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THE MORDELL-BOMBIERI-NOGUCHI CONJECTURE OVER FUNCTION FIELDS

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§1. **Introduction.** G. Faltings recently proved the Mordell conjecture [F]. The author learned from J. Noguchi that E. Bombieri made the following conjecture generalizing the conjecture above (cf. also [L]):

The set of rational points of any projective variety of general type over an algebraic number field is not Zariski dense.

Noguchi ([N1], [N2]) has obtained some results over function fields which are analogues of the Bombieri conjecture.

CONJECTURE A (Noguchi). *Let $f: X \to S$ be a proper surjective map between non-singular projective varieties over the complex number field. Let σ_λ denote the rational sections of f. Assume that a general fibre X_s of f is a variety of general type and that the union of $S_\lambda = \sigma_\lambda(S)$ is Zariski dense in X. Then X is birationally trivial, i.e., there exists a projective variety X_0 such that X is birational to $X_0 \times S$.*

We pose the following conjecture, which implies Conjecture A.

CONJECTURE B. *Let X and S be non-singular projective varieties. Then there exists an ample divisor D on S such that for any birational embedding j_λ: $S \to X$ we have $\mathcal{O}(j_\lambda^* K_X) \subset \mathcal{O}(D)$.*

Note that when X is the minimal model of a surface with $\kappa(X) \geqq 0$, Miyaoka and Umezu proved Conjecture B([MU]). We shall prove Conjecture A with additional assumptions:

MAIN THEOREM. *Let $f: X \to S$ be a proper surjective map between non-singular projective varieties over the complex number field. Let σ_λ denote the rational sections of f. Assume that a general fibre X_s of f is a variety of general type and the union of $S_\lambda = \sigma_\lambda(S)$ is Zariski dense in X. Let P denote the projective bundle $p: P(\Omega_X^s) \to X$, where $s = \dim S$. Furthermore suppose that $\mathcal{O}(\alpha) \otimes p^*\mathcal{O}(-K_X)$ is $f \circ p$-nef for some $\alpha > 0$. Then X is birationally trivial over S.*

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§ 2. Proof of Main Theorem.

We reduce the proof of Main Theorem to the next lemma ([M1], section 5(p. 115), Appendix (p. 119)), which is derived from the weak positivity of direct image sheaf of a high multiple of the relative dualizing sheaf ([KMM], [Ko1], [M2], [V]).

LEMMA 1. *Let T be a complete variety and $\phi: T\times S\to X$ be a dominant S-rational map. Then X is birationally trivial over S.*

For this purpose it suffices to show that there exists a dense bounded subfamily of the graphs of rational sections.

LEMMA 2. *We let $S_\lambda=\sigma_\lambda(S)$ and S_λ^o denote the regular part of S_λ. The natural surjection $\Omega_X^s|_{S_\lambda^o}\to\omega_{S_\lambda^o}$ gives a unique X-map $S_\lambda^o\to P$.*

Proof. The natural surjection $\Omega_X|_{S_\lambda}\to\Omega_{S_\lambda}$ induces a surjection $\Omega_X^s|_{S_\lambda^o}\to\Omega_{S_\lambda^o}^s=\omega_{S_\lambda^o}$. Hence from universality of P, we have the unique X-map $s_\lambda: S_\lambda^o\to P$ such that the pull-back by s_λ of the surjection $p^*\Omega_X^s\to\mathcal{O}(1)$ coincides with $\Omega_X^s|_{S_\lambda^o}\to\omega_{S_\lambda^o}$. Q.E.D.

We denote by G_λ the graph in $S\times P$ of the composition of the rational sections $s_\lambda: S_\lambda\to P$ and $\sigma_\lambda: S\to S_\lambda$. Let B and H be suitable ample invertible sheaves on S and X, respectively. We put

$$M=\mathrm{pr}_1^*B\otimes\mathrm{pr}_2^*((\mathcal{O}(\alpha)\otimes p^*\omega_X^{-1})^k\otimes p^*H)^\beta,$$

where the $\mathrm{pr}_{i=1,2}$ are the projections from $S\times P$ onto S and P, respectively. We estimate the Hilbert polynomials of subvarieties $\{G_\lambda\}$ of $S\times P$ with respect to M.

LEMMA 3. *For a dense subset $\{G_\lambda\}$, $\chi(G_\lambda, M^m)$ are a finite number of polynomials in m.*

Proof. From our assumption, $\mathcal{O}(\alpha)\otimes p^*\omega_X^{-1}$ is $f\circ p$-nef for some $\alpha>0$. Since $\mathcal{O}(\alpha)\otimes p^*\omega_X^{-1}$ is p-ample, we can take H such that $\mathcal{O}(\alpha)\otimes p^*\omega_X^{-1}\otimes p^*H$ is ample. Hence for any $k>0$, $(\mathcal{O}(\alpha)\otimes p^*\omega_X^{-1})^{k-1+1}\otimes p^*H$ is $f\circ p$-ample. We choose k such that $\kappa((\omega_X\otimes f^*A)^k\otimes H^{-1})\geqq 0$, where A is an ample invertible sheaf on S. In fact, by the Viehweg formula $\kappa(\omega_X\otimes f^*A)=\kappa(\omega_{X_s})+\dim S$([V]), $\omega_X\otimes f^*A$ is big. We take $\beta>0$ so that $((\omega_X\otimes f^*A)^k\otimes H^{-1})^\beta$ is effective. Let $\mu_\lambda: F_\lambda\to G_\lambda$ be a desingularization of G_λ such that $a_\lambda: F_\lambda\to S_\lambda$ and $\nu_\lambda: F_\lambda\to S$ are at the same time resolutions of a birational map $G_\lambda\to S_\lambda$, i.e., the inverse of the composition of $\bar{S}_\lambda=s_\lambda(S_\lambda)\to S_\lambda$ and $\mathrm{pr}_2: G_\lambda\to\bar{S}_\lambda$ and a birational map $G_\lambda\to S$, i.e., the inverse of the composition of $S_\lambda\to S$, $\bar{S}_\lambda\to S_\lambda$ and $\mathrm{pr}_2: G_\lambda\to\bar{S}_\lambda$.

We have the following diagram:

$$\begin{tikzcd}
F_\lambda \arrow[r, "\mu_\lambda"] \arrow[dr, "\phi_\lambda"'] \arrow[ddr, bend right, "a_\lambda"'] \arrow[dddrr, bend right=60, "\nu_\lambda"'] & G_\lambda \arrow[d] & \subset & S\times P \arrow[d, "\mathrm{pr}_2"] \\
 & \bar{S}_\lambda \arrow[d] & \subset & P \arrow[d, "p"] \\
 & S_\lambda & \subset & X \arrow[d, "f"] \\
 & & & S . \arrow[ull, "\sigma_\lambda"']
\end{tikzcd}$$

Here we denote by ϕ_λ the composition of the projection $\mathrm{pr}_2|_{G_\lambda}: G_\lambda \to \bar{S}_\lambda$ and $\mu_\lambda: F_\lambda \to G_\lambda$. We have the natural homomorphism $\theta_\lambda: a_\lambda^* \Omega_X^s \to \Omega_{F_\lambda}^s = \omega_{F_\lambda}$ induced from $a_\lambda^* \Omega_X \to \Omega_{F_\lambda}$. Then the image $\mathrm{im}\,\theta_\lambda$ of θ_λ is torsion free. Therefore there exists an open immersion $F_\lambda^\circ \subset F_\lambda$ with $\mathrm{codim}(F_\lambda \backslash F_\lambda^\circ) \geqq 2$ such that $\mathrm{im}\,\theta_\lambda$ is invertible over F_λ°. By Lemma 2, θ_λ is nothing but $\phi_\lambda^* p^* \Omega_X^s \to \phi_\lambda^* \mathcal{O}(1)$ over F_λ°. Hence the double dual $(\mathrm{im}\,\theta)^{\vee\vee}$ coincides with $\phi_\lambda^* \mathcal{O}(1)$. Moreover we have

$$(\mathrm{im}\,\theta_\lambda)^{\vee\vee} = i_*(\mathrm{im}\,\theta_\lambda|_{F_\lambda^\circ}) \longrightarrow i_*(\omega_{F_\lambda^\circ}) = \omega_{F_\lambda}.$$

Thus $\mathrm{im}\,\theta_\lambda = \phi_\lambda^* \mathcal{O}(1)$. We obtain

$$(1) \quad \phi_\lambda^* \mathcal{O}(1) \subset \omega_{F_\lambda}.$$

We shall estimate the following polynomials $\chi(G_\lambda, \mu_{\lambda *}\omega_{F_\lambda} \otimes M^m)$ in m, which equals $\dim H^0(F_\lambda, \omega_{F_\lambda} \otimes M^m)$ for $m \geqq 1$ by the Kollár vanishing ([Kol]). We have for a dense subset $\{G_\lambda\}$

$$\dim H^0(F_\lambda, \omega_{F_\lambda} \otimes M^m) = h^0(F_\lambda, \omega_{F_\lambda} \otimes \nu_\lambda^* B^m \otimes \phi_\lambda^*((\mathcal{O}(\alpha) \otimes p^* \omega_X^{-1})^k \otimes p^* H)^{\beta m})$$

$$\leqq h^0(F_\lambda, \omega_{F_\lambda} \otimes \nu_\lambda^* B^m \otimes \omega_{F_\lambda}^{km\beta} \otimes p^* f^* A^{km\beta}).$$

By the projection formula, we get

$$\begin{aligned} & h^0(F_\lambda, \omega_{F_\lambda} \otimes \nu_\lambda^* B^m \otimes \omega_{F_\lambda}^{km\beta} \otimes p^* f^* A^{km\beta}) \\ & = h^0(F_\lambda, \omega_{F_\lambda}^{1+km\beta} \otimes \nu_\lambda^*(B^m \otimes A^{km\beta})) \\ & = h^0(S, \omega_S^{1+km\beta} \otimes B^m \otimes A^{km\beta}) \qquad \text{for} \quad m \geqq 1. \end{aligned}$$

Therefore there exist only a finite number of polynomials in m in the form $\chi(G_\lambda, \mu_{\lambda *}\omega_{F_\lambda} \otimes M^m)$. From Kleiman ([KL]), $\chi(G_\lambda, M^m)$ are also a finite number of polynomials in m. Q.E.D.

We now return to the proof of Main Theorem. The next lemma implies Main Theorem since its assumption is already shown in Lemma 3.

LEMMA 4. *Let $\{G_\lambda\} \subset S \times P$ be a set of graphs of rational sections from S to P with a finite number of Hilbert polynomial $\chi(G_\lambda, M^m)$. Suppose that there exist rational sections corresponding to $\{G_\lambda\}$ which form a Zariski dense subset* *in X. Then X is birationally trivial over S.*

Proof. By the same argument as in [M1], we find an open subset T^o of the Hilbert scheme with a Hilbert polynomial such that all the points of T^o correspond to the graphs of rational sections from S into P. From our assumption that there exist only a finite number of Hilbert polynomials, we find T^o such that T^o parametrizes a dense set of rational sections in X. Let G denote the universal family restricted on T^o. Then we obtain a T^o-birational morphism $G \to S \times T^o$ and a T^o-morphism $G \to P \times T^o$, which induces a T^o-map $S \times T^o \to P \times T^o$. Since T^o contains points associated to rational sections $\{\sigma_\lambda\}$ forming a Zariski dense subset, we obtain a dominant S-map $F^o: S \times T^o \to X$, composing S-maps $S \times T^o \to P \times T^o$ and $P \times T^o \to P \to X$. Taking a smooth compactification T of T^o, we have a dominant S-rational map $S \times T \to X$. Thus Lemma 1 implies that X is birationally trivial over S. Q.E.D.

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