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A UNIQUENESS THEOREM FOR MINIMAL SURFACES IN S^3

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§1. Introduction.

R^3 and S^3 have some similar properties. First both of them have congruent translations such as the parallel translations or the rotations, and secondly they have the concept of the convex hull. ([1]).

W. H. Meeks III states some uniqueness theorems for minimal surfaces in R^3 and one of the theorems is the following.

THEOREM 1 (Meeks III [3]). *Suppose γ is a C^2-Jordan curve on a plane. Then there exists a positive number ε so that any Jordan curve in R^3 which is ε close to γ in the C^2-norm is the boundary curve of a unique compact minimal surface. Furthermore, this minimal surface is a graph over the plane.*

We will show anologous theorem paying attention to the next paragraph for minimal surfaces in S^3.

THEOREM 2. *Suppose that γ is a C^2-Jordan curve on a geodesic 2-sphere in S^3 and belongs to some open hemisphere of S^3. Then there exists a positive number ε such that any Jordan curve in S^3 which is ε close to γ in the C^2-norm is the boundary curve of a unique compact minimal surface in the open hemisphere. And also this minimal surface can be represented as a graph over the geodesic 2-sphere.*

§2. Preparation.

We introduce for our arguement the following model for S^3 which is found in [2].

We identify R^3 which S^3 by the stereographic projection from the point $(0, 0, 0, -1)$. The origin $O=(0, 0, 0)$ corresponds to the south pole of the projection. The geodesics of S^3 correspond to the all straight lines through O, or all great circles of the unit sphere S centered at O, or all plane circles meeting S in antipodal points. The geodesic 2-spheres of S^3 correspond to all planes through O, or the sphere S, or all Euclidean spheres which meet S in a great

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circle of S. All plane circles correspond to the small circles in S^3. So consider all plane circles which intersect orthogonally with xy-plane P and S. Then we find some translation along this flow of circles corresponds to a rotation which leaves C=the intersection of P and S fixed.

§ 3. Proof of the theorem 2.

Proof. We use the same method as [3]. By the assumption of the theorem we shall assume that γ is on P and belongs to the interior of C, which is the argument in the model of S^3. Let $F=\{\gamma_t \mid 0 \leqq t \leqq 1\}$ be a C^2-foliation by Jordan curves of the annular region in P bounded by $\gamma_0=\gamma$ and $\gamma_1=C$.

Consider a point which goes from each point on γ_t along the geodesic m on P in the direction of the inner normal by β. Here β is a small positive number so that for each t we obtain another Jordan curve α_t as a set of the points above. Suppose that on a geodesic 2-sphere two geodesics intersect orthogonally at some point Q, and let $\delta(\beta)$ be the distance between the two points each of which is on each of the two geodesics and is far from Q by β. Consider a point which goes above (below) by β from each point on α_t along the geodesic n through the point that is orthogonal to P, and around this point construct a geodesic circle of radius $\delta(\beta)$ on the geodesic 2-sphere made by m and n. By these constructions we obtain the torus $T_t^+(T_t^-)$ which is around α_t and contains γ_t for each t. Here we put a new restriction on β. The number β should be small enough for T_t^+ (or T_t^-) to have positive mean curvature with respect to the inner normal.

Take two points X^+, X^- on z-axis which are far from O by $\delta(\beta)-\beta$ and construct two geodesic 2-spheres P^+, P^- including C and X^+ or X^-. Let S_t be a piecewise smooth sphere constructed by T_t^+, T_t^-, P^+ and P^-, which contains γ_t. Then there are subsets A_t^+, A_t^-, D_t^+ and D_t^- of T_t^-, T_t^+, P^+ and P^- such that S_t is the union of A_t^+, A_t^-, D_t^+ and D_t^-.

Let $G=\{f: S^3 \to S^3 \mid f$ is a C^2-diffeomorphism$\}$ and

$$N=\{f \mid f \text{ is contained in } G \text{ and satisfies (1) (2) (3)}\}.$$

(1) $\|f(x)-x\|<\xi$ for all points x in S^3.

(2) $(Df_x(v)/\|Df_x(v)\|, v)>\cos\eta$ for all x in S^3 and v in T_xS^3 with $\|v\|=1$.

(3) $f(\operatorname{int} A_t^+)$ and $f(\operatorname{int} A_t^-)$ have positive mean curvature for all t.

Here we take $\xi>0$ such that ξ-neighborhood of P^+ (or P^-) and the convex hull of ξ-neighborhood of γ are disjoint and set

$$\eta=\min\{\text{the angle between } v \text{ and the tangent vector at } x \text{ of the small circle through } x \text{ which is orthogonal to } P \text{ and } S\}.$$

Here we take minimum among all x in A_0^+ and v in T_xS_0. Construct the coordinate system in S^3 by the flow of small circles orthogonal to P and S, and by the coordinate system on the geodesic 2-spheres orthogonal to the flow. The inner product in (2) is the inner product of R^3 in such coordinate. N is an open neighborhood of identity map in G.

Let M be a minimal surface in S with boundary $f(\gamma)$ where f is contained in N. We will first show that M is contained in the ball B bounded by $f(S_0)$. If M is not contained in B, then $\sigma=\max\{t \mid 0 \leqq t \leqq 1$, the intersection of M and $f(S_t)$ is not empty$\}$ is greater than 0. By condition (1), the convex hull of $f(\gamma)$ is contained between $f(P^+)$ and $f(P^-)$. This shows that the intersection of M and $f(S_\sigma)$ is contained in the union of $f(A_\sigma^+)$ and $f(A_\sigma^-)$. And also by condition (1), M is contained in the interior of the ball bounded by $f(S_1)$. Thus $\sigma<1$. Condition (2) implies that the interior angles along the surfaces $f(A_\sigma^+)$ and $f(A_\sigma^-)$ are less than π. Hence the intersection of M and $f(S_\sigma)$ is contained in the union of $f(\text{int } A_\sigma^+)$ and $f(\text{int } A_\sigma^-)$. We find this fact is a contradiction by condition (3) and by maximum principle. So we obtain $\sigma=0$ and M is contained in B, and also $\text{int } M$ is contained in $\text{int } B$.

We regard the flow of small circles orthogonal to P and S as vertical lines. Condition (2) implies that f(union of A_0^+ and D_0^+) and f(union of A_0^- and D_0^-) are graphs over P in the above sence. This fact and the fact that $\text{int } M$ is contained in $\text{int } B$ show that the nontrivial rotation of M along the flow of small circles orthogonal to P and S is disjoint from $f(\gamma)$, which is essential.

If M is not a graph over P, then some nontrivial rotation of $\text{int } M$ intersects $\text{int } M$ and locally above it. By using maximum principle we can lead a contradiction. Thus M is a graph over P. The same method leads a contradiction if there are two distinct minimal surfaces in S with boundary $f(\gamma)$. Thus $f(\gamma)$ is the boundary of a unique compact minimal surface in S. q. e. d.

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