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ASYMPTOTIC BEHAVIOR OF CERTAIN SMALL SUBHARMONIC FUNCTIONS IN {Re $z>0$}

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1. Notation.

Let $\boldsymbol{C}$ be the complex plane. If $u(z)$ is subharmonic in a region $\Omega\subset\boldsymbol{C}$, we put

$$M(r,\ u)=\sup_{\substack{|z|=r\\ z\in\Omega}} u(z).$$

Let $\partial\Omega$ be the boundary of Ω. If $\zeta\in\partial\Omega$ and $u(z)$ is subharmonic in Ω, we define

$$u(\zeta)=\limsup_{\substack{z\to\zeta\\ z\in\Omega}} u(z).$$

2. Statement of Theorem.

In our previous paper [4], the following result is proved.

THEOREM A. *Let $u(z)$ be subharmonic in* $\{\mathrm{Re}\, z>0\}$. *If $u(z)$ satisfies the conditions*

(2.1) $$u(0)<\infty$$

and

(2.2) $$u(iy)\leqq M^{+}(|y|,\ u)-\pi^{2}\sigma \quad (-\infty<y<+\infty,\ y\neq 0;\ \sigma:\ a\ positive\ constant),$$

then either $u(z)\leqq-\pi^{2}\sigma$ in $\{\mathrm{Re}\, z>0\}$ *or*

(2.3) $$\lim_{r\to\infty}\frac{M(r,\ u)-4\sigma(\log r)^{2}}{\log r}=\alpha \qquad (-\infty<\alpha\leqq+\infty).$$

It seems to be interesting to investigate the asymptotic behavior of the subharmonic functions in $\{\mathrm{Re}\, z>0\}$ satisfying the conditions (2.1), (2.2) and (2.3) with a finite number α. In this note we prove

THEOREM. *Suppose that $u(z)$ is subharmonic in* $\{\mathrm{Re}\, z>0\}$ *and satisfies* (2.1), (2.2) *and* (2.3) (*where α is finite*) *with a suitable positive number σ. Suppose further that for any $r>0$ there exists z_r such that*

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(2.4) $$|z_r|=r,\quad u^+(z_r)=M^+(r,\ u),\quad |\arg z_r|\leqq\delta<\pi/2,$$

where δ is independent of r. Then

(2.5) $$\lim_{\substack{r\to\infty\\ re^{i\theta}\notin E}}\frac{u(re^{i\theta})-4\sigma(\log r)^2}{\log r}=\alpha,$$

uniformly for $\theta\in(-\pi/2,\pi/2)$, where the exceptional set E can be covered by disks $\{B_i\}$ such that if r_i is the radius of B_i and R_i is the distance from the center of B_i to the origin, then

(2.6) $$\sum_{i=1}^{\infty}(r_i/R_i)<\infty.$$

3. Introduction of several functions.

In what follows we assume that $M(e,u)\leqq 0$. But this may be achieved without loss of generality to our result by replacing u by $u-M(e,u)$, if necessary. As we have shown in [4, Lemma 2], the assumptions (2.1) and (2.2) imply that $M^+(r,u)$ is nondecreasing for $r>0$. From this and (2.3) with a finite number α we deduce that

(3.1) $$\Phi(x)=\frac{M^+(e^x,\ u)-4\sigma(x^+)^2}{x}\in L^\infty(-\infty,+\infty).$$

Now, set

(3.2) $$\Psi(x)=M^+(e^x,\ u)-4\sigma(x^+)^2=x\Phi(x),$$

(3.3) $$K_1(x)=\frac{1}{\pi}\frac{1}{\cosh x},\qquad K_2(x)=xK_1(x),$$

and

(3.4) $$N_j(x)=\begin{cases}\displaystyle\int_x^\infty K_j(y)dy & (x>0)\\ \displaystyle-\int_{-\infty}^x K_j(y)dy & (x<0)\end{cases}\qquad (j=1,\ 2).$$

Then by (3.3) and (3.4)

(3.5) $$N_j(x)\in L^1(-\infty,\ +\infty)\qquad (j=1,\ 2).$$

This together with (3.1) yields

(3.6) $$\begin{aligned}|\Phi*N_j(x)|&=\left|\int_{-\infty}^{+\infty}\Phi(x-y)N_j(y)dy\right|\\ &\leqq\sup_{-\infty<x<+\infty}|\Phi(x)|\cdot\int_{-\infty}^{+\infty}|N_j(y)|dy=C_j<\infty\qquad (j=1,\ 2).\end{aligned}$$

4. Two lemmas on convolution inequalities.

First, concerning $\Psi * K_1(x)$ we have the following estimate.

LEMMA 1. *$\Psi * K_1(x) > \Psi(x)$ for all large x.*

Proof. Set

$$v(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x}{x^2+(y-t)^2}\{M^+(|t|,\, u) - \pi^2\sigma\}\, dt \qquad (z = x+iy), \tag{4.1}$$

where we interpret $M^+(|t|, u)$ for $t=0$ as 0. From Lemma 2 in [4] and (2.3) we see that $M^+(|t|, u) - \pi^2\sigma$ is continuous for $-\infty < t < +\infty$ and that $\{M^+(|t|, u) - \pi^2\sigma\}/(1+t^2)$ is integrable for $-\infty < t < +\infty$. Thus $v(z)$ is the harmonic function in $\{\operatorname{Re} z > 0\}$ taking boundary values $v(iy) = M^+(|y|,\, u) - \pi^2\sigma$ $(-\infty < y < +\infty)$. From (4.1) $v(z)$ clearly satisfies

$$M(r,\, -v) \leqq \pi^2\sigma\,. \tag{4.2}$$

Let $W(z)$ be the harmonic function in $H = \{\operatorname{Re} z > 0\} \cap \{|z| < e\}$ whose boundary values are $W(e^{1+i\theta}) = 0$ $(-\pi < \theta < +\pi)$, $W(iy) = -\pi^2\sigma$ $(-e < y < +e)$. Then $u(z) - W(z)$ is subharmonic in H and satisfies $\sup_{z\in H}(u(z) - W(z)) < +\infty$, $(u-W)(\zeta) \leqq 0$ $(\zeta \in \partial H,\ \zeta \neq 0,\ \zeta \neq \pm ie)$. Hence the Phragmén-Lindelöf maximum principle gives $u(z) \leqq W(z)$ $(z \in H)$, so in particular,

$$u(0) = \limsup_{\substack{z\to 0\\ z\in H}} u(z) \leqq \limsup_{\substack{z\to 0\\ z\in H}} W(z) = -\pi^2\sigma = v(0)\,. \tag{4.3}$$

Now, we put $p(z) = v(z) - u(z)$. Clearly $-p(z)$ is subharmonic in $\{\operatorname{Re} z > 0\}$. From (2.2) and (4.3) $-p(iy) \leqq 0$ $(-\infty < y < +\infty)$. Also, from (2.3) and (4.2) $\liminf_{r\to\infty} M(r,\, -p)/r = 0$. Thus the Phragmén-Lindelöf theorem (cf. [3, p 111]) gives

$$u(z) \leqq v(z) \qquad (\operatorname{Re} z > 0)\,. \tag{4.4}$$

As in the proof of (1.9) in [4, p 150], we note from Lemma 2 in [4] that

$$M(r,\, v) = v(r) \qquad (r > 0)\,. \tag{4.5}$$

In view of (4.1), (4.4) and (4.5)

$$M(r,\, u) \leqq v(r) = \frac{2}{\pi}\int_0^{\infty} \frac{r}{r^2+t^2}\{M^+(t,\, u) - \pi^2\sigma\}\, dt\,. \tag{4.6}$$

Moreover a residue calculation yields

$$\int_0^{\infty} \frac{r}{r^2+t^2}(\log t)^2 dt = \frac{\pi}{2}(\log r)^2 + \frac{\pi^3}{8}\,, \tag{4.7}$$

and combining (4.6) and (4.7), we obtain

$$(4.8)\qquad M(r,\ u)-4\sigma(\log r)^2 \leqq v(r)-4\sigma(\log r)^2 < \frac{2}{\pi}\int_0^\infty \frac{r}{r^2+t^2}\{M^+(t,\ u)-4\sigma(\log^+ t)^2\}\,dt.$$

Hence by (2.3) for all large r

$$(4.9)\qquad M^+(r,\ u)-4\sigma(\log^+ r)^2 < \int_0^\infty \frac{2}{\pi}\frac{1}{(r/t)+(t/r)}\{M^+(t,\ u)-4\sigma(\log^+ t)^2\}\,dt/t.$$

By the change of variables $r=e^x$, $t=e^y$, we deduce from (3.2), (3.3), (4.8) and (4.9) that for all large x

$$(4.10)\qquad \Psi(x)=M^+(e^x,\ u)-4\sigma(x^+)^2 \leqq v(e^x)-4\sigma(x^+)^2 < K_1 * \Psi(x)=\Psi * K_1(x).$$

Our second lemma is

LEMMA 2. $\left|\int_x^y \{\Phi * K_1(t)-\Phi(t)\}\,dt\right| < 2C_1$, $\left|\int_x^y \Phi * K_2(t)\,dt\right| < 2C_2$, *where* C_1 *and* C_2 *are constants which appear in* (3.6).

Proof. We compute $\int_x^y \Phi * K_j(t)\,dt$ $(j=1,\ 2)$. Set $F(x)=\int_0^x \Phi(t)\,dt$. Then, using the Fubini's theorem, we have

$$(4.11)\qquad \begin{aligned}\int_x^y \Phi * K_j(t)\,dt &= \int_{-\infty}^{+\infty} K_j(u)\left(\int_x^y \Phi(t-u)\,dt\right)du \\ &= \int_{-\infty}^{+\infty} K_j(u)\{(F(y-u)-F(y))-(F(x-u)-F(x))\}\,du \\ &\quad + \int_x^y \Phi(t)\,dt \cdot \int_{-\infty}^{+\infty} K_j(u)\,du.\end{aligned}$$

From (3.3) we see at once that

$$(4.12)\qquad \int_{-\infty}^{+\infty} K_1(t)\,dt=1, \qquad \int_{-\infty}^{+\infty} K_2(t)\,dt=0.$$

The first term of the right hand side of (4.11)—which we denote by $I_j(x,\ y)$—requires further attention. Using the Fubini's theorem again, we deduce from (3.4) that

$$(4.13)\qquad \begin{aligned} I_j(x,\ y) &= \int_{-\infty}^0 K_j(u)\left(\int_u^0 \Phi(y-t)\,dt\right)du - \int_0^{+\infty} K_j(u)\left(\int_0^u \Phi(y-t)\,dt\right)du \\ &\quad - \int_{-\infty}^0 K_j(u)\left(\int_u^0 \Phi(x-t)\,dt\right)du + \int_0^{+\infty} K_j(u)\left(\int_0^u \Phi(x-t)\,dt\right)du \\ &= -\int_{-\infty}^0 \Phi(y-t)\left(-\int_{-\infty}^t K_j(u)\,du\right)dt - \int_0^{+\infty} \Phi(y-t)\left(\int_t^{+\infty} K_j(u)\,du\right)dt \\ &\quad + \int_{-\infty}^0 \Phi(x-t)\left(-\int_{-\infty}^t K_j(u)\,du\right)dt + \int_0^{+\infty} \Phi(x-t)\left(\int_t^{+\infty} K_j(u)\,du\right)dt \\ &= \Phi * N_j(x) - \Phi * N_j(y).\end{aligned}$$

Combining (4.11)-(4.13), we obtain

$$\int_x^y \{\Phi * K_1(t) - \Phi(t)\} dt = \Phi * N_1(x) - \Phi * N_1(y),$$

$$\int_x^y \Phi * K_2(t) dt = \Phi * N_2(x) - \Phi * N_2(y),$$

and so (3.6) yields

$$\left\{ \begin{array}{l} \left| \int_x^y \{\Phi * K_1(t) - \Phi(t)\} dt \right| < 2C_1, \\ \left| \int_x^y \Phi * K_2(t) dt \right| < 2C_2. \end{array} \right. \tag{4.14}$$

5. Preliminary study on the behavior of $p(z)$ at infinity.

The following lemma is the key to the proof of the Theorem.

LEMMA 3. *Let $p(z)$ be the function defined in the proof of Lemma* 1, *and let $\delta \in (0, \pi/2)$ be the number which appears in* (2.4). *Then, if* $w(r) = \inf_{|\theta| \leq \delta} p(re^{i\theta})$,

$$\int_{t_0}^{+\infty} \frac{w(t)}{t \log t} dt < \infty, \tag{5.1}$$

where t_0 is a positive constant such that $M(t_0, u) > 0$.

Proof. Set $h(x) = \Psi * K_1(x) - \Psi(x)$. From (2.4), (4.5) and (4.10) it follows that

$$w(t) \leqq p(z_t) = v(z_t) - u(z_t) \leqq v(t) - M^+(t, u) < \Psi * K_1(y) - \Psi(y) = h(y) \quad (e^y = t)$$

for $t \geqq t_0$ $(y \geqq y_0 = \log t_0 > 1)$. Thus, in order to show (5.1) it is enough to show that

$$\int_{y_0}^{+\infty} \frac{h(y)}{y} dy < \infty. \tag{5.2}$$

In view of (3.2) and (3.3)

$$\begin{aligned} h(y) &= \int_{-\infty}^{+\infty} (y-t)\Phi(y-t) K_1(t) dt - y\Phi(y) \\ &= y\{\Phi * K_1(y) - \Phi(y)\} - \Phi * K_2(y). \end{aligned}$$

Hence for $x > y_0$

$$\int_{y_0}^{x} \frac{h(y)}{y} dy = \int_{y_0}^{x} \{\Phi * K_1(y) - \Phi(y)\} dy - \int_{y_0}^{x} \frac{\Phi * K_2(y)}{y} dy. \tag{5.3}$$

By the mean value theorem there is a number $z \in [y_0, x]$ such that

$$\int_{y_0}^{x} \frac{\Phi * K_2(y)}{y} dy = \frac{1}{y_0} \int_{y_0}^{z} \Phi * K_2(y) dy. \tag{5.4}$$

Combining (4.14), (5.3) and (5.4), we obtain

$$\int_{y_0}^{x} \frac{h(y)}{y}\,dy < 2C_1 + 2C_2 \qquad (x > y_0).$$

This together with (4.10) implies (5.2). This completes the proof of Lemma 2.
Next, we give the following two estimates.

LEMMA 4.

$$\int_{\rho}^{+\infty} \frac{dr}{r^2 \log r} > \frac{1}{\rho \log \rho} - \frac{1}{\rho (\log \rho)^2} \qquad \text{for} \quad \rho > 1. \tag{5.5}$$

$$\int_{t_0}^{+\infty} \frac{dr}{(r^2+t^2)\log r} > \frac{\pi}{4t \log t} - \frac{1}{t \log t_0} \tan^{-1}\left(\frac{t_0}{t}\right) \qquad \text{for} \quad t \geqq t_0, \tag{5.6}$$

where t_0 ($>e$) is the constant which appears in Lemma 3.

Proof. A change of variable in the integral yields

$$\int_{\rho}^{+\infty} \frac{dr}{r^2 \log r} = \int_{\log \rho}^{+\infty} u^{-1} e^{-u}\,du.$$

Integrating by parts twice, we have

$$\int_{\rho}^{+\infty} \frac{dr}{r^2 \log r} = \frac{1}{\rho \log \rho} - \frac{1}{\rho (\log \rho)^2} + 2\int_{\log \rho}^{+\infty} u^{-3} e^{-u}\,du$$
$$> \frac{1}{\rho \log \rho} - \frac{1}{\rho (\log \rho)^2},$$

since $\rho > 1$. This shows (5.5).
Next, we integrate by parts to get

$$\int_{t_0}^{+\infty} \frac{dr}{(r^2+t^2)\log r} = -\frac{1}{t \log t_0} \tan^{-1}\left(\frac{t_0}{t}\right) + \frac{1}{t}\int_{t_0}^{+\infty} \frac{1}{r(\log r)^2} \tan^{-1}\left(\frac{r}{t}\right) dr. \tag{5.7}$$

For $r > t$, $\tan^{-1}(r/t) > \pi/4$, and since $t \geqq t_0$,

$$\int_{t_0}^{+\infty} \frac{1}{r(\log r)^2} \tan^{-1}\left(\frac{r}{t}\right) dr > \frac{\pi}{4}\int_{t}^{+\infty} \frac{dr}{r(\log r)^2} = \frac{\pi}{4 \log t}. \tag{5.8}$$

Combining (5.7) and (5.8), we deduce (5.6).

Now, from (4.4) the subharmonic function $u(z) - v(z)$ is nonpositive in $\{\operatorname{Re} z > 0\}$. Using two representation theorems, one of F. Riesz and one of Herglotz (cf. Heins [3, Theorem 4.2]), we obtain

$$p(z) = v(z) - u(z) = \int_{-\infty}^{+\infty} \frac{x}{x^2 + (y-t)^2}\,d\gamma(t) + \int_{\operatorname{Re}\zeta > 0} g(z, \zeta)\,d\mu(\zeta) \quad (z = x + iy), \tag{5.9}$$

where $\gamma(t)$ is an increasing function, μ is the Riesz measure of $-p(z)$ in $\{\operatorname{Re} z > 0\}$ and $g(z, \zeta)$ is the Green's function for $\{\operatorname{Re} z > 0\}$ with pole at ζ, namely

$$g(z,\ \zeta)=\log\left|\frac{z+\bar{\zeta}}{z-\zeta}\right|.$$

The following notation will be preserved throughout the rest of this note: $z=x+iy=re^{i\theta}$ $(x>0)$, $\zeta=\xi+i\eta=\rho e^{i\varphi}$ $(\xi\geqq 0)$. Then it is easy to check that

$$g(z,\ \zeta)=\frac{1}{2}\log\left\{1+\frac{4x\xi}{|z-\zeta|^2}\right\}\geqq\frac{2}{9}\frac{x\xi}{|z-\zeta|^2}\geqq\frac{2}{9}\frac{x\xi}{(\rho+r)^2}\geqq\frac{8}{81}\frac{x\xi}{r^2}\quad(\rho<r/2) \tag{5.10}$$

and

$$g(z,\ \zeta)\leqq\frac{2x\xi}{|z-\zeta|^2}\leqq\frac{2x\xi}{(\rho-r)^2}\leqq\begin{cases}\dfrac{8x\xi}{r^2} & (\rho<r/2)\\[2mm] \dfrac{8x\xi}{\rho^2} & (\rho>2r)\end{cases} \tag{5.11}$$

Here we claim the following

LEMMA 5.

$$\int_{\{\rho<t_0\}\cap\{\xi>0\}}\xi d\mu(\zeta)+\int_{\{\rho\geqq t_0\}\cap\{\xi>0\}}\frac{\xi}{\rho\log\rho}d\mu(\zeta)<\infty, \tag{5.12}$$

$$\int_{|t|<t_0}d\gamma(t)+\int_{|t|\geqq t_0}\frac{1}{|t|\log|t|}d\gamma(t)<\infty. \tag{5.13}$$

Proof. From (5.1) and (5.9) we have

$$\int_{t_0}^{+\infty}\frac{1}{r\log r}\left\{\inf_{|\theta|\leqq\delta}\int_{\{\rho<r/2\}\cap\{\xi>0\}}g(z,\ \zeta)d\mu(\zeta)\right\}dr<\infty. \tag{5.14}$$

and

$$\int_{t_0}^{+\infty}\frac{1}{r\log r}\left\{\inf_{|\theta|\leqq\delta}\int_{-\infty}^{+\infty}\frac{x}{x^2+(y-t)^2}d\gamma(t)\right\}dr<\infty. \tag{5.15}$$

After (5.5) and (5.10) are taken into account, (5.14) implies

$$\begin{aligned}\infty&>\int_{t_0}^{+\infty}\frac{1}{r\log r}\left\{\int_{\{\rho<r/2\}\cap\{\xi>0\}}\frac{\xi}{r}d\mu(\zeta)\right\}dr\\&=\int_{\{\rho<t_0/2\}\cap\{\xi>0\}}\xi\left\{\int_{t_0}^{+\infty}\frac{dr}{r^2\log r}\right\}d\mu(\zeta)+\int_{\{\rho\geqq t_0/2\}\cap\{\xi>0\}}\xi\left\{\int_{2\rho}^{+\infty}\frac{dr}{r^2\log r}\right\}d\mu(\zeta)\\&>\text{Const.}\left\{\int_{\{\rho<t_0/2\}\cap\{\xi>0\}}\xi d\mu(\zeta)+\int_{\{\rho\geqq t_0/2\}\cap\{\xi>0\}}\frac{\xi}{\rho\log\rho}d\mu(\zeta)\right\},\end{aligned}$$

which gives (5.12). For $|\theta|\leqq\delta$ $(<\pi/2)$, $x/(x^2+(y-t)^2)\geqq((\cos\delta)/2)(r/(r^2+t^2))$ holds, and as $t\to\infty$ $(\tan^{-1}(t_0/t))\log t\to 0$. Hence from (5.6) and (5.15) we deduce that

$$\infty>\int_{t_0}^{+\infty}\frac{1}{r\log r}\left\{\int_{-\infty}^{+\infty}\frac{r}{r^2+t^2}d\gamma(t)\right\}dr$$
$$=\int_{-\infty}^{+\infty}\left\{\int_{t_0}^{+\infty}\frac{dr}{(r^2+t^2)\log r}\right\}d\gamma(t)$$
$$>\text{Const.}\int_{|t|<t_0}d\gamma(t)+\text{Const.}\int_{|t|\geqq t_0}\frac{1}{|t|\log|t|}d\gamma(t),$$

which gives (5.13).

6. Study on the behavior of $p(z)$ at infinity.

Let

$$d\nu(\zeta)=\begin{cases}\dfrac{\xi}{1+\rho\log^+\rho}d\mu(\zeta) & (\xi>0),\\[2mm] \dfrac{1}{1+|\eta|(\log^+|\eta|)}d\gamma(\eta) & (\xi=0),\end{cases}$$

$$K(z,\zeta)=\begin{cases}(1+\rho\log^+\rho)g(z,\zeta)\xi^{-1} & (\xi>0),\\[2mm] \dfrac{x(1+|\eta|\log^+|\eta|)}{|z-i\eta|^2} & (\xi=0),\end{cases}$$

and set $\bar{D}=\{\operatorname{Re} z\geqq 0\}=\bar{D}_1(z)\cup\bar{D}_2(z)\cup\bar{D}_3(z)$ for $|z|=r>e$, where

$$\bar{D}_1(z)=\bar{D}\cap\{\zeta;\rho\leqq\log r\},$$
$$\bar{D}_2(z)=\bar{D}\cap\{\zeta;\log r<\rho<2r\},$$
$$\bar{D}_3(z)=\bar{D}\cap\{\zeta;\rho\geqq 2r\}.$$

Further, define for $|z|>e$

$$p_j(z)=\int_{\bar{D}_j(z)}K(z,\zeta)d\nu(\zeta) \qquad (j=1,2,3).$$

Then it is easily verified that

$$p(z)=p_1(z)+p_2(z)+p_3(z) \qquad (|z|>e), \tag{6.1}$$

and we deduce from Lemma 5 that

$$\int_{\bar{D}}d\nu(\zeta)<\infty. \tag{6.2}$$

We first show

LEMMA 6.

$$\lim_{r\to\infty}p_1(z)/(\log r)=0 \tag{6.3}$$

uniformly for $\theta\in(-\pi/2, \pi/2)$.

Proof. Assume first that $r=|z|>e$, $\zeta\in\bar{D}_1(z)$ and $\xi>0$. Since $\rho\leqq\log r<r/2$ in this case, we have from (5.11) $g(z, \zeta)\leqq 8x\xi/r^2$, and so $K(z, \zeta)\leqq 8(1+\log r\log\log r)x/r^2\leqq 8(1+\log r\log\log r)/r$. Assume next that $r=|z|>e$, $\zeta\in\bar{D}_1(z)$ and $\xi=0$. Since $|\eta|\leqq\log r$, $K(z, \zeta)<x(1+\log r\log\log r)/(r-\log r)^2<4(1+\log r\log\log r)/r$. Hence

$$(0\leqq)p_1(z)\leqq\frac{8}{r}(1+\log r\log\log r)\int_{\bar{D}_1(z)}d\nu(\zeta),$$

and thus from (6.2) we deduce that

$$0\leqq\frac{p_1(z)}{\log r}\leqq\text{Const.}\frac{\log\log r}{r}\longrightarrow 0 \qquad (\text{as} \quad r\to\infty).$$

Similarly we have for $p_3(z)$

LEMMA 7.

$$\lim_{r\to\infty}p_3(z)/(\log r)=0 \tag{6.4}$$

uniformly for $\theta\in(-\pi/2, \pi/2)$.

Proof. Assume first that $r>e$, $\zeta\in\bar{D}_3(z)$ and $\xi>0$. From (5.11) it follows that $g(z, \zeta)\leqq 8x\xi/\rho^2$, and so $K(z, \zeta)\leqq 8x(1+\rho\log\rho)/\rho^2<12x(\log\rho)/\rho<12x(\log r)/r\leqq 12\log r$. Assume next that $r>e$, $\zeta\in\bar{D}_3(z)$ and $\xi=0$. Since $|\eta|\geqq 2r$, $K(z, \zeta)\leqq 4x(1+|\eta|\log|\eta|)/|\eta|^2<6\log r$. Hence we deduce from (6.2) that

$$0\leqq\frac{p_3(z)}{\log r}\leqq 12\int_{\bar{D}_3(z)}d\nu(\zeta)\longrightarrow 0 \qquad (\text{as} \quad r\to\infty).$$

It remains to consider $p_2(z)$. Following Hayman [2], if $\varepsilon>0$ and $z\in\{\operatorname{Re}z>0\}$ are given, we say that the z is ε-normal (with respect to ν) provided that

$$\int_{\bar{D}\cap\{\zeta,\,|\zeta-z|<h\}}d\nu(\zeta)<\varepsilon h/r \tag{6.5}$$

for $0<h\leqq r/2$.

LEMMA 8. *If* $z\in\{\operatorname{Re}z>0, r=|z|>e\}$ *is* ε-*normal* (*with respect to* ν), *then*

$$p_2(z)<\text{Const.}\left\{\varepsilon+\int_{\bar{D}_2(z)}d\nu(\zeta)\right\}\log r. \tag{6.6}$$

Proof. Let

$$\Omega_n=\{\zeta\in\bar{D}_2(z)\,;\,2^{n-1}x\leqq|z-\zeta|<2^n x\} \qquad (n=0, \pm1, \pm2, \cdots). \tag{6.7}$$

Since z is ε-normal, $\nu(z)=0$, and thus

$$p_2(z)=\sum_{n=-\infty}^{+\infty}\int_{\Omega_n}K(z, \zeta)d\nu(\zeta)=\sum_{n=-\infty}^{+\infty}q_n(z). \tag{6.8}$$

Suppose first that $n\leqq-1$ and $\zeta\in\Omega_n$. From (6.7) we have $|x-\xi|\leqq|z-\zeta|<x/2$ and $|z+\bar{\zeta}|\leqq|z-\zeta|+|\zeta+\bar{\zeta}|<x/2+2\xi<7x/2$ in turn. Using these estimates, we have $g(z, \zeta)<\log(7/2^n)$ and $\xi>x/2$. Hence $K(z, \zeta)<(1+\rho\log\rho)(2/x)\log(7/2^n)<(6r/x)(\log 2r)\log(7/2^n)$. Also by (6.5), $\int_{\Omega_n}d\nu(\zeta)<\varepsilon 2^n x/r$. Thus

$$q_n(z)<6\varepsilon(\log 2r)2^n\log(7/2^n)\qquad(n\leqq-1). \tag{6.9}$$

Suppose next that $n\geqq 0$ and $\zeta\in\Omega_n$. If $\xi>0$, using (5.11) and (6.7), we have $K(z,\zeta)\leqq(1+\rho\log\rho)2x/|z-\zeta|^2\leqq 6r(\log 2r)/(2^{2n-2}x)$. If $\xi=0$, $K(z,\zeta)\leqq x(1+2r\log 2r)/(2^{2n-2}x^2)\leqq 3r(\log 2r)/(2^{2n-2}x)$. Then, if $2^n x\leqq r/2$. we deduce from (6.5) that

$$q_n(z)\leqq 6r(\log 2r)/(2^{2n-2}x)\cdot(\varepsilon 2^n x/r)=24\varepsilon(\log 2r)/2^n. \tag{6.10}$$

On the other hand, if $2^n x>r/2$, then

$$q_n(z)\leqq 6r\log 2r\frac{1}{2^{2n-2}x}\int_{\Omega_n}d\nu(\zeta)<\frac{48}{2^n}(\log 2r)\int_{\bar{D}_2(z)}d\nu(\zeta). \tag{6.11}$$

Combining (6.8)-(6.11), we obtain (6.6).

Now, from (6.2) and a result of Azarin [1] it follows that the set $\Delta(\varepsilon)$ of points not ε-normal (with respect to ν) may be covered by a system $F(\varepsilon)$ of disks $\{B_k\}$ whose radii $\{r_k\}$ and distances $\{R_k\}$ from their centers to the origin satisfy $\sum_{k=1}^{\infty}(r_k/R_k)<\infty$. Choose an increasing unbounded sequence $\{t_n\}$ of positive numbers such that $\int_{\bar{D}_2(z)}d\nu(\zeta)<1/n$ for $|z|=r>t_n$ and $\sum_{R_k>t_n}(r_k/R_k)<1/2^n$ for a system $F(1/n)$ of disks $\{B_k\}$. If $F(1/n, t_n)$ is the set of disks whose radii appear in this sum, we put $F_0=\bigcup_{n=1}^{\infty}F(1/n, t_n)$. Clearly the system F_0 of disks satisfies (2.6). From (6.6) we deduce that

$$p_2(z)\leqq\text{Const.}(\log r/n)\qquad(|z|>t_n,\ z\in\{\operatorname{Re}z>0\}\setminus F_0). \tag{6.12}$$

Thus (4.4), (5.9), (6.1), (6.3), (6.4) and (6.12) yield

$$\lim_{\substack{r\to\infty\\ z\notin E}}p(z)/(\log r)=0 \tag{6.13}$$

uniformly for $\theta\in(-\pi/2, \pi/2)$, where the exceptional set E can be covered by F_0

7. Completion of the proof of the Theorem.

Given $\varepsilon>0$, define $U_\varepsilon(z)=v(z)-4\sigma\{(\log r)^2-\theta^2\}-(\alpha+\varepsilon)\log r-K_1$, where K_1 is a large positive constant. Clearly $U_\varepsilon(z)$ is harmonic in $\{\operatorname{Re}z>0\}$. In view of (2.3) and (4.1) $U_\varepsilon(iy)\leqq 0$ $(-\infty<y<+\infty)$. Also $\liminf_{r\to\infty}M(r, U_\varepsilon)/r=0$. Hence from the Phragmén-Lindelöf theorem, we have

$$U_\varepsilon(z)\leqq 0\qquad(\operatorname{Re}z>0). \tag{7.1}$$

Similarly, if $\varepsilon>0$ is given and if we define $V_{\varepsilon}(z)=-v(z)+\max\{4\sigma(\log r)^2-4\sigma\theta^2, 0\}+(\alpha-\varepsilon)\log^+ r-K_2$ with a large positive number K_2, then

$$V_{\varepsilon}(z)\leqq 0 \qquad (\mathrm{Re}\, z>0). \tag{7.2}$$

Combining (7.1) and (7.2), we have

$$\lim_{r\to\infty}\frac{v(re^{i\theta})-4(\log r)^2}{\log r}=\alpha \tag{7.3}$$

uniformly for $\theta\in(-\pi/2, \pi/2)$. Thus (2.5) follows from (5.9), (6.13) and (7.3). This completes the proof of the Theorem.

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