H. ISHIHARA AND W. TAKAHASHI
KODAI MATH. J.
10 (1987), 197—208

MODULUS OF CONVEXITY, CHARACTERISTIC OF CONVEXITY AND FIXED POINT THEOREMS

BY HAJIME ISHIHARA AND WATARU TAKAHASHI

§1. Introduction.

Let C be a bounded closed convex subset of a Banach space E and let T be a nonexpansive mapping from C into itself. Browder [2] and Göhde [10] showed that if E is uniformly convex then T has a fixed point, while Kirk [13] proved that if E is reflexive and if C has normal structure then T has a fixed point. On the other hand, Goebel [7] defined the characteristic ε_0 of convexity of E and showed that E is uniformly convex if and only if $\varepsilon_0=0$, if $\varepsilon_0<1$ then E has normal structure and if $\varepsilon_0<2$ then E is reflexive. Also, Bynum [3] defined the normal structure coefficient $N(E)$ of E, and then Maluta [17] and Bae [1] proved that if $N(E)^{-1}<1$ then E is reflexive and has normal structure. Using these coefficients, Goebel and Kirk [8], Goebel, Kirk and Thele [9] and Casini and Maluta [4] proved the fixed point theorems for uniformly k-lipschitzian mappings. (For the results on Hilbert space, see [5], [12], [14].) But it seems natural to define these coefficients for a convex set, since for any Banach space E, a nonexpansive mapping has a fixed point if C is weakly compact and has normal structure.

In this paper, we introduce the modulus $\delta(C, \varepsilon)$ of convexity, the characteristic $\varepsilon_0(C)$ of convexity and the constant $\tilde{N}(C)$ of uniformity of normal structure for a convex subset C of a Banach space and prove some results similar to [3], [7], [11], [17]. For example, we show that if $\tilde{N}(C)<1$ then C is boundedly weakly compact. Further, by using these coefficients, we prove three fixed point theorems. All of these proofs are given by explicitly constructing a sequence which converges to a fixed point. We first show a fixed point theorem for nonexpansive semigroups. Secondly, we obtain a fixed point theorem for uniformly k-lipschitzian semigroups on C under $k<\gamma$, where γ is determined by the modulus of convexity of C. Also, using our results, we evaluate γ as $1<\gamma\leqq 1+(1-\varepsilon_0(C))/2$. Finally, we prove that Casini and Maluta's result [4] is valid under more general semigroups.

§2. Preliminaries.

Let E be a real Banach space and let B be a bounded subset of E. For a

Received November 12, 1986

nonempty subset C of E define,

$$R(B, x)=\sup\{\|x-y\|: y\in B\};$$

$$R(B, C)=\inf\{R(B, x): x\in C\};$$

$$\mathcal{C}(B, C)=\{x\in C: R(B, x)=R(B, C)\}.$$

We call the number $R(B, C)$ the *Chebyshev radius* of B in C and the set $\mathcal{C}(C, B)$ the *Chebyshev center* of B in C.

Let $\{B_\alpha: \alpha\in\Lambda\}$ be a decreasing net of bounded subsets of E. For a nonempty subset C of E define,

$$r(\{B_\alpha\}, x)=\inf_\alpha R(B_\alpha, x);$$

$$r(\{B_\alpha\}, C)=\inf\{r(\{B_\alpha\}, x): x\in C\};$$

$$\mathcal{A}(\{B_\alpha\}, C)=\{x\in C: r(\{B_\alpha\}, x)=r(\{B_\alpha\}, C)\}.$$

The number $r(\{B_\alpha\}, C)$ and the set $\mathcal{A}(\{B_\alpha\}, C)$ are called the *asymptotic radius* and the *asymptotic center* of $\{B_\alpha: \alpha\in\Lambda\}$ in C, respectively. We also know that $R(B, \cdot)$ and $r(\{B_\alpha\}, \cdot)$ are continuous convex functions on E which satisfy the following:

$$|R(B, x)-R(B, y)|\leqq\|x-y\|\leqq R(B, x)+R(B, y);$$

$$|r(\{B_\alpha\}, x)-r(\{B_\alpha\}, y)|\leqq\|x-y\|\leqq r(\{B_\alpha\}, x)+r(\{B_\alpha\}, y)$$

for each $x, y\in E$, cf. [16].

A nonempty subset C of E is *boundedly weakly compact* if its intersection with every closed ball is weakly compact. It is easy to see that if C is boundedly weakly compact and convex, then $\mathcal{C}(B, C)$ and $\mathcal{A}(\{B_\alpha\}, C)$ are nonempty.

For a subset D of E, we denote by $d(D)$ the diameter of D and by $\overline{co}D$ the closure of the convex hull of D. A convex set C of E is said to have *normal structure* if each bounded convex subset D of C with $d(D)>0$ contains a point y such that $R(D, y)<d(D)$.

The *modulus of convexity* of E is the function

$$\delta_E(\varepsilon)=\inf\left\{1-\left\|\frac{x+y}{2}\right\|: \|x\|\leqq 1, \|y\|\leqq 1, \|x-y\|\geqq\varepsilon\right\}$$

defined for $0\leqq\varepsilon\leqq 2$.

Let S be a semitopological semigroup, i. e., S is a semigroup with a Hausdorff topology such that for each $a\in S$ the mappings $s\to a\cdot s$ and $s\to s\cdot a$ from S to S are continuous. Let C be a nonempty closed convex subset of E. Then a family $\mathcal{S}=\{T_t: t\in S\}$ of mappings from C into itself is said to be a *uniformly k-lipschitzian semigroup* on C if $\mathcal{S}$ satisfies the following:

(1) $T_{ts}(x)=T_tT_s(x)$ *for* $t, s\in S$ *and* $x\in C$;

(2) *the mapping* $(s, x)\to T_s(x)$ *from* $S\times C$ *into* C *is continuous when* $S\times C$ *has*

the product topology;

(3) $\|T_s(x)-T_s(y)\|\leqq k\|x-y\|$ *for* $x, y\in C$ *and* $s\in S$.

In particular, a uniformly 1-lipschitzian semigroup on C is said to be a *nonexpansive semigroup* on C. A semitopological semigroup S is *left reversible* if any two closed right ideals of S have nonvoid intersection. In this case, $(S, \leqq)$ is a directed system when the binary relation "$\leqq$" on S is defined by $a\leqq b$ if and only if $\{a\}\cup\overline{aS}\supseteqq\{b\}\cup\overline{bS}$.

§ 3. Modulus of convexity and characteristic of convexity.

We first define the modulus of convexity, the characteristic of convexity and the constant of uniformity of normal structure for a nonempty convex subset of a Banach space.

DEFINITION 3.1. Let C be a nonempty convex subset of a real Banach space E with $d(C)>0$. Then we define, for ε with $0\leqq\varepsilon\leqq 2$,

$$\delta(C, \varepsilon)=\inf\left\{1-\frac{1}{r}\left\|z-\frac{x+y}{2}\right\| : x, y, z\in C, 0<r\leqq d(C),\right.$$

$$\left.\|z-x\|\leqq r, \|z-y\|\leqq r, \|x-y\|\geqq r\varepsilon\right\};$$

$$\varepsilon_0(C)=\sup\{\varepsilon : 0\leqq\varepsilon\leqq 2, \delta(C, \varepsilon)=0\};$$

$$\tilde{N}(C)=\sup\left\{\frac{R(D, D)}{d(D)} : D \text{ is a nonempty bounded convex}\right.$$

$$\left.\text{subset of } C \text{ with } d(D)>0\right\}.$$

Remark 3.1. It follows from Definition 3.1 that $\delta(C, 0)=0$, $0\leqq\delta(C, \varepsilon)\leqq 1$, $\delta(C, \varepsilon)$ is nondecreasing in ε and $\delta(E, \varepsilon)=\delta_E(\varepsilon)$. Further for a nonempty convex subset D of C with $d(D)>0$ it follows that $\delta(C, \varepsilon)\leqq\delta(D, \varepsilon)$, $\varepsilon_0(D)\leqq\varepsilon_0(C)$ and $\tilde{N}(D)\leqq\tilde{N}(C)$.

Remark 3.2. Let C and D be convex subsets of E. For $a\in E$, it is easy to see that $\delta(\overline{C}, \varepsilon)=\delta(C, \varepsilon)$, $\delta(C+a, \varepsilon)=\delta(C, \varepsilon)$, and $\delta(C\cap D, \varepsilon)=\max\{\delta(C, \varepsilon), \delta(D, \varepsilon)\}$. Similarly we have $\tilde{N}(\overline{C})=\tilde{N}(C)$, $\tilde{N}(C+a)=\tilde{N}(C)$, and $\tilde{N}(C\cap D)=\min\{\tilde{N}(C), \tilde{N}(D)\}$.

Example 3.1. Let $C[0, 1]$ be a Banach space of all continuous real functions on $[0, 1]$ with supremum norm and let A be a subspace of all affine functions in $C[0, 1]$. Since $C[0, 1]$ is not reflexive, we have $\tilde{N}(C[0, 1])=1$. But it is easy to see that A is isomorphic to $l_\infty^2=(\boldsymbol{R}^2, \|\cdot\|_\infty)$ and hence $\tilde{N}(A)=\tilde{N}(l_\infty^2)=\frac{1}{2}$, cf. [17], [1].

It is well known that $\delta_E(\varepsilon)$ is continuous on $[0, 2)$, cf [11]. We can also prove an inequality concerning the continuity of $\delta(C, \varepsilon)$. Before proving it we need the following lemma.

LEMMA 3.1. *Let C be a nonempty convex subset of a real Banach space E with $d(C)>0$, let $u, v\in E$ and let $0<r\leqq d(C)$. For $z\in C$ and ε with $0\leqq\varepsilon\leqq 2$ define a set $N_{r,u,v}(z)$ and a function $\delta_{r,u,v}(\varepsilon)$ as follows:*

$$N_{r,u,v}(z)=\Big\{(x, y): x, y\in C, \|z-x\|\leqq r, \|z-y\|\leqq r,$$

$$x-y=au, z-\frac{x+y}{2}=bv \text{ for some } a, b\geqq 0\Big\};$$

$$\delta_{r,u,v}(\varepsilon)=\inf\Big\{1-\frac{1}{r}\Big\|z-\frac{x+y}{2}\Big\|: z\in C, (x, y)\in N_{r,u,v}(z), \|x-y\|\geqq r\varepsilon\Big\}.$$

Then $\delta_{r,u,v}$ is a nondecreasing convex function from $[0, 2]$ to $[0, 1]$ with

$$\delta(C, \varepsilon)=\inf\{\delta_{r,u,v}(\varepsilon): u, v\in E, 0<r\leqq d(C)\}.$$

Proof. Since it is obvious that $\delta_{r,u,v}$ is nondecreasing and

$$\delta(C, \varepsilon)=\inf\{\delta_{r,u,v}(\varepsilon): u, v\in E, 0<r\leqq d(C)\},$$

we only prove that $\delta_{r,u,v}$ is convex.

For arbitrary $z_1, z_2\in C$ and $(x_1, y_1)\in N_{r,u,v}(z_1)$ and $(x_2, y_2)\in N_{r,u,v}(z_2)$ with $\|x_1-y_1\|\geqq r\varepsilon_1$ and $\|x_2-y_2\|\geqq r\varepsilon_2$, there exist $a_1, a_2, b_1, b_2\geqq 0$ such that

$$x_1-y_1=a_1u, z_1-\frac{x_1+y_1}{2}=b_1v,$$

and

$$x_2-y_2=a_2u, z_2-\frac{x_2+y_2}{2}=b_2v.$$

For λ with $0\leqq\lambda\leqq 1$, define $x_3=\lambda x_1+(1-\lambda)x_2$, $y_3=\lambda y_1+(1-\lambda)y_2$ and $z_3=\lambda z_1+(1-\lambda)z_2$. Then, we have

$$x_3-y_3=\lambda(x_1-y_1)+(1-\lambda)(x_2-y_2)=(\lambda a_1+(1-\lambda)a_2)u,$$

$$z_3-\frac{x_3+y_3}{2}=\lambda\Big(z_1-\frac{x_1+y_1}{2}\Big)+(1-\lambda)\Big(z_2-\frac{x_2+y_2}{2}\Big)$$

$$=(\lambda b_1+(1-\lambda)b_2)v.$$

Since $\|z_3-x_3\|\leqq r$ and $\|z_3-y_3\|\leqq r$, we have $(x_3, y_3)\in N_{r,u,v}(z_3)$. We also obtain

$$\|x_3-y_3\|=\lambda\|x_1-y_1\|+(1-\lambda)\|x_2-y_2\|\geqq\lambda\varepsilon_1+(1-\lambda)\varepsilon_2$$

and

$$\delta_{r,u,v}(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)\leqq 1-\frac{1}{r}\left\|z_3-\frac{x_3+y_3}{2}\right\|$$

$$=\lambda\left(1-\frac{1}{r}\left\|z_1-\frac{x_1+y_1}{2}\right\|\right)+(1-\lambda)\left(1-\frac{1}{r}\left\|z_2-\frac{x_2+y_2}{2}\right\|\right)$$

for arbitrary $z_1, z_2\in C$, $(x_1, y_1)\in N_{r,u,v}(z_1)$ and $(x_2, y_2)\in N_{r,u,v}(z_2)$. Therefore we have

$$\delta_{r,u,v}(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)\leqq\lambda\delta_{r,u,v}(\varepsilon_1)+(1-\lambda)\delta_{r,u,v}(\varepsilon_2).$$

THEOREM 3.1. *Let C be a nonempty convex subset of a real Banach space E with $d(C)>0$. Then for all ε_1 and ε_2 with $0\leqq\varepsilon_1<\varepsilon_2\leqq 2$,*

$$\delta(C, \varepsilon_2)-\delta(C, \varepsilon_1)\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}(1-\delta(C, \varepsilon_1))\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}.$$

Proof. For any real number with $\eta>0$, there exist $u, v\in E$ and r with $0<r<d(C)$ such that $\delta_{r,u,v}(\varepsilon_1)\leqq\delta(C, \varepsilon_1)+\eta$ and hence we obtain

$$\delta_{r,u,v}(\varepsilon_2)=\delta_{r,u,v}\left(\left(\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}\right)2+\left(1-\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}\right)\varepsilon_1\right)$$

$$\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}\delta_{r,u,v}(2)+\left(1-\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}\right)\delta_{r,u,v}(\varepsilon_1)$$

or

$$\delta_{r,u,v}(\varepsilon_2)-\delta_{r,u,v}(\varepsilon_1)\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}(\delta_{r,u,v}(2)-\delta_{r,u,v}(\varepsilon_1))$$

$$\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}(1-\delta(C, \varepsilon_1)).$$

Then we have

$$\delta(C, \varepsilon_2)-\delta(C, \varepsilon_1)\leqq\delta_{r,u,v}(\varepsilon_2)-\delta_{r,u,v}(\varepsilon_1)+\eta$$

$$\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}(1-\delta(C, \varepsilon_1))+\eta.$$

Since $\eta>0$ is arbitrary, we have

$$\delta(C, \varepsilon_2)-\delta(C, \varepsilon_1)\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}(1-\delta(C, \varepsilon_1))\leqq\frac{\varepsilon_2-\varepsilon_1}{2-\varepsilon_1}.$$

The following lemma can be proved as in [16].

LEMMA 3.2. *Let C be a convex subset of a real Banach space E. Let B be a bounded subset of C and let $\{B_\alpha : \alpha\in A\}$ be a decreasing net of bounded subsets of C. For each $x, y\in C$, if $R(B, x)\leqq t$, $R(B, y)\leqq t$ and $\|x-y\|\geqq t\cdot\varepsilon$ then*

$$R\left(B, \frac{x+y}{2}\right) \leqq t(1-\delta(C, \varepsilon))$$

and if $r(\{B_\alpha\}, x) \leqq t$, $r(\{B_\alpha\}, y) \leqq t$ *and* $\|x-y\| \geqq t\varepsilon$ *then*

$$r\left(\{B_\alpha\}, \frac{x+y}{2}\right) \leqq t(1-\delta(C, \varepsilon)).$$

It was proved by Bynum [3] that $\tilde{N}(E) \leqq 1-\delta_E(1)$. By using Theorem 3.1, Lemma 3.2 and the method of [3], we can also obtain the following: *Let C be a nonempty convex subset of a real Banach space E with* $d(C)>0$. *Then* $\tilde{N}(C) \leqq 1-\delta(C, 1)$.

Maluta [17] and Bae [1] proved that if $\tilde{N}(E)<1$ then E is reflexive. We can prove the following:

THEOREM 3.2. *Let C be a nonempty convex subset of a real Banach space E with* $d(C)>0$. *If* $\tilde{N}(C)<1$ *then C is boundedly weakly compact and has normal structure.*

Proof. It is obvious from $\tilde{N}(C)<1$ that C has normal structure. We may assume that C is bounded. Let $\{C_n\}$ be an arbitrary decreasing sequence of nonempty closed convex subsets of C. If we show $\{C_n\}$ has nonempty intersection then we complete the proof, cf. [p. 433, 4]. If $d(C_n)=0$ for some $n \geqq 1$ then it is obvious that $\{C_n\}$ has nonempty intersection. So we assume $d(C_n)>0$ for all $n \geqq 1$. Let η be a real number with $\tilde{N}(C)<\eta<1$ and define by induction:

$$C_{n,0}=C_n;$$

$$x_{n,m} \in C_{n,m} \text{ such that } R(C_{n,m}, x_{n,m}) \leqq \eta d(C_{n,m});$$

$$C_{n,m+1}=\overline{co}\{x_{k,m} : k \geqq n\}.$$

Then, we have $C_{n,m}$ is nonempty, $C_{n,m} \supseteqq C_{n+1,m}$, $C_{n,m} \supseteqq C_{n,m+1}$ and

$$\begin{aligned} d(C_{n,m}) &= \sup\{\|x_{i,m-1}-x_{j,m-1}\| : i, j \geqq n\} = \sup_{i \geqq n} \sup_{j \geqq i} \|x_{i,m-1}-x_{j,m-1}\| \\ &\leqq \sup_{i \geqq n} R(C_{i,m}, x_{i,m-1}) \leqq \sup_{i \geqq n} R(C_{i,m-1}, x_{i,m-1}) \\ &\leqq \sup_{i \geqq n} \eta d(C_{i,m-1}) \leqq \eta d(C_{n,m-1}) \leqq \eta^m d(C_n) \end{aligned}$$

for all $n, m \geqq 1$. Hence $\lim_{m\to\infty} d(C_{n,m})=0$. Since $\bigcap_{m=1}^{\infty} C_{n,m} \supseteqq \bigcap_{m=1}^{\infty} C_{n+1,m}$ for all $n \geqq 1$, there exists $y \in E$ such that $\bigcap_{m=1}^{\infty} C_{n,m}=\{y\}$ for all $n \geqq 1$. Therefore $\bigcap_{n=1}^{\infty} C_n$ is nonempty.

COROLLARY 3.1 (Maluta [17] and Bae [1]). *Let E be a real Banach space with* $\tilde{N}(E)<1$. *Then E is reflexive and has normal structure.*

§ 4. Fixed point theorems.

In this section, we prove three fixed point theorems by using the results obtained in section 3. The following lemma is crucial in the proofs.

LEMMA 4.1. *Let C be a convex subset of a real Banach space E. Let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of C and let D be a boundedly weakly compact convex subset of C. Let r be the asymptotic radius and A be the asymptotic center of $\{B_\alpha\}$ in D. Then*

$$d(A) \leqq \varepsilon_0(C) r .$$

Further let $\varepsilon_0(C) < 1$ and let γ be a real number such that $\gamma(1-\delta(C, 1/\gamma))=1$. For a real number k with $1 \leqq k < \gamma$, define $A_k = \{x \in D : r(\{B_\alpha\}, x) \leqq kr\}$. Then

$$d(A_k) \leqq \frac{k}{\gamma} r .$$

Proof. In case $r=0$, the inequality is true. In fact, if $x, y \in A$ then

$$\|x-y\| \leqq r(\{B_\alpha\}, x) + r(\{B_\alpha\}, y) = 0$$

and hence $d(A)=0$. So we assume $r>0$ and $d(A)>0$. For any real number η with $0<\eta<d(A)$, there exist $x, y \in A$ such that $\|x-y\| \geqq d(A)-\eta$. By Lemma 3.2 and convexity of A, we have

$$r = r\left(\{B_\alpha\}, \frac{x+y}{2}\right) \leqq r\left(1-\delta\left(C, \frac{d(A)-\eta}{r}\right)\right).$$

This implies

$$\delta\left(C, \frac{d(A)-\eta}{r}\right) = 0$$

and hence $d(A) \leqq \varepsilon_0(C) r$.

We may also assume $r>0$ and $d(A_k)>0$. For any real number η with $0<\eta<d(A_k)$, there exist $x, y \in A_k$ such that $\|x-y\| \geqq d(A_k)-\eta$. Then, we have

$$r \leqq r\left(\{B_\alpha\}, \frac{x+y}{2}\right) \leqq kr\left(1-\delta\left(C, \frac{d(A_k)-\eta}{kr}\right)\right).$$

Since $\eta>0$ is arbitrary and δ is continuous, it follows that

$$\delta\left(C, \frac{d(A_k)}{kr}\right) \leqq 1-\frac{1}{k}.$$

Suppose that $\frac{1}{\gamma} \leqq \frac{d(A_k)}{kr}$. Then we have

$$1-\frac{1}{\gamma} = \delta\left(C, \frac{1}{\gamma}\right) \leqq \delta\left(C, \frac{d(A_k)}{kr}\right) \leqq 1-\frac{1}{k} < 1-\frac{1}{\gamma}.$$

This is a contradiction.

Remark 4.1. From Lemma 4.1, we have immediately the similar inequality concerning the Chebyshev radius and center. In fact, putting $B_\alpha = B$, we have

$$d(\mathcal{C}(B, D)) \leqq \varepsilon_0(C) R(B, C).$$

The following theorem is a special case of results of Lim [15] and Takahashi [18], while the proof is constructive.

THEOREM 4.1. *Let C be a closed convex subset of a real Banach space E with $\varepsilon_0(C) < 1$ and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Suppose that S is left reversible and $\{T_t y : t \in S\}$ is bounded for some $y \in C$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.*

Proof. Let $B_s(x) = \{T_t x : t \geqq s\}$ for $s \in S$ and $x \in C$. Define $\{x_n : n \geqq 0\}$ by induction as follows:

$$x_0 = y;$$

$$x_n \in \mathcal{A}(\{B_s(x_{n-1})\}, C) \quad \text{for} \quad n \geqq 1.$$

Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$, $r_n = r(\{B_s(x_{n-1})\}, C)$ and $A_n = \mathcal{A}(\{B_s(x_{n-1})\}, C)$ for $n \geqq 1$. Then we have

$$\begin{aligned} r_n(T_t x_n) &= \lim_s \sup \|T_s x_{n-1} - T_t x_n\| \leqq \lim_s \sup \|T_t T_s x_{n-1} - T_t x_n\| \\ &\leqq \lim_s \sup \|T_s x_{n-1} - x_n\| = r_n \end{aligned}$$

for all $t \in S$ and $n \geqq 1$ and hence $T_t A_n \subseteqq A_n$ for $t \in S$ and $n \geqq 1$. By Lemma 4.1, we obtain

$$\begin{aligned} r_{n+1} &= r_{n+1}(x_{n+1}) \leqq r_{n+1}(x_n) \leqq \sup_s \|T_s x_n - x_n\| \\ &\leqq d(A_n) \leqq \varepsilon_0(C) r_n \leqq (\varepsilon_0(C))^n r_1 \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - x_n\| &\leqq r(\{B_s(x_n)\}, x_{n+1}) + r(\{B_s(x_n)\}, x_n) = r_{n+1} + r_{n+1}(x_n) \\ &\leqq 2(\varepsilon_0(C))^n r_1 \end{aligned}$$

for all $n \geqq 1$. So, $\{x_n\}$ is a Cauchy sequence and hence $\{x_n\}$ converges to a point $z \in C$. Therefore we have

$$\begin{aligned} \|z - T_s z\| &= \lim_{n \to \infty} \|x_n - T_s x_n\| \leqq \lim_{n \to \infty} (r_n(x_n) + r_n(T_s x_n)) \\ &\leqq \lim_{n \to \infty} 2(\varepsilon_0(C))^{n-1} r_1 = 0 \end{aligned}$$

for all $s \in S$.

By the method of Theorem 4.1, we can prove the following fixed point theorem which is slightly different from [9].

THEOREM 4.2. *Let C be a closed convex subset of a real Banach space E with $\varepsilon_0(C)<1$ and let γ be a real number such that $\gamma(1-\delta(C, 1/\gamma))=1$. Let $\mathcal{S}=\{T_t : t\in S\}$ be a uniformly k-lipschitzian semigroup on C with $1\leqq k<\gamma$. Suppose that S is left reversible and $\{T_t y : t\in S\}$ is bounded for some $y\in C$. Then there exists a $z\in C$ such that $T_s z=z$ for all $s\in S$.*

Proof. Let $B_s(x)=\{T_t x : t\geqq s\}$ for $s\in S$ and $x\in C$. Define $\{x_n : n\geqq 0\}$ by induction as follows:

$$x_0=y\,;$$
$$x_n\in \mathcal{A}(\{B_s(x_{n-1})\}, C) \qquad \textit{for} \quad n\geqq 1.$$

Let $r_n(x)=r(\{B_s(x_{n-1})\}, x)$, $r_n=r(\{B_s(x_{n-1})\}, C)$ and $A_n=\{x\in C : r_n(x)\leqq kr_n\}$ for $n\geqq 1$. Then since $r_n(x_n)=r_n\leqq kr_n$ and

$$r_n(T_t x_n)=\lim_s \sup\|T_s x_{n-1}-T_t x_n\|\leqq k \lim_s \sup\|T_s x_{n-1}-x_n\|=kr_n$$

for all $t\in S$ and $n\geqq 1$, we have $x_n, T_t x_n\in A_n$ for all $t\in S$ and $n\geqq 1$. By Lemma 4.1, we obtain

$$\begin{aligned} r_{n+1}=r_{n+1}(x_{n+1})&\leqq r_{n+1}(x_n)\leqq \sup_s\|T_s x_n-x_n\| \\ &\leqq d(A_n)\leqq \frac{k}{\gamma}r_n\leqq \left(\frac{k}{\gamma}\right)^n r_1 \end{aligned}$$

for all $n\geqq 1$. Therefore, as in the proof of Theorem 4.1, $\{x_n\}$ converges to a point $z\in C$. So, we have

$$\begin{aligned} \|z-T_s z\|&=\lim_{n\to\infty}\|x_n-T_s x_n\|\leqq \lim_{n\to\infty}(r_n(x_n)+r_n(T_s x_n)) \\ &\leqq \lim_{n\to\infty}(1+k)r_n=0 \end{aligned}$$

for all $s\in S$.

Remark 4.2. Let C and γ be defined as in Theorem 4.2. Then we have

$$1<\gamma\leqq 1+\frac{1-\varepsilon_0(C)}{2}.$$

In fact, let $\eta=1/\gamma$ and if $\delta(C, \eta)=1-\eta=0$. Then we have $1>\varepsilon_0(C)\geqq\eta=1$. This is a contradiction. Hence $\varepsilon_0(C)\leqq\eta<1$. So, from Theorem 3.1,

$$1-\eta=\delta(C, \eta)\leqq \frac{\eta-\varepsilon_0(C)}{2-\varepsilon_0(C)}.$$

Therefore we have

$$1<\gamma\leqq 1+\frac{1-\varepsilon_0(C)}{2}.$$

We can also obtain a generalization of Casini and Maluta's fixed point theorem [4].

LEMMA 4.2. *Let C be a boundedly weakly compact convex subset of a real Banach space E. Let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded closed convex subsets of C and let $B=\bigcap_\alpha B_\alpha$. Then*

$$r(\{B_\alpha\},\ B) \leqq \tilde{N}(C) \inf_\alpha d(B_\alpha).$$

Proof. Let $u_\beta \in \mathcal{C}(B_\beta, B_\beta)$ for each $\beta \in \Lambda$. Then we have

$$r(\{B_\alpha\},\ u_\beta) \leqq R(B_\beta,\ u_\beta) = R(B_\beta,\ B_\beta) \leqq \tilde{N}(C) d(B_\beta).$$

Let $\{u_{\beta_\gamma}\}$ be a subnet of $\{u_\beta\}$ which converges weakly to a point $u_0 \in B$. By weakly lower semicontinuity of r and monotonicity of $d(B_\beta)$, we have

$$\begin{aligned} r(\{B_\alpha\},\ B) &\leqq r(\{B_\alpha\},\ u_0) \leqq \liminf_\gamma r(\{B_\alpha\},\ u_{\beta_\gamma}) \leqq \liminf_\gamma \tilde{N}(C) d(B_{\beta_\gamma}) \\ &= \tilde{N}(C) \inf_\gamma d(B_{\beta_\gamma}) = \tilde{N}(C) \inf_\alpha d(B_\alpha). \end{aligned}$$

THEOREM 4.3. *Let C be a closed convex subset of a real Banach space E with $\tilde{N}(C)<1$ and let $\mathcal{S}=\{T_t : t \in S\}$ be a uniformly k-lipschitzian semigroup on C with $k<\tilde{N}(C)^{-1/2}$. Suppose that S is left reversible and $\{T_t y : t \in S\}$ is bounded for some $y \in C$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.*

Proof. Let $B_s(x)=\overline{co}\{T_t x : t \geqq s\}$ and let $B(x)=\bigcap_s B_s(x)$ for $s \in S$ and $x \in C$. Define $\{x_n : n \geqq 0\}$ by induction as follows:

$$x_0 = y;$$

$$x_n \in \mathcal{A}(\{B_s(x_{n-1})\},\ B(x_{n-1})) \qquad \text{for} \quad n \geqq 1.$$

Let $r_n(x)=r(\{B_s(x_{n-1})\},\ x)$ and $r_n = r(\{B_s(x_{n-1})\},\ B(x_{n-1}))$ for $n \geqq 1$. Then from $x_n \in B(x_{n-1}) = \bigcap_t B_t(x_{n-1})$ for $n \geqq 1$, we have

$$\begin{aligned} r_{n+1}(x_n) &= \limsup_s \|T_s x_n - x_n\| \leqq \limsup_s (\inf_t R(B_t(x_{n-1}),\ T_s x_n)) \\ &= \limsup_s r_n(T_s x_n) = \limsup_s (\limsup_t \|T_t x_{n-1} - T_s x_n\|) \\ &\leqq \limsup_s (k \limsup_t \|T_t x_{n-1} - x_n\|) = k r_n \\ &\leqq k \tilde{N}(C) \inf_s d(B_s(x_{n-1})) \end{aligned}$$

and

$$\inf_s d(B_s(x_{n-1})) = \inf_s \sup\{\|T_a x_{n-1} - T_b x_{n-1}\| :\ a,\ b \geqq s\}$$

$$\leqq \limsup_t (\limsup_s \|T_s x_{n-1} - T_t x_{n-1}\|)$$
$$= \limsup_t r_n(T_t x_{n-1})$$
$$\leqq k r_n(x_{n-1}).$$

Hence we have

$$r_{n+1}(x_n) \leqq k r_n \leqq k^2 \tilde{N}(C) r_n(x_{n-1}) \leqq (k^2 \tilde{N}(C))^n r_1(x_0).$$

Therefore, as in the proof of Theorem 4.2, $\{x_n\}$ converges to a common fixed point.

REFERENCES

[1] BAE, J.S., Reflexivity of a Banach space with a uniformly normal structure, Proc. Amer. Math. Soc. **90** (1984), 269-270.

[2] BROWDER, F.E., Nonexpansive nonlinear operators in Banach space, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041-1044.

[3] BYNUM, W.L., Normal structure coefficients for Banach spaces, Pacific J. Math. **86** (1980), 427-436.

[4] CASINI, E. AND E. MALUTA, Fixed points of uniformly lipschitzian mappings in spaces with uniformly normal structure, Nonlinear Analysis **9** (1985), 103-108.

[5] DOWNING, D.J. AND W.O. RAY, Uniformly lipschitzian semigroup in Hilbert space, Canad. Math. Bull. **25** (1982), 210-214.

[6] DUNFORD, N. AND J.T. SCHWARTZ, Linear operators, Part 1., Interscience, New York (1958).

[7] GOEBEL, K., Convexity of balls and fixed-point theorems for mappings with non-expansive square, Compositio Math. **22** Fasc. **3** (1970), 269-274.

[8] GOEBEL, K. AND W.A. KIRK, A fixed point theorem for transformations whose iterates have uniform Lipschitz constant, Studia Math. **47** (1973), 135-140.

[9] GOEBEL, K., W.A. KIRK AND R.L. THELE, Uniformly lipschitzian families of transformations in Banach space, Can. J. Math. **26** (1974), 1245-1256.

[10] GÖHDE, D., Zum prinzip der kontraktiven abbildung, Math. Nachr. **30** (1965), 251-258.

[11] GURARII, V.I., On the differential properties of the modulus of convexity in a Banach space, Mat. Issled. **2** (1967), 141-148.

[12] ISHIHARA, H. AND W. TAKAHASHI, Fixed point theorems for uniformly lipschitzian semigroups in Hilbert spaces, to appear in J. Math. Anal. Appl. **126** (1987).

[13] KIRK, W.A., A fixed point theorem for mappings which do not increase distance, Amer. Math. Monthly **72** (1965), 1004-1006.

[14] LIFSCHITZ, E.A., Fixed point theorems for operators in strongly convex spaces, Voronez Gos. Univ. Trudy Math. Fak. **16** (1975), 23-28.

[15] LIM, T.C., Characterizations of normal structure, Proc. Amer. Math. Soc. **43** (1974), 313-319.

[16] LIM, T.C., On asymptotic centers and fixed points of nonexpansive mappings, Can. J. Math. **32** (1980), 421-430.

[17] MALUTA, E., Uniformly normal structure and related coefficients, Pacific J. Math. **111** (1984), 357-369.

[18] TAKAHASHI, W., Fixed point theorems for families of nonexpansive mappings on unbounded sets, J. Math. Soc. Japan **36** (1984), 545-553.

DEPARTMENT OF INFORMATION SCIENCE,
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU, TOKYO 152, JAPAN