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# A NOTE ON POISSON APPROXIMATION IN MULTIVARIATE CASE

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## 0. Introduction.

It has been studied in recent years to show the Poisson approximation for the sum of independent Bernoulli random variables which may or may not be identically distributed (see [1], [2], [4]).

In paper [3], K. Kawamura has derived sufficient conditions of a Poisson approximation for the sum of independent identically multivariate Bernoulli random variables. In this paper, we are going to extend the result of paper [2] and generalize the result of paper [3] to the multivariate case.

## 1. Notations and Definitions.

**a.** Suffix and $n$-dimensional vectors.

1. $j, k, m, n$ : positive integers,
2. $\lambda_{\boldsymbol{i}}$ : parameter of Poisson distribution for every $\boldsymbol{i} \in \boldsymbol{E}$,
3. $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ : base of $n$-dimensional vectors,
4. $\boldsymbol{E}=\{0, 1\}^n-\{\boldsymbol{O}\}$ and $\boldsymbol{EO}=\{0, 1\}^n$,
5. $\boldsymbol{O}$ : $n$-dimensional zero vector.
6. $\boldsymbol{i}=(i_1, i_2, \cdots, i_n)$ : $n$-dimensional vector belonging to $\boldsymbol{E}$,
7. $\boldsymbol{k}=(k_1, k_2, \cdots, k_n)$ : $n$-dimensional vector belonging to $\boldsymbol{E}$,
8. $\boldsymbol{s}=(s_1, s_2, \cdots, s_n)$ : $n$-dimensional vector with nonnegative integer components.

**b.** Sum of Bernolli vectors.

1. $\{\boldsymbol{X}_{kj}=(X1_{kj}, X2_{kj}, \cdots, X_{n_{kj}}), j=1, 2, \cdots, n_k, k\geqq 1\}$ be a sequence of independent multivariate Bernoulli vectors with

$$P_{kj}(\boldsymbol{i})=P(\boldsymbol{X}_{kj}=\boldsymbol{i}), \qquad \text{for all} \quad \boldsymbol{i}\in \boldsymbol{EO},$$

where

$$\sum_{\boldsymbol{i}\in\boldsymbol{EO}} P_{kj}(\boldsymbol{i})=1,$$

2. $P_j(\boldsymbol{i})$ : $P_{kj}(\boldsymbol{i})$ expressed in the notation b.1. will be replaced by $P_j(\boldsymbol{i})$ for simplicity if we don't need any information about fixed $k$,

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3. $\boldsymbol{S}_k=\sum_{j=1}^{n_k}\boldsymbol{X}_{kj}$ : the sum of Bernoulli vectors,
4. $n_k$ : positive integer with $\lim_{k\to\infty} n_k=\infty$.

**c.** Probability of the sum of Bernoulli vectors.

1. $\alpha_i$ : frequence of the observation $\boldsymbol{i}$ in $n_k$ trials of $\{\boldsymbol{X}_{kj}=(X1_{kj}, X2_{kj}, \cdots, Xn_{kj}), j=1, 2, \cdots, n_k, k\geqq 1\}$,
2. $\boldsymbol{\alpha}=(\alpha_{e_1}, \alpha_{e_2}, \cdots, \alpha_i, \cdots, \alpha_{e_1+\cdots+e_n})$ : $2^n-1$ dimensional vector,
3. $\boldsymbol{i}\cdot\boldsymbol{e}_r$ : inner product of $\boldsymbol{i}$ and $\boldsymbol{e}_r$,
4. $[C]=[\boldsymbol{\alpha};\ \sum_{\boldsymbol{i}\cdot\boldsymbol{e}_r=1}\alpha_i=s_r, r=1, 2, \cdots, n, \boldsymbol{i}\in\boldsymbol{E}]$, where $[C]$ is a set of $\boldsymbol{\alpha}$ uniquely defined by the given vector $\boldsymbol{s}$,
5. $f_t(\boldsymbol{i})$ : trial number for the $t$-th occurrence of observation $\boldsymbol{i}$ in $n_k$ trials with $f_t(\boldsymbol{i})\in\{1, 2, \cdots, n_k\}$ and $t=1, 2, \cdots, \alpha_i$,
6. $F_i=\{(f_1(i), f_2(i), \cdots, f_{\alpha_i}(\boldsymbol{i}));\ 1\leqq f_1(\boldsymbol{i})<f_2(\boldsymbol{i})<\cdots<f_{\alpha_i}(\boldsymbol{i})\leqq n_k\}$
7. $G_i$ : the set of integers expressed in $(f_1(\boldsymbol{i}), f_2(\boldsymbol{i}), \cdots, f_{\alpha_i}(\boldsymbol{i}))$ belonging to $F_i$, with $G_i=\{f_1(\boldsymbol{i}), f_2(\boldsymbol{i}), \cdots, f_{\alpha_i}(\boldsymbol{i})\}$,
8. $T(\boldsymbol{i})=i_1\cdot 2^0+i_2\cdot 2^1+\cdots+i_m\cdot 2^{m-1}+\cdots+i_n\cdot 2^{n-1}$ : one to one correspondence on $\boldsymbol{E}$ and $S^n$, where $S^n=\{1, 2, \cdots, 2^n-1\}$,
9. $\boldsymbol{i}'<\boldsymbol{i}\overset{def}{\Longleftrightarrow}T(\boldsymbol{i}')<T(\boldsymbol{i})$,
10. $H_i=\bigcup_{i'<i}G_{i'}$ where $\boldsymbol{i}'<\boldsymbol{i}$ is defined in the notation c. 9.,
11. $Q_t(\boldsymbol{i})=P_{f_t(i)}(\boldsymbol{i})$ for simplicity,
12. $Q_t'(\boldsymbol{i})=P_{f_t(i)}(\boldsymbol{i})/P_{f_t(i)}(\boldsymbol{O})$ for simplicity,
13. $$P[\boldsymbol{S}_k=\boldsymbol{s}]=\sum_{[C]}\{\sum_{F_{e_1}}\prod_{t=1}^{\alpha_{e_1}}Q_t(\boldsymbol{e}_1)\cdots\sum_{\substack{F_i\\ G_i\cap H_i=\varnothing}}\prod_{t=1}^{\alpha_i}Q_t(\boldsymbol{i})\cdots \sum_{\substack{F_{e_1+\cdots+e_n}\\ G_{e_1+\cdots+e_n}\cap H_{e_1+\cdots+e_n}=\varnothing}}\prod_{t=1}^{\alpha_{e_1+\cdots+e_n}}Q_t(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n)\}\prod_{\substack{j=1\\ j\notin\cup G_i\\ i\in E}}^{n_k}P_j(\boldsymbol{O})$$

$P[\boldsymbol{S}_k=\boldsymbol{s}]$ will appear in section 2 in detail.

**d.** Variation forms of the probability $P[\boldsymbol{S}_k=\boldsymbol{s}]$

Let us express two variation forms of $P[\boldsymbol{S}_k=\boldsymbol{s}]$ for the proof of Poisson approximation.

1. Let us put

$$B_{n_k}(\boldsymbol{\alpha})=\sum_{F_{e_1}}\prod_{t=1}^{\alpha_{e_1}}Q_t(\boldsymbol{e}_1)\cdots\sum_{\substack{F_i\\ G_i\cap H_i=\varnothing}}\prod_{t=1}^{\alpha_i}Q_t(\boldsymbol{i})\cdots \sum_{\substack{F_{e_1+\cdots+e_n}\\ G_{e_1+\cdots+e_n}\cap H_{e_1+\cdots+e_n}=\varnothing}}\prod_{t=1}^{\alpha_{e_1+\cdots+e_n}}Q_t(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n)\},$$

then we have from notation c. 13.

$$P[\boldsymbol{S}_k=\boldsymbol{s}]=\sum_{[C]}\{B_{n_k}(\boldsymbol{\alpha})\}\prod_{\substack{j=1\\ j\notin\cup G_i\\ i\in E}}^{n_k} P_j(\boldsymbol{O})$$

2. Let us put

$$A_{n_k}(\boldsymbol{\alpha})=\sum_{F_{e_1}}\prod_{t=1}^{\alpha_{e_1}}\boldsymbol{o}_t'(\boldsymbol{e}_1)\cdots\sum_{\substack{F_i\\ G_i\cap H_i=\varnothing}}\prod_{t=1}^{\alpha_i}Q_t'(\boldsymbol{i})\cdots$$

$$\sum_{\substack{F_{e_1+\cdots+e_n}\\ G_{e_1+\cdots+e_n}\cap H_{e_1+\cdots+e_n}=\varnothing}}\prod_{t=1}^{\alpha_{e_1+\cdots+e_n}}Q_t'(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n)\},$$

then we have

$$P[\boldsymbol{S}_k=\boldsymbol{s}]=\sum_{j=1}^{n_k}\{A_{n_k}(\boldsymbol{\alpha})\}\prod_{j=1}^{n_k}P_j(\boldsymbol{O}),$$

because $P[\boldsymbol{S}_k=\boldsymbol{s}]$ (see notation c. 13.) can be rewritten by

$$P[\boldsymbol{S}_k=\boldsymbol{s}]=\sum_{[C]}\{\sum_{F_{e_1}}\prod_{t=1}^{\alpha_{e_1}}Q_t'(\boldsymbol{e}_1)\cdots\sum_{\substack{F_i\\ G_i\cap H_i=\varnothing}}\sum_{t=1}^{\alpha_i}Q_t'(\boldsymbol{i})\cdots$$

$$\sum_{\substack{F_{e_1+\cdots+e_n}\\ G_{e_1+\cdots+e_n}\cap H_{e_1+\cdots+e_n}=\varnothing}}\prod_{t=1}^{\alpha_{e_1+\cdots+e_n}}Q_t'(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n)\}\prod_{j=1}^{n_k}P_j(\boldsymbol{O}),$$

3. Let us denote $C_{n_k}(\alpha)$ from $B_{n_k}(\alpha)$ by removing all the restriction in the sums (see notation d. 1.) as follows,

$$C_{n_k}(\alpha)=\sum_{F_{e_1}}\prod_{t=1}^{\alpha_{e_1}}Q_t(\boldsymbol{e}_1)\cdots\sum_{F_i}\prod_{t=1}^{\alpha_i}Q_t(\boldsymbol{i})\cdots\sum_{F_{e_1+\cdots+e_n}}\prod_{t=1}^{\alpha_{e_1+\cdots+e_n}}Q_t(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n)\}.$$

## 2. Conditions sufficient for Poisson approximation.

Let $\{\boldsymbol{X}_{kj}=(X1_{kj}, X2_{kj}, \cdots, Xn_{kj}),\ j=1, 2, \cdots, n_k,\ k\geqq 1\}$ be a sequence of independent multivariate Bernoulli vectors with

$$P_{kj}(\boldsymbol{i})=P(X_{kj}=\boldsymbol{i}),\qquad \text{for all}\quad \boldsymbol{i}\in \boldsymbol{EO},$$

where

$$\sum_{\boldsymbol{i}\in\boldsymbol{EO}}P_{kj}(\boldsymbol{i})=1,$$

and denote the sum of multivariate Bernoulli vectors by $\boldsymbol{S}_k=\sum_{j=1}^{n_k}\boldsymbol{X}_{kj}$. In the following discussion, $P_{kj}(\boldsymbol{i})$ expressed in notation b. 1. will be replaced by $P_j(\boldsymbol{i})$ for simplicity and we can easily see that

$$P[\boldsymbol{S}_k=\boldsymbol{s}]=\sum_{[C]}\{\sum_{F_{e_1}}\prod_{t=1}^{\alpha_{e_1}}Q_t(\boldsymbol{e}_1)\cdots\sum_{\substack{F_i\\ G_i\cap H_i=\varnothing}}\prod_{t=1}^{\alpha_i}Q_t(\boldsymbol{i})\cdots$$

$$\sum_{\substack{F_{e_1+\cdots+e_n}\\ G_{e_1+\cdots+e_n}\cap H_{e_1+\cdots+e_n}=\varnothing}}\prod_{t=1}^{\alpha_{e_1+\cdots+e_n}}Q_t(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n)\}\prod_{\substack{j=1\\ j\notin\cup G_i\\ i\in E}}^{n_k}P_j(\boldsymbol{O})$$

THEOREM. *If the following relations* (2.1) *and* (2.2) *are satisfied for the sequence of independent n-variate Bernoulli distribution which may or may not be identically distributed, that is, for every* $\boldsymbol{i}\in\boldsymbol{E}$

(2.1) $\sum_{j=1}^{n_k}P_{kj}(\boldsymbol{i})\to\lambda_i$ *as* $k\to\infty$,

(2.2) $\min_{1\leq j\leq n_k}P_{kj}(\boldsymbol{i})\to 1$ *as* $k\to\infty$,

*then we have*

(2.3) $\lim_{k\to\infty}P[\boldsymbol{S}_k=\boldsymbol{s}]=\sum_{[C]}[\prod_{i\in E}(\lambda_i^{\alpha_i}/\alpha_i!)]\exp(-\sum_{i\in E}\lambda_i)$

*for all* $\boldsymbol{s}$ ($\boldsymbol{s}\geqq 0$), *where* $[C]$ *is uniquely defined by the vector* $\boldsymbol{s}$ *as*

$$[C]=[\boldsymbol{\alpha};\sum_{i\cdot e_r=1}\alpha_i=s_r,\ r=1,2,\cdots,n,\ ,\ \boldsymbol{i}\in\boldsymbol{E}].$$

In order to prove the theorem, we are going to show lemma 1, lemma 2 and lemma 3.

LEMMA 1. *If the conditions* (2.1) *and* (2.2) *are satisfied then we have*

$$\sum_{g=1}^{n_k}P_g(\boldsymbol{O})\to\exp(-\sum_{i\in E}\lambda_i)\quad as\quad k\to\infty.$$

*Proof*. Consider the inequality

$$1+y\leqq\exp(y),\qquad y\in[-1,\infty),$$

putting $y=-x$ and $y=x/(1-x)$ with $x\in[0,1]$ we obtain

$$\exp(-x/(1-x))\leqq 1-x\leqq\exp(-x),\ x\in[0,1).$$

Now putting $\Delta_g=\sum_{i\in E}P_g(\boldsymbol{i})=1-P_g(\boldsymbol{O})$,
where $0\leqq\Delta_g<1$ (by (2.2)) for sufficiently large $k\,(1\leqq g\leqq n_k)$, and using the last inequality, we get

$$\exp\Big[-\Big(\sum_{g=1}^{n_k}\Delta_g\Big)/\min_g P_g(\boldsymbol{O})\Big]\leqq\prod_{g=1}^{n_k}P_g(\boldsymbol{O})\leqq\exp\Big(-\sum_{g=1}^{n_k}\Delta_g\Big),$$

and from (2.1), (2.2) we can prove that

(2.4) $\sum_{g=1}^{n_k} P_g(\boldsymbol{O}) \to \exp(-\sum_{\iota\in E}\lambda_\iota)$ as $k\to\infty$. ■

LEMMA 2. *If the conditions* (2.1) *and* (2.2) *are satisfied then we have*

(2.5) $C_{n_k}(\boldsymbol{\alpha}) \to \prod_{\iota\in E}(\lambda_\iota^{\alpha_\iota}/\alpha_\iota!)$ *as* $k\to\infty$.

*Proof.* It is sufficient to prove that

(2.5.1) $\sum_{F_i}[\prod_{t=1}^{\alpha_i} P_{f_t(i)}(\boldsymbol{i})] \to \lambda_\iota^{\alpha_\iota}/\alpha_\iota!$, for every $\boldsymbol{i}\in\boldsymbol{E}$.

The proof of (2.5.1) is given by induction with respect to $\alpha_\iota$.

(1) $\alpha_i=1$. It is obvious by (2.1) that

$$\sum_{f_1(i)=1}^{n_k} P_{f_1(i)}(\boldsymbol{i}) \to \lambda_i \quad \text{as} \quad k\to\infty.$$

(2) $\alpha_i=2$. By (2.1) and (2.2), we have

$$\sum_{f_1(i)<f_2(i)} P_{f_1(i)}(\boldsymbol{i})P_{f_2(i)}(\boldsymbol{i}) \to (\lambda_\iota)^2/2!$$

because

$$0\leqq \sum_{j=1}^{n_k} P_j^2(\boldsymbol{i}) \leqq [1-\min_j P_j(\boldsymbol{O})]\sum_{j=1}^{n_k} P_j(\boldsymbol{i})$$

and by (2.1), (2.2) the right hand side of the inequality tends to 0, so we have

$$2\sum_{f_1(i)<f_2(i)} P_{f_1(i)}(\boldsymbol{i})P_{f_2(i)}(\boldsymbol{i}) = [\sum_{j=1}^{n_k} P_j(\boldsymbol{i})]^2 - \sum_{j=1}^{n_k} P_j^2(\boldsymbol{i}) \to \lambda_\iota^2 \quad \text{as} \quad k\to\infty.$$

(3) Assume that (2.5.1) is correct as $\alpha_i=m-1$, that is,

$$\sum_{f_1(i)<\cdots<f_{m-1}(i)} \prod_{t=1}^{m-1} P_{f_t(i)}(\boldsymbol{i}) \to (\lambda_\iota)^{m-1}/(m-1)! \quad \text{as} \quad k\to\infty.$$

In order to finish the induction, let us prove (2.5.1) to be also true as $\alpha_i=m$. Multiply the left hand side of the last relation by $\sum_{f_m(i)=1}^{n_k} P_{f_m(i)}(\boldsymbol{i})$ which tends to $\lambda_i$ (by (2.1)), we obtain

$$\begin{aligned} &\sum_{f_1(i)<\cdots<f_{m-1}(i)} P_{f_1(i)}(\boldsymbol{i})\prod_{t=1}^{m-1} P_{f_t(i)}(\boldsymbol{i}) \\ &+\sum_{f_1(i)<\cdots<f_{m-1}(i)} P_{f_2(i)}(\boldsymbol{i})\prod_{t=1}^{m-1} P_{f_t(i)}(\boldsymbol{i})+\cdots \\ &+\sum_{f_1(i)<\cdots<f_{m-1}(i)} P_{f_{m-1}(i)}(\boldsymbol{i})\prod_{t=1}^{m-1} P_{f_t(i)}(\boldsymbol{i}) \\ &+\sum_{f_m(i)<f_1(i)<\cdots<f_{m-1}(i)} \prod_{t=1}^{m} P_{f_t(i)}(\boldsymbol{i}) \end{aligned} \tag{2.5.2}$$

$$+\sum_{f_1(i)<f_m(i)<f_2(i)<\cdots<f_{m-1}(i)}\prod_{t=1}^{m}P_{f_t(i)}(\boldsymbol{i})+\cdots$$

$$+\sum_{f_1(i)<\cdots<f_{m-1}(i)<f_m(i)}\prod_{t=1}^{m}P_{f_t(i)}(\boldsymbol{i})$$

Each of the first $(m-1)$ terms of (2.5.2) may be nonnegative and estimated by

$$[1-\min P_j(\boldsymbol{O})]\sum_{f_1(i)<\cdots<f_{m-1}(i)}\prod_{t=1}^{m-1}P_{f_t(i)}(\boldsymbol{i})$$

which is an upper bound of these terms and by (2.2) tends to 0, that is,

$$0\leqq[\text{each of the first } (m-1) \text{ terms of } (2.5.2)]$$

$$\leqq[1-\min P_j(\boldsymbol{O})]\sum_{f_1(i)<\cdots<f_{m-1}(i)}\prod_{t=1}^{m-1}P_{f_t(i)}(\boldsymbol{i})$$

So each of the first $(m-1)$ terms tends to 0, and each of the last $m$ terms has the same value, then we can obtain the limiting value of (2.5.2) to be

$$m\sum_{f_1(i)<\cdots<f_{m-1}(i)}\prod_{j=1}^{m}P_{f_t(i)}(\boldsymbol{i})\to\lambda_i(\lambda_i)^{m-1}/(m-1)\,!,$$

that is, (2.5.1) is correct as $\alpha_i=m$ and we finish the proof of (2.5.1) by induction. Then by (2.5.1), we have

$$(2.5)\quad C_{n_k}(\boldsymbol{\alpha})\to\prod_{\boldsymbol{i}\in\boldsymbol{E}}(\lambda_i^{\alpha_i}/\alpha_i\,!)\quad\text{as}\quad k\to\infty.$$

this is the result of lemma 2. ■

LEMMA 3. *Three values (defined in the notation d.)* $A_{n_k}(\boldsymbol{\alpha})$, $B_{n_k}(\boldsymbol{\alpha})$ *and* $C_{n_k}(\boldsymbol{\alpha})$ *have the same limiting value, that is,*

$$(2.8)\quad B_{n_k}(\boldsymbol{\alpha})\to\prod_{\boldsymbol{i}\in\boldsymbol{E}}(\lambda_i^{\alpha_i}/\alpha_i\,!)\quad\textit{as}\quad k\to\infty,$$

*and*

$$(2.9)\quad A_{n_k}(\boldsymbol{\alpha})\to\prod_{\boldsymbol{i}\in\boldsymbol{E}}(\lambda_i^{\alpha_i}/\alpha_i\,!)\quad\textit{as}\quad k\to\infty.$$

*Proof*. For the proof of lemma 3, we consider the following three steps. (step 1) Let us define

$$\operatorname{Rem}(\boldsymbol{i})=\sum_{F_i}\prod_{t=1}^{\alpha_i}P_{f_t(i)}(\boldsymbol{i})-\sum_{\substack{F_i\\ G_i\cap H_i=\emptyset}}\prod_{t=1}^{\alpha_i}P_{f_t(i)}(\boldsymbol{i}),$$

$$=\sum_{\substack{F_i\\ G_i\cap H_i\neq\emptyset}}\prod_{t=1}^{\alpha_i}P_{f_t(i)}(\boldsymbol{i}),$$

for every $\boldsymbol{i}\in\boldsymbol{E}-\{\boldsymbol{e}_1\}$. In this step, we are going to prove that

(2.6) $\operatorname{Rem}(\boldsymbol{i}) \to 0$ as $k \to \infty$,

for every $\boldsymbol{i} \in \boldsymbol{E} - \{\boldsymbol{e}_1\}$.

Proof of (2.6): It is easy to see that

$$(2.7) \quad \sum_{F_i} \prod_{t=1}^{n} P_{f_t(i)}(\boldsymbol{i}) \leqq \Big[\sum_{f_t(i)=1}^{n_k} P_{f_t(i)}(\boldsymbol{i})\Big]^n \leqq (\lambda_i + \varepsilon)^n .$$

It is obvious that $\operatorname{Rem}(\boldsymbol{i})$ is nonnegative and estimated as follows:

$$\begin{aligned} \operatorname{Rem}(\boldsymbol{i}) &\leqq \sum_{r=1}^{d(i)} \sum_{s=1}^{\alpha_i} P_{f_s(i)}(\boldsymbol{i}) \sum_{F_i} \prod_{\substack{t=1 \\ \neq s}}^{\alpha_i} P_{f_t(i)}(\boldsymbol{i}) \\ &\leqq d(\boldsymbol{i})\alpha_i[1 - \min_j P_j(\boldsymbol{O})](\lambda_i + \varepsilon)^{\alpha_i - 1} \end{aligned}$$

with $f_s(\boldsymbol{i}) = f_r(\boldsymbol{k})$, where $d(\boldsymbol{i}) = \sum_{i<k} \alpha_k$.

By (2.2) and (2.7) for $n = \alpha_i - 1$ the right hand side of the last inequality tends to zero as $k \to \infty$, and we finish step 1.

(step 2) By the definition of $C_{n_k}(\boldsymbol{\alpha})$ we can obtain

$$\begin{aligned} C_{n_k}(\boldsymbol{\alpha}) &= \prod_{i \in E} \sum_{F_i} \prod_{t=1}^{\alpha_i} P_{f_t(i)}(\boldsymbol{i}) \\ &= \sum_{F_{e_1}} \prod_{t=1}^{\alpha_{e_1}} P_{f_t(e_1)}(\boldsymbol{e}_1) \Big\{ \prod_{i \in E - \{e_1\}} \Big[\operatorname{Rem}(\boldsymbol{i}) + \sum_{\substack{F_i \\ G_i \cap H_i = \varnothing}} \prod_{t=1}^{\alpha_i} P_{f_t(i)}(\boldsymbol{i})\Big]\Big\} \end{aligned}$$

and by the definitions of $\operatorname{Rem}(\boldsymbol{i})$, $B_{nk}(\boldsymbol{\alpha})$ and using (2.6), we can obtain that $B_{n_k}(\boldsymbol{\alpha})$ and $C_{n_k}(\boldsymbol{\alpha})$ have the same limiting value as $k \to \infty$. Then from lemma 2, we have

(2.8) $B_{n_k}(\boldsymbol{\alpha}) \to \prod_{i \in E} (\alpha_i^{\alpha_i} / \alpha_i !)$ as $k \to \infty$.

(step 3) It is easy to see by the definition of $A_{n_k}(\boldsymbol{\alpha})$, $B_{n_k}(\boldsymbol{\alpha})$ that

$$B_{n_k}(\boldsymbol{\alpha}) \leqq A_{n_k}(\boldsymbol{\alpha}) \leqq (1/\min P_j(\boldsymbol{O}))^h B_{n_k}(\boldsymbol{\alpha}),$$

where $h = \sum_{i \in E} \alpha_i$ and by (2.2), (2.8), we have

(2.9) $A_{n_k}(\boldsymbol{\alpha}) \to \prod_{i \in E} (\lambda_i^{\alpha_i} / \alpha_i !)$ as $k \to \infty$. ■

**Proof of the theorem.**

Summarize lemma 1 and lemma 3, we finish the theorem. ■

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