two distinct points are contained in one and only one line. If $\Pi=E_6/SO(10)\times SO(2)$ or $E_7/SO(12)\times SO(3)$, we have $\Delta_0\neq\emptyset$ and $\Delta-\Delta_0\neq\emptyset$. Then, the singular position can be characterized by the shortest closed geodesics or by 3-dimensional tori with the minimal volume respectively (cf. [2], [3]).

DEFINITION. (i) Two distinct points P and Q in Π are said to be in the general position if P is a regular point with respect to Q in the sense of the symmetric space. If not so, they are said to be in the singular position. (ii) Two distinct lines $L(P)$ and $L(Q)$ are said to be in the general (resp. singular) position if and only if P and Q are in the general (resp. singular) position.

THEOREM 4.16. *Π is a projective plane in the wider sense:*
(i) *For two distinct points there exist exactly* 135 *lines passing through them if the points are in the general position. If in the singular position, the set of such lines becomes one of the* 65 *cases, except for the cases* (9) *and* (67), *given in the table of* §5.
(ii) *The correspondence L asserts the duality of* (i) *for two distinct lines.*

Proof. The second (ii) is a direct consequence from Proposition 4.3. We show (i). By the transitivity we may take $P(K_1)$ and $P\in T_0$ as two distinct points. Then, from Corollary 4.15, the set of lines passing through them is diffeomorphic to the orbit $N_0(P)$ of S_0 by $U(P)_0$. Especially, if the points are in the general position, $N_0(P)=S_0$ holds because the group $\exp(\mathrm{ad}\,\mathfrak{U}(\mathfrak{T}_0))$ leaves S_0 fixed. In the table of §5, (9) is the case that $P(K_1)=P$ and (67) is the case that $P(K_1)$ and P are in the general position.

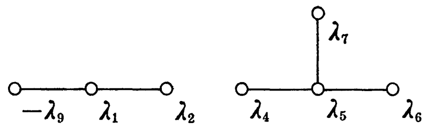
§5. The classification of $N_0(P)$.

In this section we list up the results of the classification of $N_0(P)$ for $P\in S(\Pi)$, where we set $S(\Pi)=\exp(\mathrm{ad}\,\mathfrak{S}(\Pi))\cdot P(K_1)$. The orbit $N_0(P)$ of S_0 is determined by the isotropy group $U(P)_0$. By the similar proof to Proposition 2.8 [8], we can see that $U(P)_0$ is also determined by the fundamental roots λ_i and the highest root $-\lambda_9$ such that $\{\lambda_i, -\lambda_9\}\subset\{\lambda\}$, where $\mathfrak{U}(P)=\mathfrak{U}(\mathfrak{T}_0)\oplus\sum\mathfrak{U}_\lambda$. Hence we calculate the 2^9 cases of $U(P)_0$ and classify $N_0(P)$. The results are obtained by direct calculations. The number of the kinds of orbits is 67 if we count the cases (9) and (67).

For $P\in S(\Pi)$, let $R(P)$ denote the set of the fundamental or the highest roots λ_i such that $P\in S_{\lambda_i}$. We can then construct from $R(P)$ a subdiagram $D(P)$ of the extended Dynkin diagram of $\mathfrak{G}$. In the notation of $R(P)$, the number 9 stands for $-\lambda_9$ and the star * means being empty.

Example.

$$R(P)=(9,\ 1,\ 2,\ *,\ 4,\ 5,\ 6,\ 7,\ *) \Longleftrightarrow$$



$D(P)$

the representative point P:

$$P=\exp(\mathrm{ad}\, z)\cdot P(K_1) \quad \text{with} \quad z=(x_3+x_8)/2.$$

Let Ξ denote the family of all subsets R of $\{-\lambda_9, \lambda_1, \cdots, \lambda_8\}$. Set $\Xi_s=\{R\in\Xi \mid R\supset\{-\lambda_9, \lambda_2, \lambda_4, \lambda_7\}\}$ and $\Xi_r=\Xi-\Xi_s$ respectively. For two distinct points P, $Q\in S(\Pi)$, we consider the subdiagrams $D(P)$, $D(Q)$ and the sets $R(P)$, $R(Q)$ of roots. Then we add the following facts to the results of the classification:

(i) Let $R(P)$, $R(Q)\in\Xi_s$ or $R(P)$, $R(Q)\in\Xi_r$ hold. Then $D(P)$ and $D(Q)$ have the same figure, as sets of points, if and only if the orbits $N_0(P)$ and $N_0(Q)$ are diffeomorphic to each other by some isometry of Π.

(ii) If $R(P)\supset R(Q)$ holds, the orbit $N_0(P)$ contains $N_0(Q)$ as a totally geodesic submanifold.

In the table we list in turn:

(i) the set $R(P)$ of the roots λ_i such that $P\in S_{\lambda_i}$,
(ii) the Lie algebra $\mathfrak{U}(P)$ of $U(P)_0$,
(iii) the types of all connected components C of the orbit $N_0(P)$,
(iv) the number of components with the same type as each C,
(v) the number of points of $C\cap S_0$.

Note that the representative point P satisfying a given $R(P)$ can be obtained easily by the same way as the above example.

(NOTATIONS). T^n is an n-dimensional torus. I stands for one-dimensional center of Lie algebras. A_7, B_n, C_4 and D_n mean the Lie algebras or the Lie groups with such types respectively. $G^C(4,4)=SU(8)/S(U(4)\times U(4))$, $G^H(2,2)=Sp(4)/Sp(2)\times Sp(2)$, $G(m,n)=SO(m+n)/SO(m)\times SO(n)$, $AI(8)=SU(8)/SO(8)$ and $CI(4)=Sp(4)/U(4)$.

Table

(1)	$R=(9, *, 2, 3, 4, 5, 6, 7, 8)$	:	$A_7\oplus I$
	$G^C(4,4)$, $AI(8)\times T^1$		
	$(1, 1)$	:	$(63, 72)$
(2)	$R=(9, 1, *, 3, 4, 5, 6, 7, 8)$	:	$C_4\oplus B_1$
	$G^H(2,2)$, $CI(4)\times G(2,1)$		
	$(1, 1)$	:	$(27, 108)$

(3)	$R=(9,1,2,*,4,5,6,7,8)$	:	$2B_2\oplus D_2$
	{one point}, $G(4,1)\times G(4,1)$, $G(3,2)\times G(3,2)\times G(2,2)$, $G(3,1)\times B_2$		
	$(1,1,1,1)$	:	$(1,10,60,64)$
(4)	$R=(9,1,2,3,*,5,6,7,8)$	:	$2B_2$
	$G(4,1)$, $G(4,1)\times G(4,1)$, $G(3,2)\times G(3,2)$		
	$(2,1,1)$	:	$(5,25,100)$
(5)	$R=(9,1,2,3,4,*,6,7,8)$	:	$I\oplus B_1\oplus D_3$
	$G(2,1)$, $T^1\times G(5,1)$, $G(4,2)$, $G(2,1)\times G(4,2)$, $T^1\times G(2,1)\times G(3,3)$		
	$(1,1,1,1,1)$	:	$(3,12,15,45,60)$
(6)	$R=(9,1,2,3,4,5,*,7,8)$	:	$D_4\oplus I$
	T^1, $G(6,2)$, $G(4,4)$, $G(4,4)\times T^1$		
	$(1,1,1,1)$	:	$(2,28,35,70)$
(7)	$R=(9,1,2,3,4,5,6,*,8)$	:	B_4
	$G(8,1)$, $G(5,4)$		
	$(1,1)$	:	$(9,126)$
(8)	$R=(9,1,2,3,4,5,6,7,*)$	:	$2D_4$
	{one point}, D_4, $G(4,4)\times G(4,4)$		
	$(1,1,1)$	:	$(1,64,70)$
(9)	$R=(9,1,2,3,4,5,6,7,8)$	:	D_8
	$G(8,8)$		
	(1)	:	(135)
(10)	$R=(9,1,2,3,4,5,*,7,*)$	:	D_4
	{one point}, $G(6,2)$, $G(4,4)$		
	$(2,1,3)$	:	$(1,28,35)$
(11)	$R=(9,1,2,*,4,5,*,7,8)$	:	$2D_2\oplus I$
	{one point}, T^1, $G(2,2)$, $G(3,1)\times G(3,1)$, $G(2,2)\times G(2,2)$, $G(3,1)\times G(3,1)\times T^1$, $G(2,2)\times G(2,2)\times T^1$		
	$(1,2,2,2,1,1,1)$	:	$(1,2,6,16,18,32,36)$
(12)	$R=(9,*,2,3,4,5,6,7,*)$	:	$2D_3\oplus I$
	{one point}, $G(4,2)\times G(4,2)$, $D_3\times T^1$, D_3, $G(3,3)\times G(3,3)\times T^1$		
	$(1,1,1,1,1)$	:	$(1,30,32,32,40)$
(13)	$R=(9,*,2,*,4,5,6,7,8)$	:	$2B_2\oplus 2I$
	{one point}, $G(4,1)\times G(4,1)$, $G(3,2)\times G(3,2)$, $B_2\times T^1$, $G(3,2)\times G(3,2)\times T^2$		
	$(1,1,1,2,1)$	:	$(1,10,20,32,40)$
(14)	$R=(9,1,2,*,4,*,6,7,8)$	:	$2I\oplus B_1\oplus D_2$
	{one point}, $G(2,1)$, T^2, $G(2,2)$, $T^1\times G(3,1)$, $G(2,1)\times G(2,2)$, $T^1\times G(2,1)\times G(3,1)$, $T^2\times G(2,1)\times G(2,2)$		
	$(1,2,1,1,2,1,2,1)$	:	$(1,3,4,6,8,18,24,36)$

(15)	$R=(9,1,2,*,4,5,6,7,*)$	:	$3D_2$
	{one point}, D_2, $D_2\times G(2,2)$, $D_2\times G(3,1)$, $G(2,2)\times G(2,2)\times G(2,2)$		
	$(3,1,1,2,1)$	:	$(1,8,24,32,36)$
(16)	$R=(9,1,2,3,4,*,6,7,*)$	:	$2I\oplus D_3$
	{one point}, T^1, $T^1\times G(5,1)$, $G(4,2)$, $T^1\times G(3,3)$, $T^1\times G(4,2)$, $T^2\times G(3,3)$		
	$(1,1,1,2,1,1,1)$	:	$(1,2,12,15,20,30,40)$
(17)	$R=(9,1,*,3,4,*,6,7,8)$	:	$I\oplus 3B_1$
	$G(2,1)$, $T^1\times G(2,1)$, $G(2,1)\times G(2,1)$, $G(2,1)\times G(2,1)\times G(2,1)$, $T^1\times G(2,1)\times G(2,1)\times G(2,1)$		
	$(3,3,3,1,1)$	:	$(3,6,9,27,54)$
(18)	$R=(*,*,2,3,4,5,6,7,8)$	:	A_7
	$G^C(4,4)$, $AI(8)$		
	$(1,1)$	:	$(63,72)$
(19)	$R=(9,*,*,3,4,5,6,7,8)$	:	$C_4\oplus I$
	$G^H(2,2)$, $CI(4)$, $CI(4)\times T^1$		
	$(1,1,1)$	:	$(27,36,72)$
(20)	$R=(*,1,2,*,4,5,6,7,8)$	:	$2B_2\oplus B_1$
	{one point}, $G(4,1)\times G(4,1)$, B_2, $B_2\times G(2,1)$, $G(3,2)\times G(3,2)\times G(2,1)$		
	$(1,1,1,1,1)$	:	$(1,10,16,48,60)$
(21)	$R=(*,1,2,3,4,*,6,7,8)$	:	$I\oplus B_1\oplus B_2$
	T^1, $G(2,1)$, $G(4,1)$, $G(3,2)$, $T^1\times G(4,1)$, $G(2,1)\times G(4,1)$, $G(2,1)\times G(3,2)$, $T^1\times G(2,1)\times G(3,2)$		
	$(1,1,1,1,1,1,1,1)$	:	$(2,3,5,10,10,15,30,60)$
(22)	$R=(9,1,2,3,4,*,6,*,8)$	:	$B_1\oplus D_3$
	$G(2,1)$, $G(5,1)$, $G(4,2)$, $G(2,1)\times G(4,2)$, $G(2,1)\times G(3,3)$		
	$(1,2,1,1,1)$	:	$(3,6,15,45,60)$
(23)	$R=(*,1,2,3,*,5,6,7,8)$	:	$B_2\oplus D_2$
	{one point}, $G(3,1)$, $G(4,1)$, $G(4,1)\times G(3,1)$, $G(3,2)\times G(3,1)$, $G(3,2)\times G(2,2)$		
	$(1,1,2,1,1,1)$	:	$(1,4,5,20,40,60)$
(24)	$R=(*,1,2,3,4,5,*,7,8)$	:	$B_3\oplus I$
	T^1, $G(6,1)$, $G(5,2)$, $G(4,3)$, $G(4,3)\times T^1$		
	$(1,1,1,1,1)$	:	$(2,7,21,35,70)$
(25)	$R=(*,1,2,3,4,5,6,*,8)$	:	D_4
	{one point}, $G(7,1)$, $G(5,3)$, $G(4,4)$		
	$(1,1,1,1)$	:	$(1,8,56,70)$
(26)	$R=(*,1,2,3,4,5,6,7,*)$	:	$2B_3$
	{one point}, B_3, $G(4,3)\times G(4,3)$		
	$(1,1,1)$	:	$(1,64,70)$

(27)	$R=(9,*,2,*,4,5,6,7,*)$	:	$2D_2\oplus 2I$
	{one point}, D_2, $G(2,2)\times G(2,2)$, $D_2\times T^1$, $D_2\times T^2$, $G(2,2)\times G(2,2)\times T^2$		
	$(3,2,1,4,1,1)$	:	$(1,8,12,16,16,24)$
(28)	$R=(9,*,2,*,4,*,6,7,8)$	:	$B_1\oplus 4I$
	{one point}, $G(2,1)$, T^2, $G(2,1)\times T^2$, $G(2,1)\times T^4$		
	$(3,4,6,6,1)$	:	$(1,3,4,12,24)$
(29)	$R=(9,1,2,*,4,5,*,7,*)$	:	$2D_2$
	{one point}, $G(2,2)$, $G(3,1)\times G(3,1)$, $G(2,2)\times G(2,2)$		
	$(5,2,4,3)$	:	$(1,6,16,18)$
(30)	$R=(9,*,2,3,4,5,*,7,*)$	:	$D_3\oplus I$
	{one point}, $G(5,1)\times T^1$, $G(4,2)$, $G(3,3)\times T^1$		
	$(3,1,4,3)$	:	$(1,12,15,20)$
(31)	$R=(9,1,2,*,4,*,6,7,*)$	:	$D_2\oplus 3I$
	{one point}, T^1, T^2, $G(2,2)$, $G(3,1)\times T^1$, $G(2,2)\times T^1$, $G(2,2)\times T^2$, $G(3,1)\times T^2$, $G(2,2)\times T^3$		
	$(3,2,1,2,4,1,1,2,1)$	:	$(1,2,4,6,8,12,12,16,24)$
(32)	$R=(9,1,*,3,4,*,6,*,8)$	:	$3B_1$
	$G(2,1)$, $G(2,1)\times G(2,1)$, $G(2,1)\times G(2,1)\times G(2,1)$		
	$(9,3,3)$	:	$(3,9,27)$
(33)	$R=(*,*,2,3,4,5,6,7,*)$	:	$2D_3$
	{one point}, $G(4,2)\times G(4,2)$, D_3, $G(3,3)\times G(3,3)$		
	$(1,1,2,1)$	:	$(1,30,32,40)$
(34)	$R=(*,*,*,3,4,5,6,7,8)$	:	C_4
	$G^H(2,2)$, $CI(4)$		
	$(1,3)$	:	$(27,36)$
(35)	$R=(9,*,*,*,4,5,6,7,8)$	:	$2B_2\oplus I$
	{one point}, $G(4,1)\times G(4,1)$, B_2, $G(3,2)\times G(3,2)$, $B_2\times T^1$, $G(3,2)\times G(3,2)\times T^1$		
	$(1,1,2,1,1,1)$	:	$(1,10,16,20,32,40)$
(36)	$R=(*,1,2,*,4,5,6,7,*)$	:	$2D_2\oplus B_1$
	{one point}, D_2, $D_2\times G(2,1)$, $G(2,2)\times G(2,2)\times G(2,1)$		
	$(3,3,3,1)$	:	$(1,8,24,36)$
(37)	$R=(*,1,2,*,4,*,6,7,8)$	:	$2B_1\oplus 2I$
	{one point}, T^1, $G(2,1)$, T^2, $G(2,1)\times T^1$, $G(2,1)\times G(2,1)$, $G(2,1)\times G(2,1)\times T^1$, $G(2,1)\times G(2,1)\times T^2$		
	$(1,2,3,1,4,2,2,1)$	:	$(1,2,3,4,6,9,18,36)$
(38)	$R=(*,1,2,3,*,5,*,7,8)$	:	$B_1\oplus D_2\oplus I$
	{one point}, T^1, $G(2,1)$, $G(3,1)$, $G(2,2)$, $G(3,1)\times T^1$, $G(2,1)\times G(3,1)$, $G(2,1)\times G(2,2)$, $G(2,1)\times G(3,1)\times T^1$, $G(2,1)\times G(2,2)\times T^1$		
	$(1,2,2,2,1,1,2,1,1,1)$	:	$(1,2,3,4,6,8,12,18,24,36)$

(39) $R=(*,1,2,3,4,*,6,7,*)$: $B_2\oplus 2I$
{one point}, T^1, $G(4,1)$, $G(3,2)$, $G(4,1)\times T^1$, $G(3,2)\times T^1$, $G(3,2)\times T^2$
$(1,2,2,2,2,2,1)$: $(1,2,5,10,10,20,40)$

(40) $R=(*,1,2,3,4,*,6,*,8)$: $B_1\oplus B_2$
{one point}, $G(2,1)$, $G(4,1)$, $G(3,2)$, $G(2,1)\times G(4,1)$,
$G(2,1)\times G(3,2)$
$(2,1,3,1,1,3)$: $(1,3,5,10,15,30)$

(41) $R=(*,1,2,3,*,5,6,7,*)$: $2D_2$
{one point}, $G(3,1)$, $G(3,1)\times G(3,1)$, $G(2,2)\times G(3,1)$, $G(2,2)\times G(2,2)$
$(1,4,2,2,1)$: $(1,4,16,24,36)$

(42) $R=(*,1,2,3,4,5,*,*,8)$: $D_3\oplus I$
{one point}, T^1, $G(5,1)$, $G(4,2)$, $G(3,3)$, $G(4,2)\times T^1$, $G(3,3)\times T^1$
$(1,1,2,2,1,1,1)$: $(1,2,6,15,20,30,40)$

(43) $R=(*,1,2,3,4,5,6,*,*)$: B_3
{one point}, $G(6,1)$, $G(5,2)$, $G(4,3)$
$(2,1,1,3)$: $(1,7,21,35)$

(44) $R=(9,*,2,*,4,5,*,7,*)$: $D_2\oplus 2I$
{one point}, T^2, $G(2,2)$, $G(3,1)\times T^1$, $G(2,2)\times T^2$
$(7,1,4,8,3)$: $(1,4,6,8,12)$

(45) $R=(9,*,2,*,4,*,*,7,8)$: $5I$
{one point}, T^1, T^2, T^3, T^4, T^5
$(7,4,12,6,1,1)$: $(1,2,4,8,8,16)$

(46) $R=(*,*,*,*,4,5,6,7,8)$: $2B_2$
{one point}, $G(4,1)\times G(4,1)$, B_2, $G(3,2)\times G(3,2)$
$(1,1,4,3)$: $(1,10,16,20)$

(47) $R=(9,*,*,*,4,5,6,7,*)$: $2D_2\oplus I$
{one point}, D_2, $G(2,2)\times G(2,2)$, $D_2\times T^1$, $G(2,2)\times G(2,2)\times T^1$
$(3,6,1,3,1)$: $(1,8,12,16,24)$

(48) $R=(*,1,2,*,4,*,6,7,*)$: $B_1\oplus 3I$
{one point}, T^1, $G(2,1)$, T^2, $G(2,1)\times T^1$, $G(2,1)\times T^2$, $G(2,1)\times T^3$
$(3,6,4,3,6,3,1)$: $(1,2,3,4,6,12,24)$

(49) $R=(*,1,2,3,*,5,*,*,8)$: $D_2\oplus 2I$
{one point}, T^1, $G(3,1)$, $G(2,2)$, $G(3,1)\times T^1$, $G(2,2)\times T^1$,
$G(3,1)\times T^2$, $G(2,2)\times T^2$
$(3,4,4,2,4,2,1,1)$: $(1,2,4,6,8,12,16,24)$

(50) $R=(*,1,2,*,4,5,*,*,8)$: $2B_1\oplus I$
{one point}, T^1, $G(2,1)$, $G(2,1)\times T^1$, $G(2,1)\times G(2,1)$,
$G(2,1)\times G(2,1)\times T^1$
$(3,3,8,2,4,3)$: $(1,2,3,6,9,18)$

(51)	$R=(*,1,2,3,*,5,6,*,*)$	:	$B_1\oplus D_2$
	{one point}, $G(2,1)$, $G(3,1)$, $G(2,2)$, $G(2,1)\times G(3,1)$, $G(2,1)\times G(2,2)$		
	$(5,2,4,1,4,3)$	:	$(1,3,4,6,12,18)$
(52)	$R=(*,1,2,3,4,*,6,*,*)$	:	$B_2\oplus I$
	{one point}, T^1, $G(4,1)$, $G(3,2)$, $G(4,1)\times T^1$, $G(3,2)\times T^1$		
	$(3,1,4,4,1,3)$	:	$(1,2,5,10,10,20)$
(53)	$R=(*,1,2,3,4,5,*,*,*)$	:	D_3
	{one point}, $G(5,1)$, $G(4,2)$, $G(3,3)$		
	$(3,2,4,3)$	:	$(1,6,15,20)$
(54)	$R=(9,*,2,*,4,*,*,7,*)$	:	$4I$
	{one point}, T^2, T^4		
	$(11,24,3)$	:	$(1,4,8)$
(55)	$R=(*,1,*,3,*,5,*,*,8)$	:	$4I$
	{one point}, T^1, T^2, T^3, T^4		
	$(7,16,12,4,1)$	:	$(1,2,4,8,16)$
(56)	$R=(*,*,*,*,4,5,6,7,*)$	:	$2D_2$
	{one point}, D_2, $G(2,2)\times G(2,2)$		
	$(3,12,3)$	:	$(1,8,12)$
(57)	$R=(*,1,2,*,4,*,6,*,*)$	:	$B_1\oplus 2I$
	{one point}, T^1, $G(2,1)$, T^2, $G(2,1)\times T^1$, $G(2,1)\times T^2$		
	$(7,8,8,1,8,3)$	:	$(1,2,3,4,6,12)$
(58)	$R=(*,1,2,3,*,5,*,*,*)$	:	$D_2\oplus I$
	{one point}, T^1, $G(3,1)$, $G(2,2)$, $G(3,1)\times T^1$, $G(2,2)\times T^1$		
	$(7,2,8,4,4,3)$	:	$(1,2,4,6,8,12)$
(59)	$R=(*,1,2,*,4,5,*,*,*)$	:	$2B_1$
	{one point}, $G(2,1)$, $G(2,1)\times G(2,1)$		
	$(9,12,10)$	:	$(1,3,9)$
(60)	$R=(*,1,2,3,4,*,*,*,*)$	:	B_2
	{one point}, $G(4,1)$, $G(3,2)$		
	$(5,6,10)$	:	$(1,5,10)$
(61)	$R=(*,1,*,3,*,5,*,*,*)$	:	$3I$
	{one point}, T^1, T^2, T^3		
	$(15,24,12,3)$	:	$(1,2,4,8)$
(62)	$R=(*,1,2,*,4,*,*,*,*)$	:	$B_1\oplus I$
	{one point}, T^1, $G(2,1)$, $G(2,1)\times T^1$		
	$(15,6,16,10)$	:	$(1,2,3,6)$
(63)	$R=(*,1,2,3,*,*,*,*,*)$	:	D_2
	{one point}, $G(3,1)$, $G(2,2)$		
	$(11,16,10)$	:	$(1,4,6)$

(64)	$R=(*,*,*,*,*,*,6,*,8)$	:	B_1
	{one point}, $G(2,1)$		
	(27, 36)	:	(1, 3)
(65)	$R=(*,*,*,*,*,5,*,*,8)$	:	$2I$
	{one point}, T^1, T^2		
	(31, 32, 10)	:	(1, 2, 4)
(66)	$R=(*,1,*,*,*,*,*,*,*)$	:	I
	{one point}, T^1		
	(63, 36)	:	(1, 2)
(67)	$R=(*,*,*,*,*,*,*,*,*)$	:	0
	{one point}		
	(135)	:	(1)

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KUMAMOTO INSTITUTE OF TECHNOLOGY
IKEDA, KUMAMOTO 860
JAPAN