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KODAI MATH. J.
10 (1987), 1—8

THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF S^2-BUNDLES OVER S^4, II: APPLICATIONS

Dedicated to Professor Hirosi Toda on his 60th birthday

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Statement of Results.

The present paper is the second part of [20] with the same title, and the object of this paper is to study the classification of homotopy types of certain six dimensional simply connected CW complexes as an application of the first part [20].

We shall use freely all the notations and notions defined in Part I [20].

Then the main results of this note are summerized as follows:

THEOREM 6.9. *Let X be a simply connected six dimensional CW complex of the type (m, n, ε).*

(1) *If $m \equiv 1 \pmod 2$, then $\varepsilon=0$ and X and $X(m, n, 0)$ are of the same homotopy type.*

(2) *If $m \equiv 0 \pmod 2$ and $\varepsilon=1$, then X and $X(m, n, 1)$ are of the same homotopy type.*

(3) *If $m \equiv 0 \pmod 2$ and $\varepsilon=0$, then X is homotopy equivalent to $X(m, n, 0)$ or $Y(m, n, 0)$, and $X(m, n, 0)$ and $Y(m, n, 0)$ are of the different homotopy types.*

COROLLARY 6.10 ([17], [18]). *Let X be a simply connected six dimensional CW complex of the type (m, n, ε). Then if $m \equiv 1 \pmod 2$ or $m \equiv 0 \pmod 2$ and $\varepsilon=1$, the homotopy type of X is uniquely determined.*

COROLLARY 6.17. *Let m be an even integer and M be a m-twisted CP^3 of the type $(m, 1, 0)$, and we put $p_1(M)=kx_2$, where x_2 denotes the generator of $H^4(M, Z) \cong Z$, and $p_1(M)$ the first Pontrjagin class of M.*

(1) *Then M and $X(m, 1, 0)$ are of the same homotopy type if and only if $m-4k \equiv 0 \pmod{48}$,*

and

(2) *M and $Y(m, 1, 0)$ are of the same homotopy type if and only if $m-4k \equiv 24 \pmod{48}$.*

(3) *In particular, if M' is another m-twisted CP^3 of the type $(m, 1, 0)$, then*

Received January 28, 1986

M and M' are of the same homotopy type if and only if $k\equiv k'$ (mod 48), where we put $p_1(M')=k'x_2'$ as above.

6. Applications.

In this section, we consider the classification of homotopy types of certain simply connected six dimensional manifolds as an application.

DEFINITION 6.1. Let X be a simply connected six dimensional CW complex. Then X has the type (m, n, ε) if the following conditions are satisfied:

(1) $H_i(X, Z)=\begin{cases} Z & \text{for } i=2j\,;\ j=0, 1, 2, 3. \\ 0 & \text{otherwise.} \end{cases}$

(2) If $x_i\in H^{2i}(X, Z)$ denotes the generator for $i=1, 2, 3$, then $x_1\cdot x_1=\pm mx_2$ and $x_1\cdot x_2=\pm nx_3$, where $m, n\geqq 0$.

(3) If $x_i'\in H^{2i}(X, Z_2)$ denotes the generator for $i=1, 2, 3$, then $Sq^2(x_2')=\varepsilon x_3'$,

where $\varepsilon=0$ or 1 and $Sq^2: H^4(X, Z_2)\to H^6(X, Z_2)$.

If X and Y have the same type (m, n, ε), then $H^*(X, Z)$ and $H^*(Y, Z)$ are isomorphic as a cohomology ring. Moreover, $H^*(X, Z_p)$ and $H^*(Y, Z_p)$ are isomorphic as a A_p algebra for each prime p, where A_p denotes the mod p Steenrod algebra. Let L_m be the 2-cell complex, $L_m=S^2\cup_{m\eta_2}e^4$, where $\pi_3(S^2)=Z\{\eta_2\}$.

LEMMA 6.2. *If the CW complex X has the type (m, n, ε), then $X=L_m\cup_b e^6$ for some element $b\in\pi_5(L_m)$, up to homopy types.*

LEMMA 6.3. *If $m\equiv 1$ (mod 2) and X has the type (m, n, ε), then $\varepsilon=0$.*

Proof. The assertion easily follows from the Adem relation $Sq^2Sq^2=Sq^3Sq^1$. Q.E.D.

LEMMA 6.4 (I.M. James, [3]). *Let X be a CW complex with the form, $X=L_m\cup_b e^6$, for an element $b\in\pi_5(L_m)$, and $i_{1*}: \pi_5(L_m)\to\pi_5(L_m, S^2)$ be the induced homomorphism.*

(1) *If $m\equiv 1$ (mod 2), then X has the type $(m, n, 0)$ if and only if $i_{1*}(b)=\pm n[a_m, \iota_2]_r$.*

(2) *If $m\equiv 0$ (mod 2), then X has the type (m, n, ε) if and only if $i_{1*}(b)=\pm n[a_m, \iota_2]_r+\varepsilon a_{m*}(\eta)$,*

where $\eta\in\pi_5(D^4, S^3)\cong Z_2$ denotes the generator and $a_m\in\pi_4(L_m, S^2)$ is the characteristic map of the 4-cell.

Proof. The assertion easily follows from Theorem 3.3 in [3]. Q.E.D.

DEFINITION 6.5. We define the CW complexes $X(m, n, \varepsilon)$ and $Y(m, n, 0)$ as follows:

(1) For each pair of non-negative integers (m, n), we put $X(m, n, 0)= L_m \bigcup_{nb_m} e^6$.

(2) If $m \equiv 0 \pmod 2$ and m, n are non-negative integers, then we put

$$X(m, n, 1)=L_m \bigcup_{nb_m+\gamma_m} e^6 \quad \text{and} \quad Y(m, n, 0)=L_m \bigcup_{nb_m+i_*(\eta_2^3)} e^6,$$

where

$$\pi_5(L_m)=\begin{cases} Z\{b_m\} & \text{if } m\equiv 1 \pmod 2 \\ Z\{b_m\}\oplus Z_4\{\gamma_m\} & \text{if } m\equiv 2 \pmod 4 \\ Z\{b_m\}\oplus Z_2\{\gamma_m\}\oplus Z_2\{i_*(\eta_2^3)\} & \text{if } m\equiv 0 \pmod 4 \end{cases}$$

and $2\gamma_m=i_*(\eta_2^3)$ if $m\equiv 2 \pmod 4$.
Here, $i_*: \pi_5(S^2)=Z_2\{\eta_2^3\}\to\pi_5(L_m)$ denotes the induced homomorphism.

PROPOSITION 6.6. (1) *The* 3-*cell complex* $X(m, n, 0)$ *has the type* $(m, n, 0)$.

(2) *In particular, if* $m\equiv 0 \pmod 2$, *then* $X(m, n, 1)$ *has the type* $(m, n, 1)$ *and* $Y(m, n, 0)$ *has the type* $(m, n, 0)$.

Proof. Since $i_{1*}(b_m)=[a_m, \iota_2]_r$, $i_{1*}(\gamma_m)=a_{m*}(\eta)$ and $i_{1*}(i_*(\eta_2^3))=0$, the above results follow from (6.4). Q.E.D.

COROLLARY 6.7. (1) *For each pair of non-negative integers* (m, n), *there exists a* 6-*dimensional* 3-*cell complex which has the type* $(m, n, 0)$.

(2) *If* m *is an odd positive integer, for any non-negative integer* n, *there is no* 6-*dimensional* 3-*cell complex of the type* $(m, n, 1)$.

(3) *If* m *is an even non-negative integer, then for each non-negative integer* n *there exists a* 6-*dimensional* 3-*cell complex of the type* $(m, n, 1)$.

Here we note the following well-known

LEMMA 6.8. *Let* L *be a simply connected* r *dimensional CW complex, and* X *and* Y *be* $(r+k)$-*dimensional CW complexes with the forms,*

$$X=L\bigcup_f e^{r+k} \quad \text{and} \quad Y=L\bigcup_g e^{r+k},$$

where $k\geqq 2$ *and* $f, g\in\pi_{r+k-1}(L)$.

Then X *and* Y *are of the same homotopy type if and only if there exists a homotopy equivalence* $\theta\in Eq(L)$ *satisfying* $\theta\circ f=\pm g$.

Then we have

THEOREM 6.9. *Let* X *be a simply connected six dimensional CW complex of the type* (m, n, ε).

(1) *If* $m\equiv 1 \pmod 2$, *then* $\varepsilon=0$ *and* X *and* $X(m, n, 0)$ *are of the same homotopy type.*

(2) *If* $m\equiv 0 \pmod 2$ *and* $\varepsilon=1$, *then* X *and* $X(m, n, 1)$ *are of the same homotopy type.*

(3) *If* $m\equiv 0 \pmod 2$ *and* $\varepsilon=0$, *then* X *is homotopy equivalent to* $X(m, n, 0)$ *or* $Y(m, n, 0)$, *and* $X(m, n, 0)$ *and* $Y(m, n, 0)$ *are of the different homotopy types.*

COROLLARY 6.10 ([17], [18]). *Let* X *be a simply connected six dimensional CW complex of the type* (m, n, ε). *Then if* $m\equiv 1 \pmod 2$ *or* $m\equiv 0 \pmod 2$ *and* $\varepsilon=1$, *the homotopy type of* X *is uniquely determined.*

Proof of Theorem 6.9.

It follows from (6.2) that we may assume $X=L_m\cup_b e^6$ for some element $b\in\pi_5(L_m)$. Consider the exact sequence

$$\begin{array}{ccccc}
\pi_5(S^2) & \xrightarrow{i_*} & \pi_5(L_m) & \xrightarrow{i_{1*}} & \pi_5(L_m, S^2) \\
\| & & \| & & \| \\
Z_2\{\eta_2^3\} & & Z\{b_m\}\oplus Tor & & Z\{[a_m, \iota_2]_r\}\oplus Z_2\{a_{m*}(\eta)\}
\end{array}$$

where

$$Tor=\begin{cases} 0 & \text{if } m\equiv 1 \pmod 2 \\ Z_4\{\gamma_m\} & \text{if } m\equiv 2 \pmod 4 \\ Z_2\{\gamma_m\}\oplus Z_2\{i_*(\eta_2^3)\} & \text{if } m\equiv 0 \pmod 4 \end{cases}$$

Then, from (6.4) we have

$$i_{1*}(b)=\begin{cases} \pm n[a_m, \iota_2]_r & \text{if } m\equiv 1 \pmod 2 \\ \pm n[a_m, \iota_2]_r+\varepsilon a_{m*}(\eta) & \text{if } m\equiv 0 \pmod 2 \end{cases} \tag{6.11}$$

where $\eta\in\pi_5(D^4, S^3)\cong Z_2$ denotes the generator.

First, we suppose $m\equiv 1 \pmod 2$. Then it follows from (2.3) and (2.13) that we have $b=\pm nb_m$. Since $Eq(L_m)=Z_2\{h_1\}$ and $h_1(-nb_m)=nb_m$ by (3.3) and (4.6), the assertion follows from (6.8).

Next, we suppose $m\equiv 0 \pmod 2$. Similarly, from (2.3), (2.8), (2.13) and (6.11), we obtain

$$b=\begin{cases} \pm nb_m+\varepsilon\gamma_m \\ \text{or} \\ \pm nb_m+\varepsilon\gamma_m+i_*(\eta_2^3) \end{cases} \tag{6.12}$$

Here we put

$$X_1(\varepsilon)=L_m\cup_{nb_m+\varepsilon\gamma_m}e^6=X(m, n, \varepsilon)\,,$$

$$X_2(\varepsilon)=L_m\cup_{-nb_m+\varepsilon\gamma_m}e^6\,,$$

$$X_3(\varepsilon)=L_m\cup_{nb_m+\varepsilon\gamma_m+i_*(\eta_2^3)}e^6 \quad \text{and} \quad X_4(\varepsilon)=L_m\cup_{-nb_m+\varepsilon\gamma_m+i_*(\eta_2^3)}e^6 .$$

In particular, if Y and Z are of the same homotopy type, we denote it by $Y\simeq Z$. Then for some integer k, $X\simeq X_k(\varepsilon)$.

On the other hand, it follows from (3.3) and (4.6) that we obtain the following equations:

(a) If $\varepsilon=1$ and $m\equiv 2 \pmod 4$,

$$h_1(nb_m+\gamma_m)=-nb_m+\gamma_m+i_*(\eta_2^3),$$

$$h_1(-nb_m+\gamma_m)=nb_m+\gamma_m+i_*(\eta_2^3),$$

and $$h_2(nb_m+\gamma_m)=nb_m+\gamma_m+i_*(\eta_2^3).$$

(b) If $\varepsilon=1$ and $m\equiv 0 \pmod 4$,

$$h_1(nb_m+\gamma_m)=-nb_m+\gamma_m,$$

$$h_1(nb_m+\gamma_m+i_*(\eta_2^3))=-nb_m+\gamma_m+i_*(\eta_2^3)$$

and $$h_2(nb_m+\gamma_m)=nb_m+\gamma_m+i_*(\eta_2^3).$$

(c) If $\varepsilon=0$,

$$h_1(nb_m)=-nb_m,$$

$$h_1(nb_m+i_*(\eta_2^3))=-nb_m+i_*(\eta_2^3)$$

and $$\theta(nb_m)\neq nb_m+i_*(\eta_2^3) \quad \text{for any homotopy equivalence } \theta\in Eq(L_m).$$

Hence, if $\varepsilon=1$ and $m\equiv 2 \pmod 4$, from (6.8) and (a), we have

$$X_1(1)\simeq X_4(1),\quad X_2(1)\simeq X_3(1) \quad \text{and} \quad X_1(1)\simeq X_3(1).$$

Similarly, if $\varepsilon=1$ and $m\equiv 0 \pmod 4$, we obtain

$$X_1(1)\simeq X_2(1),\quad X_3(1)\simeq X_4(1) \quad \text{and} \quad X_1(1)\simeq X_3(1).$$

Therefore, if $\varepsilon=1$, for each $1\leqq p<q\leqq 4$, $X_p(1)$ and $X_q(1)$ are of the same homotopy type. Since $X_1(m, n, 1)=X_1(1)$, X and $X(m, n, 1)$ are of the same homotopy type if $\varepsilon=1$.

On the other hand, if $\varepsilon=0$, it follows from (6.8) and (c) that $X_1(0)\simeq X_2(0)$, $X_3(0)\simeq X_4(0)$, and $X_1(0)$ and $X_3(0)$ are of the different homotopy types.

Since $X_1(0)=X(m, n, 0)$ and $X_3(0)=Y(m, n, 0)$, we obtain $X\simeq X(m, n, 0)$ or $X\simeq Y(m, n, 0)$ and that $X(m, n, 0)$ and $Y(m, n, 0)$ are of the different homotopy types. This completes the proof. Q. E. D.

DEFINITION 6.13 (M. Masuda, [9]). Let X be a six-dimensional simply-connected CW complex of the type (m, n, ε). Then we say that X is a m-twisted CP^3 if X is a closed smooth manifold.

PROPOSITION 6.14 (C. T. C. Wall, [15]). *Let X be a six-dimensional simply-connected CW complex of the type (m, n, ε). Then there exists a m-twisted CP^3 M_X which is homotopy equivalent to X, if and only if $n=1$.*

Furthermore, if $n=1$, then M_X has the spin structure if and only if $\varepsilon=0$.

Proof. It is easy to see that X is a Poincaré complex if and only if $n=1$. Since $H^3(X, Z_2)=0$, the assertions easily follow from Theorem 8 in [15] and the *Wu* formula. Q. E. D.

PROPOSITION 6.15. *Let M be a m-twisted CP^3 of the type $(m, 1, \varepsilon)$, $p_1(M)$ be its first Pontrjagin class, and we put $p_1(M)=kx_2$ for some integer k, where x_i denotes the generator of $H^{2i}(M, Z)$, $i=1, 2, 3$.*

Then the following relations hold:

(1) *If* $\varepsilon=0$, $k\equiv 4m \pmod{24}$.

(2) *If* $\varepsilon=1$, $k\equiv 9m \pmod{16}$.

Proof. Let $[M]\in H_6(M, Z)$ be the fundamental class of M. Then by the $\hat{A}$-integrality theorem, $\langle \exp(W/2)\cdot\exp(x_1)\cdot(1-p_1(M)/24), [M]\rangle\in Z$, where we put $W=\varepsilon x_1$.

Since $(x_1)^2=mx_2$ and $x_1\cdot x_2=x_3$,

$$\exp(W/2)=\begin{cases} 1+(1/2)x_1+(m/2)x_2+(m/48)x_3 & \text{if } \varepsilon=1 \\ 1 & \text{if } \varepsilon=0, \end{cases}$$

$$\exp(x_1)=1+x_1+(m/2)x_2+(m/6)x_3, \quad \text{and} \quad 1-p_1(M)/24=1-(k/24)x_2.$$

Hence an easy calculation shows the above results. Q. E. D.

PROPOSITION 6.16 (C. T. C. Wall, [15]). *Let m and k be integers. Then, if $k\equiv 4m \pmod{24}$, there exists a m-twisted CP^3 of the type $(m, 1, 0)$, M, such that $p_1(M)=kx_2$, where x_2 denotes the generator of $H^4(M, Z)$ $p_1(M)$ the first Pontrjagin class of M.*

Proof. The assertion easily follows from Theorem 5 in [15]. Q. E. D.

COROLLARY 6.17. *Let m be an even integer and M be a m-twisted CP^3 of the type $(m, 1, 0)$, and we put $p_1(M)=kx_2$, where x_2 denotes the generator of $H^4(M, Z)\cong Z$, and $p_1(M)$ the first Pontrjagin class of M.*

(1) *Then M and $X(m, 1, 0)$ are of the same homotopy type if and only if* $m-4k\equiv 0 \pmod{48}$,

and

(2) *M and $Y(m, 1, 0)$ are of the same homotopy type if and only if* $m-4k\equiv 24 \pmod{48}$.

(3) *In particular, if M' is another m-twisted CP^3 of the type $(m, 1, 0)$, then M and M' are of the same homotopy type if and only if* $k\equiv k' \pmod{48}$, *where*

we put $p_1(M')=k'x_2'$ *as above.*

Proof. The assertions easily follow from (6.9) and Theorem 2 in [21] and the detail is left to the reader. Q.E.D.

Remark 6.18 (The case $(m, n, \varepsilon)=(0, 1, 0)$). Let X and Y be the 3-cell complexes,

$$X=X(0, 1, 0)=S^2\vee S^4\cup_{[\iota_2, \iota_4]}e^6=S^2\times S^4$$

and

$$Y=Y(0, 1, 0)=S^2\vee S^4\cup_{[\iota_2, \iota_4]+i_*(\eta_2^3)}e^6 .$$

Since X and Y have the same type (0, 1, 0), $H^*(X, Z)$ and $H^*(Y, Z)$ are isomorphic as cohomology rings and moreover $H^*(X, Z_p)$ and $H^*(Y, Z_p)$ are also isomorphic as A_p-modules for each prime p, where A_p denotes the mod p Steenrod algebra. Furthermore, it follows from (6.14) that X and Y have the homotopy types of 6-dimensional closed smooth spin manifolds. However, using (6.9), X and Y are not of the same homotopy type. This fact is essentially because η^3 can not be detected by the primary cohomology operations. (In fact, η^3 is detected by the secondary cohomology operation associated with the Adem relation $Sq^2Sq^2+Sq^1Sq^2Sq^1=0$.)

Acknowledgement. I would like to take this opportunity to thank Professor M. Masuda for his sincere advice and suggestions.

In particular, the results (6.16) and (6.17) were tought by him.

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