M. SHIMURA
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MEANDERING POINTS OF TWO-DIMENSIONAL BROWNIAN MOTION

BY MICHIO SHIMURA

§1. Introduction and the result.

Let $Z_w(t)=(X_w(t), Y_w(t))$, $w\in W$, $-\infty<t<\infty$, be the *two-dimensional standard Brownian motion* with $Z_w(0)=\mathbf{0}$ on a probability space (W, β, P). Let $\mathcal{U}$ denote the set of all unit vectors in $\boldsymbol{R}^2$. For every $\boldsymbol{u}$ in $\mathcal{U}$ we set a half-plane $H(\boldsymbol{u})=\{\boldsymbol{x}\in\boldsymbol{R}^2 \mid \boldsymbol{u}\cdot\boldsymbol{x}\geqq 0\}$, where $\boldsymbol{u}\cdot\boldsymbol{x}$ denotes the inner product. For $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\mathcal{U}$ we consider the random set of all two-sided *meandering times* of the Brownian motion:

$$\mathcal{M}_w(\boldsymbol{u}_1, \boldsymbol{u}_2)=\{-\infty<t<\infty \mid {}^{\exists}h>0 \text{ such that } Z_w(s)\in H(\boldsymbol{u}_1)+Z_w(t) \quad \text{for } t-h<{}^{\forall}s<t$$
$$\& \quad Z_w(s)\in H(\boldsymbol{u}_2)+Z_w(t) \quad \text{for } t<{}^{\forall}s<t+h\}.$$

In this paper we will prove the following theorem.

THEOREM 1. *For every $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\mathcal{U}$ with $\boldsymbol{u}_1\neq\boldsymbol{u}_2$, we have $\mathcal{M}_w(\boldsymbol{u}_1, \boldsymbol{u}_2)=\emptyset$ almost surely* (a.s. for abbreviation).

Our problem arises from the following observation. By a result of Evans [2] we have $\dim\mathcal{M}_w(\boldsymbol{u}_1, \boldsymbol{u}_2)=1-\pi/2\pi-\pi/2\pi=0$ a.s.. Here we note that, in such a critical case, we do not know from the result whether the set is empty or not (a.s.). Indeed, both cases may occur: Obviously $\mathcal{M}_w(\boldsymbol{u}, \boldsymbol{u})\neq\emptyset$ a.s.; $\mathcal{M}_w(\boldsymbol{u}, -\boldsymbol{u})=\emptyset$ a.s. from Dvoretzky, Erdös and Kakutani [1] on the nonexistence of points of increase (decrease) for the one-dimensional Brownian motion. So, it may be interesting to see if the set is empty or not (a.s.) for $\boldsymbol{u}_1\neq\boldsymbol{u}_2$. By Theorem 1 we answer the problem. As will be shown in the following sections, the proof of Theorem 1 in [1] still works to ours by some modification.

The paper is organized as follows. In §2 we give preliminaries to our proof of Theorem 1. We show in §3 two lemmas which play key role in §4, where the proof of Theorem 1 is given.

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§ 2. Preliminaries.

We may take in Theorem 1 $\boldsymbol{u}_1=(0, 1)$ and $\boldsymbol{u}_2=(-\sin\theta, \cos\theta)$, $0<\theta\leqq\pi$, without loss of generality. We write $\mathcal{M}_w=\mathcal{M}_w(\boldsymbol{u}_1, \boldsymbol{u}_2)$. Besides the xy coordinate system we set the $x'y'$ one in which the y'-axis is directed toward the vector $\boldsymbol{u}_2$. We put $Z_w(t)=(X'_w(t), Y'_w(t))$ in this system. Note that each of the processes $X_w(\cdot)$, $Y_w(\cdot)$, $X'_w(\cdot)$ and $Y'_w(\cdot)$ is the one-dimensional standard Brownian motion. For $-\infty<s<t<\infty$ we define

$$\underline{Y}_w[s, t]=\min\{Y_w(u) \mid s\leqq u\leqq t\} \quad \text{and} \quad \overline{Y}_w[s, t]=\max\{Y_w(u) \mid s\leqq u\leqq t\}$$

($\underline{Y}'_w[s, t]$, $\overline{Y}'_w[s, t]$, etc are defined in the same way).

We put

$$A=\{w\in W \mid {}^{\exists}t\in[0, 1] \text{ such that } \underline{Y}_w[t-2, t]\geqq Y_w(t) \ \ \& \ \ Y'_w(t)\leqq\underline{Y}'_w[t, t+2]\}.$$

It is easy to see that Theorem 1 follows if we have $P(A)=0$. For $n\geqq 1$ and $1\leqq k\leqq 2n$ we set

$$A_k^n=\Big\{w\in W \mid \underline{Y}_w\Big[\frac{k}{n}-2, \frac{k-1}{n}\Big]\geqq\underline{Y}_w\Big[\frac{k-1}{n}, \frac{k}{n}\Big]$$
$$\& \ \ \underline{Y}'_w\Big[\frac{k-1}{n}, \frac{k}{n}\Big]\leqq\underline{Y}'_w\Big[\frac{k}{n}, \frac{k-1}{n}+2\Big]\Big\}.$$

Then $A=\bigcap_{n=1}^{\infty}\bigcup_{k=1}^{n}A_k^n$. Here we note a modification made in the definition of the sets A and A_k^n from those given in (5.5) and (5.6) in [1]. Such a change will be necessary to treat the case $\boldsymbol{u}_1\neq\boldsymbol{u}_2$. Put

$$S_k^n=S_k^n(w)=\textstyle\sum_{j=1}^{k}1_w(A_j^n),$$

where $1_w(A)$ is the indicator function on a set A ($\subseteqq W$). Since $P(A)\leqq P(S_n^n\geqq 1)$ for all n, we will prove as in [1]

$$P(S_n^n\geqq 1)\leqq\frac{E(S_{2n}^n)}{E(S_{2n}^n \mid S_n^n\geqq 1)}\to 0 \quad \text{as } n\to\infty \tag{2.1}$$

in § 4 to get Theorem 1.

§ 3. Two lemmas.

Before showing the lemmas we list some formulas which will be used often later. Put

$$\Phi_t(\xi)=\Big(\frac{2}{\pi t}\Big)^{1/2}\int_0^{\xi}\exp\Big(-\frac{x^2}{2t}\Big)dx \quad \text{for } t>0 \text{ and } \xi\geqq 0.$$

It is well-known that, for $t>0$ and $\xi\leqq 0$,

$$P(\underline{Y}_w[-t, 0]\geqq\xi)=P(\underline{Y}_w[0, t]\geqq\xi)=\Phi_t(-\xi) \tag{3.1}$$

((3.1) also holds by replacing the process Y by another one, e.g., the Y'). It holds from the inequalities $1-x \leqq \exp(-x) \leqq 1$ for $x \geqq 0$ the following:

$$\Phi_t(\xi) \leqq \left(\frac{2}{\pi t}\right)^{1/2} \int_0^{\xi} dx = \left(\frac{2}{\pi t}\right)^{1/2} \xi \quad \text{for } \xi \geqq 0 \tag{3.2}$$

and

$$\Phi_t(\xi) \geqq \left(\frac{2}{\pi t}\right)^{1/2} \int_0^{\xi} \left(1-\frac{x^2}{2t}\right) dx \geqq (2\pi t)^{-1/2} \xi \quad \text{for } 0 \leqq \xi \leqq (3t)^{1/2}. \tag{3.3}$$

Moreover

$$1-\Phi_t(\xi) < \left(\frac{2}{\pi t}\right)^{1/2} \int_{\xi}^{\infty} x \exp\left(-\frac{x^2}{2t}\right) dx = \left(\frac{2}{\pi t}\right)^{1/2} \exp\left(-\frac{\xi^2}{2t}\right) \quad \text{for } \xi \geqq 1. \tag{3.4}$$

Let $f(s) \asymp s^{-\alpha}$ $(s \to c)$ denote

$$0 < \liminf_{s \to c} s^{\alpha} f(s) \leqq \limsup_{s \to c} s^{\alpha} f(s) < \infty.$$

Firstly we show the following lemma.

LEMMA 1. *We have* $P(A_1^n) \asymp n^{-1}$ $(n \to \infty)$.

Proof. Set $H^n(d\xi, d\eta) = P(\underline{Y}_w[0, 1/n] \in d\xi \ \&\ \underline{Y}'_w[0, 1/n] - Y'_w(1/n) \in d\eta)$. Since the Brownian motion has independent and stationary increments, the following identity holds:

$$\begin{aligned} P(A_1^n) = P\Big(&\underline{Y}_w\left[\frac{1}{n}-2, 0\right] \geqq \underline{Y}_w\left[0, \frac{1}{n}\right] \ \& \\ &\underline{Y}'_w\left[0, \frac{1}{n}\right] - Y'_w\left(\frac{1}{n}\right) \leqq \underline{Y}'_w\left[\frac{1}{n}, 2\right] - Y'_w\left(\frac{1}{n}\right)\Big) \\ = \int_{-\infty}^{0}\int_{-\infty}^{0} &P\left(\underline{Y}_w\left[\frac{1}{n}-2, 0\right] \geqq \xi\right) P\left(\underline{Y}'_w\left[0, 2-\frac{1}{n}\right] \geqq \eta\right) H^n(d\xi, d\eta). \end{aligned}$$

Therefore, together with the scaling relation $H^n(d\xi, d\eta) = H^1(n^{1/2}d\xi, n^{1/2}d\eta)$, we conclude from (3.1), (3.2) and from (3.3) the following:

$$P(A_1^n) \leqq \int_{-\infty}^{0}\int_{-\infty}^{0} \Phi_1(-\xi)\Phi_1(-\eta) H^n(d\xi, d\eta) \leqq \frac{2}{\pi n} \int_{-\infty}^{0}\int_{-\infty}^{0} |\xi\eta| H^1(d\xi, d\eta)$$

and

$$\begin{aligned} P(A_1^n) &\geqq \int_{-6^{1/2}}^{0}\int_{-6^{1/2}}^{0} \Phi_2(-\xi)\Phi_2(-\eta) H^n(d\xi, d\eta) \\ &\geqq \frac{1}{4\pi n} \int_{-(6n)^{1/2}}^{0}\int_{-(6n)^{1/2}}^{0} |\xi\eta| H^1(d\xi, d\eta) \end{aligned}$$

$\left(\text{note } 0 < \int_{-\infty}^{0}\int_{-\infty}^{0} |\xi\eta| H^1(d\xi, d\eta) < \infty\right)$. This proves the lemma.

Let $G(\varDelta, \delta)$, $0 \leqq \varDelta, \delta$, denote the probability

$$P(\underline{Y}'_w[0, 1] \geqq -\varDelta \ \& \ Y'_w(1) \geqq 1 \ \& \ \underline{Y}_w[0, 1] \geqq Y_w(1) - \delta \ \& \ Y_w(1) \leqq -1).$$

Next we show the following lemma.

LEMMA 2. *There exists a positive constant K such that the following holds:*

$$G(\varDelta, \delta) \geqq K \varDelta \delta \quad \textit{for every } 0 \leqq \varDelta, \delta \leqq 1. \tag{3.5}$$

Proof. Let us consider the case $0 < \theta \leqq \pi/2$. Take the point A where the lines $y = -2$ and $y' = -1$ cross each other. Note that the x' co-ordinate of A, say a, is less than -1.73. Let B denote the point of intersection of the lines $y = -4$ and $y' = 1$, and put by b its x co-ordinate. Take a disc D of diameter $1/2$ contained in the region $\{z = (x, y) \mid x < b\} \cap \{z = (x', y') \mid a < x' < 0\}$. Then, an elementary geometric consideration, together with the independence and the stationarity of increments of the Brownian motion, lead us to the following estimate (see Figure 1 below):

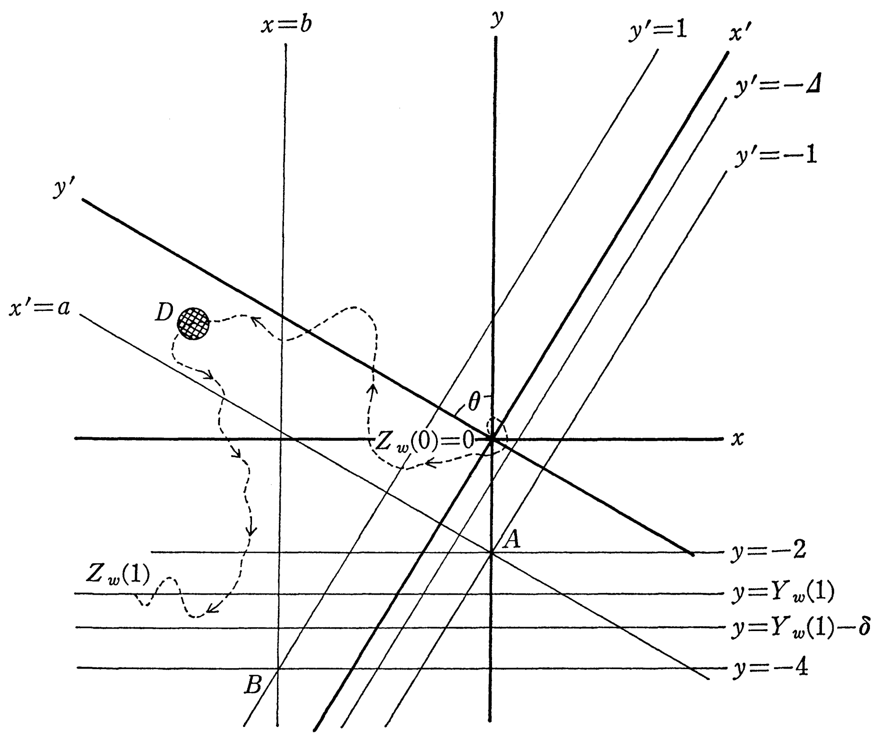

Figure 1

$$G(\varDelta, \delta)>P\Big(\underline{Y}'_w\Big[0, \frac{1}{2}\Big]\geqq-\varDelta \ \& \ \underline{X}'_w\Big[0, \frac{1}{2}\Big]\geqq a \ \&$$

$$Z_w\Big(\frac{1}{2}\Big)\in D \ \& \ \underline{Y}_w\Big[\frac{1}{2}, 1\Big]\geqq Y_w(1)-\delta \ \& \ -3\leqq Y_w(1)\leqq-2 \ \&$$

$$\bar{X}_w\Big[\frac{1}{2}, 1\Big]\leqq b\Big)=\iint_D P\Big(\underline{Y}'_w\Big[0, \frac{1}{2}\Big]\geqq-\varDelta \ \& \ \underline{X}'_w\Big[0, \frac{1}{2}\Big]\geqq a \ \&$$

$$\Big(X_w\Big(\frac{1}{2}\Big), Y_w\Big(\frac{1}{2}\Big)\Big)\in dx\times dy\Big)P\Big(\underline{Y}_w\Big[0, \frac{1}{2}\Big]\geqq Y_w\Big(\frac{1}{2}\Big)-\delta \ \&$$

$$-3-y\leqq Y_w\Big(\frac{1}{2}\Big)\leqq-2-y \ \& \ \bar{X}_w\Big[0, \frac{1}{2}\Big]\leqq b-x\Big).$$

Put $\min\{b-x\,|\,(x, y)\in D\}=b'$ (>0) and $\min\{-2-y\,|\,(x, y)\in D\}=c$, Then, noting $\max\{-3-y\,|\,(x, y)\in D\}=c-1/2$, we have

$$G(\varDelta, \delta)>P\Big(\underline{Y}'_w\Big[0, \frac{1}{2}\Big]\geqq-\varDelta \ \& \ \underline{X}'_w\Big[0, \frac{1}{2}\Big]\geqq a \ \& \tag{3.6}$$

$$Z_w\Big(\frac{1}{2}\Big)\in D\Big)P\Big(\underline{Y}_w\Big[0, \frac{1}{2}\Big]\geqq Y_w\Big(\frac{1}{2}\Big)-\delta \ \& \ c-\frac{1}{2}\leqq Y_w\Big(\frac{1}{2}\Big)\leqq c \ \&$$

$$\bar{X}_w\Big[0, \frac{1}{2}\Big]\leqq b'\Big)$$

(say IJ). In terms of conditional probability

$$I=P\Big(\underline{Y}'_w\Big[0, \frac{1}{2}\Big]\geqq-\varDelta\Big)P\Big(\underline{X}'_w\Big[0, \frac{1}{2}\Big]\geqq a \ \& \ Z_w\Big(\frac{1}{2}\Big)\in D\,|\,\underline{Y}'_w\Big[0, \frac{1}{2}\Big]\geqq-\varDelta\Big)$$

(say I_1I_2). Making use of the fact that $\tilde{Z}_w(t)=Z_w(1/2-t)-Z_w(1/2)$, $-\infty<t<\infty$, is also a standard Brownian motion, we have

$$J=P\Big(\underline{Y}_w\Big[0, \frac{1}{2}\Big]\geqq-\delta \ \& \ -c\leqq Y_w\Big(\frac{1}{2}\Big)\leqq\frac{1}{2}-c \ \&$$

$$\bar{X}_w\Big[0, \frac{1}{2}\Big]\leqq b'+X_w\Big(\frac{1}{2}\Big)\Big)=P\Big(\underline{Y}_w\Big[0, \frac{1}{2}\Big]\geqq-\delta\Big)P\Big(-c\leqq Y_w\Big(\frac{1}{2}\Big)\leqq\frac{1}{2}-c \ \&$$

$$\bar{X}_w\Big[0, \frac{1}{2}\Big]\leqq b'+X_w\Big(\frac{1}{2}\Big)\Big|\underline{Y}_w\Big[0, \frac{1}{2}\Big]\geqq-\delta\Big)$$

(say J_1J_2). Then, by (3.1), (3.2) and by (3.3) we get $I_1\asymp\varDelta$ $(\varDelta\to+0)$ and $J_1\asymp\delta$ $(\delta\to+0)$. Moreover, it follows from the limit theorem of conditioned Brownian motion (see, Shimura [3]) both I_2 and I_3 tend to positive numbers as $\varDelta\to+0$ and $\delta\to+0$ respectively. Hence we have (3.5) from (3.6).

We can show (3.5) for the case $\pi/2<\theta\leqq\pi$ in a similar way, so we omit it here.

§ 4. Proof of Theorem 1.

In this section $K_1, K_2, \cdots$ will denote some positive constants. Note that $P(A_k^n)=P(A_1^n)$ for $1 \leqq k \leqq 2n$, because the Brownian motion has stationary increments. Then we have from Lemma 1

$$E(S_{2n}^n)=\sum_{k=1}^{2n} P(A_k^n) \leqq 2n(K_1/n)=2K_1<\infty . \tag{4.1}$$

Set $B_k^n=A_k^n-\bigcup_{j=1}^{k-1} A_j^n$, and denote by C_k^n the event

$$\bigcap_{j=1}^{k-1}\left\{w \in W \mid \underline{Y}_w\left[\frac{j}{n}-2, \frac{j-1}{n}\right]<\underline{Y}_w\left[\frac{j-1}{n}, \frac{j}{n}\right] \text{ or } \underline{Y}_w'\left[\frac{j-1}{n}, \frac{j}{n}\right]>\underline{Y}_w'\left[\frac{j}{n}, \frac{k}{n}\right]\right\}.$$

Let $F_k^n(x)$ denote the conditional probability distribution function

$$P\left(\underline{Y}_w\left[\frac{k}{n}-2, \frac{k-1}{n}\right] \geqq \underline{Y}_w\left[\frac{k-1}{n}, \frac{k}{n}\right] \;\&\; Y_w'\left(\frac{k}{n}\right)-\underline{Y}_w'\left[\frac{k-1}{n}, \frac{k}{n}\right] \leqq x \mid C_k^n\right).$$

Note that $B_k^n=C_k^n \cap A_k^n$ and that C_k^n is an event given in terms of $Z_w(s)$, $-\infty<s \leqq k/n$. Then, making use of the independence and the stationarity of the increments, we have

$$P(B_k^n)=P(C_k^n)\left\{\int_0^{(j-1/n)^{1/2}}+\int_{(j-1/n)^{1/2}}^{\infty}\right\} dF_k^n(x) P\left(\underline{Y}_w'\left[0, 2-\frac{1}{n}\right] \geqq -x\right).$$

Then, as was shown in [1] (7.23), we conclude from (3.1), (3.2) and from (3.4) the following:

$$P(B_k^n)<2P(C_k^n) \int_0^{(j-1/n)^{1/2}} x\, dF_k^n(x) \tag{4.2}$$

for every k and j satisfying

$$1 \leqq k \leqq n \text{ with } P(B_k^n) \geqq 2/n^2 \text{ and } 6 \log n+1 \leqq j \leqq 2n-k \tag{4.3}$$

(note $P(B_1^n)=P(A_1^n) \geqq 2/n^2$ for almost all n by Lemma 1).

Noting $B_k^n \cap A_{k+j}^n=C_k^n \cap A_k^n \cap A_{k+j}^n$, we have

$$\begin{aligned} P(B_k^n \cap A_{k+j}^n) \geqq P\Big(C_k^n \;\&\; & \underline{Y}_w\left[\frac{k}{n}-2, \frac{k-1}{n}\right] \geqq \underline{Y}_w\left[\frac{k-1}{n}, \frac{k}{n}\right] \;\& \\ & \underline{Y}_w'\left[\frac{k-1}{n}, \frac{k}{n}\right] \leqq \underline{Y}_w'\left[\frac{k}{n}, \frac{k+j}{n}\right] \;\& \\ & \underline{Y}_w'\left[\frac{k+j-1}{n}, \frac{k+j}{n}\right] \geqq Y_w'\left(\frac{k+j-1}{n}\right)-\left(\frac{j-1}{n}\right)^{1/2} \geqq Y_w'\left(\frac{k}{n}\right) \;\& \\ & \underline{Y}_w\left[\frac{k-1}{n}, \frac{k+j-1}{n}\right] \geqq \underline{Y}_w\left[\frac{k+j-1}{n}, \frac{k+j}{n}\right] \;\& \end{aligned} \tag{4.4}$$

$$\underline{Y}_w\Big[\frac{k-1}{n}, \frac{k}{n}\Big] \geqq Y_w\Big(\frac{k}{n}\Big) - \Big(\frac{j-1}{n}\Big)^{1/2} \geqq Y_w\Big(\frac{k+j-1}{n}\Big) \;\&$$

$$\underline{Y}'_w\Big[\frac{k+j-1}{n}, \frac{k+j}{n}\Big] \leqq \underline{Y}'_w\Big[\frac{k+j}{n}, \frac{k+j-1}{n}+2\Big]\Big)$$

$$> P\Big(C_k^n \;\&\; \underline{Y}_w\Big[\frac{k}{n}-2, \frac{k-1}{n}\Big] \geqq \underline{Y}_w\Big[\frac{k-1}{n}, \frac{k}{n}\Big] \;\&$$

$$\underline{Y}'_w\Big[\frac{k-1}{n}, \frac{k}{n}\Big] \leqq \underline{Y}'_w\Big[\frac{k}{n}, \frac{k+j-1}{n}\Big] \;\&\; Y'_w\Big(\frac{k+j-1}{n}\Big) - Y'_w\Big(\frac{k}{n}\Big) \geqq \Big(\frac{j-1}{n}\Big)^{1/2} \;\&$$

$$\underline{Y}_w\Big[\frac{k}{n}, \frac{k+j-1}{n}\Big] \geqq \underline{Y}_w\Big[\frac{k+j-1}{n}, \frac{k+j}{n}\Big] \;\&\; Y_w\Big(\frac{k+j-1}{n}\Big) - Y_w\Big(\frac{k}{n}\Big)$$

$$\leqq -\Big(\frac{j-1}{n}\Big)^{1/2} \;\&\; \underline{Y}'_w\Big[\frac{k+j-1}{n}, \frac{k+j}{n}\Big] \leqq \underline{Y}'_w\Big[\frac{k+j}{n}, \frac{k+j-1}{n}+2\Big]\Big)$$

$$-P\Big(\underline{Y}_w\Big[\frac{k-1}{n}, \frac{k}{n}\Big] < Y_w\Big(\frac{k}{n}\Big) - \Big(\frac{j-1}{n}\Big)^{1/2}\Big) - P\Big(\underline{Y}'_w\Big[\frac{k+j-1}{n}, \frac{k+j}{n}\Big]$$

$$< Y'_w\Big(\frac{k+j-1}{n}\Big) - \Big(\frac{j-1}{n}\Big)^{1/2}\Big)$$

(say L_1-L_2-L_3). We apply the independence and the stationarity of the increments repeatedly to get the following estimate of L_1:

$$L_1 > P(C_k^n)\int_0^\infty dF_k^n(x) P\Big(\underline{Y}'_w\Big[0, \frac{j-1}{n}\Big] \geqq -x \;\&\; Y'_w\Big(\frac{j-1}{n}\Big) \geqq \Big(\frac{j-1}{n}\Big)^{1/2} \;\& \tag{4.5}$$

$$\underline{Y}_w\Big[0, \frac{j-1}{n}\Big] - Y_w\Big(\frac{j-1}{n}\Big) \geqq -n^{-1/2} \;\&\; \underline{Y}_w\Big[\frac{j-1}{n}, \frac{j}{n}\Big] - Y_w\Big(\frac{j-1}{n}\Big) \leqq -n^{-1/2}$$

$$\&\; Y_w\Big(\frac{j-1}{n}\Big) \leqq -\Big(\frac{j-1}{n}\Big)^{1/2} \;\&\; \underline{Y}'_w\Big[\frac{j-1}{n}, \frac{j}{n}\Big] \leqq \underline{Y}'_w\Big[\frac{j}{n}, \frac{j-1}{n}+2\Big]\Big)$$

$$> P(C_k^n)\int_0^{(j/n-1/n)^{1/2}} dF_k^n(x) G\Big(\Big(\frac{n}{j-1}\Big)^{1/2} x, (j-1)^{-1/2}\Big) P\Big(\underline{Y}_w[0, 1] \leqq -1 \;\&$$

$$\underline{Y}'_w[0, 1] \leqq -1 \;\&\; Y'_w(1) \geqq 1\Big) P\Big(\underline{Y}'_w\Big[0, 2-\frac{1}{n}\Big] \geqq -2n^{-1/2}\Big).$$

By (3.1), (3.2) and by (3.3)

$$P\Big(\underline{Y}'_w\Big[0, 2-\frac{1}{n}\Big] \geqq -2n^{-1/2}\Big) \asymp n^{-1/2} \quad (n \to \infty),$$

and by (3.4)

$$L_2 = L_3 = P(\underline{Y}_w[0, 1] < -(j-1)^{1/2}) \leqq n^{-3} \quad \text{for } j \geqq 6 \log n + 1.$$

Then it follows from (4.4), (4.5) and from Lemma 2 the following:

$$P(B_k^n \cap A_{k+j}^n) > K_2 \frac{P(C_k^n)}{j-1} \int_0^{(j/n-1/n)^{1/2}} x\,dF_k^n(x) - 2n^{-3} \quad \text{for } j \geqq 6 \log n + 1. \tag{4.6}$$

Therefore, from (4.2) and (4.6) we have

$$P(A_{k+j}^n \mid B_k^n) > K_3(j-1)^{-1} - n^{-1} \tag{4.7}$$

for every k and j satisfying (4.3) (see (7.24) in [1]).

Once we have (4.7), we can show in the same way to [1] p. 115 the following:

$$E(S_{2n}^n \mid S_n^n \geqq 1) > K_4 \log n \quad \text{for all } n,$$

from which, together with (4.1), we conclude (2.1). This proves the theorem.

Concluding Remark. Suppose that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ $(= \boldsymbol{u})$. Then $G(\Delta, \delta) = 0$ for every Δ and δ, because the processes Y and Y' are coincident. So, in the case, we note that the right hand side of (4.5) vanishes and that, as a result, we do not have $\lim_{n\to\infty} E(S_{2n}^n \mid S_n^n \geqq 1) = \infty$ which would lead an contradictory assertion $\mathcal{M}_w(\boldsymbol{u}, \boldsymbol{u}) = \emptyset$ a.s..

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