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EXPLOSION AND ASYMPTOTIC BEHAVIOR OF NONLINEAR ITÔ TYPE STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS

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Abstract

The present paper deals with the study of explosion and growth order of solutions of a general class of Itô-type stochastic integrodifferential equations which contain as a special case the study of Itô type stochastic differential equations. Sufficient conditions for infinite explosion time and asymptotic behavior of solutions are investigated.

Key Wcrds and Phrases: Explosion, Asymptotic behavior, Itô type stochastic integrodifferential equations, infinite explosion time, concave functions, stochastic integral isometry and Jenson's inequality.

1. Introduction

In this paper we consider a general class of Itô type stochastic integrodifferential equations of the form

$$(1.1)\qquad dx(t)=F\Big(t,\ x(t),\ \int_{t_0}^{t} f_1(t,\ s,\ x(s))ds,\ \int_{t_0}^{t} f_2(t,\ s,\ x(s))d\xi(s)\Big)dt$$
$$+H\Big(t,\ x(t),\ \int_{t_0}^{t} h_1(t,\ s,\ x(s))ds,\ \int_{t_0}^{t} h_2(t,\ s,\ x(s))d\xi(s)\Big)d\xi(t)$$

where $\xi(t)$ is a Brownian motion process on a probability space $(\Omega,\ \zeta,\ P)$ and $f_i(t,\ s,\ x)$, $h_i(t,\ s,\ x)$, $i=1,\ 2$ are Borel measurable functions defined on $R^{+2}XR$ into R and F, H are Borel measurable functions defined on $R^{+}XR^{3}$ into R where $R^{+}=[0,\ \infty)$ and $R=(-\infty,\ \infty)$. In a recent paper [10] the present authors have studied the problems of existence and uniqueness of solutions of a more general class of Itô type stochastic Volterra integral equations having continuous sample paths with probability one, which contains as a special case the study of equation (1.1). The equation (1.1) is a further generalization of the stochastic integrodifferential equations recently studied by Berger and Mizel [2] and Pachpatte [13] and it contains as a special case the well known Itô type stochastic differential equation studied by many authors in the literature (see [1, 3-9, 14]).

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In many fields of science and engineering there are large number of problems which are intrinsically nonlinear and complex in nature involving deterministic and stochastic excitations. For instance we refer to a second order stochastic differential equation

$$\ddot{y}+\alpha(t)\beta(t)f(y)=g(t, y, \dot{y})\dot{\xi} \tag{1.2}$$

where $\xi(t)$ is a Brownian motion process which is the outcome of the effect of "white noise" random forces on the system

$$\ddot{y}+\alpha(t)\beta(t)f(y)=0. \tag{1.3}$$

Equation (1.3) is extensively referred in the literature and represents a characteristic of many systems of control engineering. The system (1.2) can be represented as a pair of the following equations

$$\begin{aligned} &dy(t)=u(t)dt \\ &du(t)=-\alpha(t)\beta(t)f(y(t))dt+g(t, y(t), u(t))d\xi(t). \end{aligned} \tag{1.4}$$

The system (1.4) with initial conditions $y(0)=u(0)=c$ is equivalent to the stochastic integrodifferential equation

$$\begin{aligned} du(t)=&-\alpha(t)\beta(t)f\left(c+\int_0^t u(s)ds\right)dt \\ &+g\left(t, c+\int_0^t u(s)ds, u(t)\right)d\xi(t). \end{aligned} \tag{1.5}$$

The systems of this type commonly come across in almost all phases of physics, control theory and other areas of applied mathematics.

Recently, Narita [11, 12] has studied the explosion phenomenon and asymptotic behavior of solutions of special form of equation (1.1) when $f_i=h_i=0$, $i=1, 2$, where F and H are vector functions. In view of the general form of equation (1.5) occurring in physical applications, the study of asymptotic behavior and explosion theory for the general class of stochastic integrodifferential equations of the type (1.1) is more interesting and challenging. In fact our results in the present paper are motivated by the recent work of Narita [11] and the general form of Itô type stochastic equations recently studied by Berger and Mizel [2] and Murge and Pachpatte [10].

The purpose of the present study is to investigate sufficient conditions for infinite explosion time and asymptotic behavior of the solutions of stochastic integrodifferential equation (1.1). The method used in our analysis is an extension of the method recently used by Narita [11] for Itô type stochastic differential equations. In Section 2, we shall deal with the preliminary lemmas needed in our subsequent discussion and prove a theorem on infinite explosion time. In Section 3, we first obtain a moment estimate for random solution process of equation (1.1) and using this estimate we establish a theorem on the order of growth of solutions of equation (1.1). Finally in Section 4 we give two examples to illustrate the hypotheses used in our main results. Throughout we shall write stochastic processes and functions by suppressing the argument ω, $\omega\in\Omega$, without further mention.

2. Infinite explosion time.

In this section we shall establish our main theorem concerning infinite explosion time. For convenience we first list the following assumption.

(A) The functions $f_i(t, s, x)$, $h_i(t, s, x)$, $i=1, 2$, $F(t, x, y, u)$ and $H(t, x, y, u)$ are continuous and for any $T>0$, $R>0$, $t\leqq T$, $|x_j|\leqq R$, $|y_j|\leqq R$, $|u_j|\leqq R$, $j=1,2$ there exists a constant $C_{TR}>0$ such that

$$|f_i(t, s, x_1)-f_i(t, s, x_2)|^2+|h_i(t, s, x_1)-h_i(t, s, x_2)|^2$$
$$\leqq C_{TR}^2(|x_1-x_2|^2),$$
$$|F(t, x_1, y_1, u_1)-F(t, x_2, y_2, u_2)|^2$$
$$+|H(t, x_1, y_1, u_1)-H(t, x_2, y_2, u_2)|^2$$
$$\leqq C_{TR}^2(|x_1-x_2|^2+|y_1-y_2|^2+|u_1-u_2|^2).$$

For any natural number n, let $g_n(x)$ denote the function defined on R such that $g_n(x)=1$, for $|x|\leqq n$, $g_n(x)=2-\dfrac{|x|}{n}$, for $n<|x|\leqq 2n$ and $g_n(x)=0$ for $2n<|x|$. Define

$$f_i^{(n)}(t, s, x)=g_n(x)f_i(t, s, x),$$
$$h_i^{(n)}(t, s, x)=g_n(x)h_i(t, s, x), \qquad i=1, 2,$$
$$F^{(n)}(t, x, y, u)=g_n(x)F(t, x, y, u)$$

and

$$H^{(n)}(t, x, y, u)=g_n(x)H(t, x, y, u).$$

It is easy to observe that the following conditions hold:

$$|f_i^{(n)}(t, s, x_1)-f_i^{(n)}(t, s, x_2)|^2+|h_i^{(n)}(t, s, x_1)-h_i^{(n)}(t, s, x_2)|^2 \tag{2.1}$$
$$\leqq M_n^2(|x_1-x_2|^2), \qquad i=1, 2,$$

$$|F^{(n)}(t, x_1, y_1, u_1)-F^{(n)}(t, x_2, y_2, u_2)|^2 \tag{2.2}$$
$$+|H^{(n)}(t, x_1, y_1, u_1)-H^{(n)}(t, x_2, y_2, u_2)|^2$$
$$\leqq M_n^2(|x_1-x_2|^2+|y_1-y_2|^2+|u_1-u_2|^2),$$

$$|f_i^{(n)}(t, s, x)|^2+|h_i^{(n)}(t, s, x)|^2\leqq M_n^2(1+|x|^2), \qquad i=1, 2, \tag{2.3}$$

$$(2.4)\qquad |F^{(n)}(t, x, y, u)|^2+|H^{(n)}(t, x, y, u)|^2 \leqq M_n^2(1+|x|^2+|y|^2+|u|^2),$$

for $t \leqq n$, x_j, y_j, $u_j \in R$, $j=1$, 2 and x, y, $u \in R$ where $M_n>0$ is a constant depending only on n. If follows from Theorem 3 in Murge and Pachpatte [10] that there exists pathwise unique solution $x^{(n)}(t)$ defined upto $t \leqq n$ of the stochastic integrodifferential equation

$$(2.5)\qquad dx^{(n)}(t)=F^{(n)}\Big(t, x^{(n)}(t), \int_{t_0}^t f_1^{(n)}(t, s, x^{(n)}(s))ds, \int_{t_0}^t f_2^{(n)}(t, s, x^{(n)}(s))d\xi(s)\Big)dt$$

$$+H^{(n)}\Big(t, x^{(n)}(t), \int_{t_0}^t h_1^{(n)}(t, s, x^{(n)}(s))ds, \int_{t_0}^t h_2^{(n)}(t, s, x^{(n)}(s))d\xi(s)\Big)d\xi(t).$$

We shall denote the solution of (2.5) with the initial condition $x^{(n)}(t_0)=x_0$, $x_0 \in R$ $(t_0 \geqq 0)$ by $x^{(n)}(t, t_0, x_0)$. Let us define $\tau_n(t_0, x_0)$ and $e_n(t_0, x_0)$ by

$$\tau_n(t_0, x_0)=\inf\{t, \ |x^{(n)}(t, t_0, x_0)| \geqq n\}$$

(and $\tau_n(t_0, x_0)=\infty$ if there is no such time) and $e_n(t_0, x_0)=\min\{n, \tau_n(t_0, x_0)\}$ respectively. Thus $\{e_n(t_0, x_0), n \geqq 1\}$ is a monotonic increasing sequence of stopping times for which

$$\sup_{t_0 \leqq t \leqq e_n(t_0, x_0)} |x^{(n)}(t, t_0, x_0)-x^{(m)}(t, t_0, x_0)|=0$$

holds with probability one, if $m>n$. For $n \geqq 1$, $t<e_n(t_0, x_0)$, we define a random process $x(t, t_0, x_0)$ by $x(t, t_0, x_0)=x^{(n)}(t, t_0, x_0)$ which is called the solution of equation (1.1) with initial condition $x(t_0)=x_0$. Let us define a random time $e(t_0, x_0)$ by $e(t_0, x_0)=\lim_{n\to\infty} e_n(t_0, x_0)$ which is called the explosion time of $x(t, t_0, x_0)$.

We need the following lemmas in our subsequent discussion. Lemma 1 is the modified version of the corollary on Theorem 1 given in Murge and Pachpatte [10] and Lemma 2 is a slight variant of Lemma 1.

LEMMA 1. *Let the conditions* (2.1)-(2.4) *hold and* $E[|x_0|^2]<\infty$. *Then, for* $t \in [t_0, n]$,

$$E[|x^{(n)}(t)|^2] \leqq (3E[|x_0|^2]+1)\exp[C(M_n, n, t_0)t]-1$$

holds, where $C(M_n, n, t_0)$ *is a constant depending on* n *and* t_0.

LEMMA 2. *Suppose the conditions* (2.1)-(2.4) *hold and* $E[|x_0|^2]<\infty$. *Then, for* $t \in [t_0, n]$,

$$(2.6)\qquad E(\sup_{t_0 \leqq v \leqq t} |x^{(n)}(v)|^2) \leqq C_1(M_n, n, t_0)(3E[|x_0|^2]+1)$$

holds, where $C_1(M_n, n, t_0)$ *is a constant depending on* n *and* t_0.

Proof. From (2.5) we get,

$$(2.7)\qquad x^{(n)}(t)=x_0+\int_{t_0}^t F^{(n)}\Big(s, x^{(n)}(s), \int_{t_0}^s f_1^{(n)}(s, \tau, x^{(n)}(\tau))d\tau,$$

$$\int_{t_0}^{s} f_2^{(n)}(s, \tau, x^{(n)}(\tau))d\xi(\tau)\Big)ds$$

$$+\int_{t_0}^{t} H^{(n)}\Big(s, x^{(n)}(s), \int_{t_0}^{s} h_1^{(n)}(s, \tau, x^{(n)}(\tau))d\tau,$$

$$\int_{t_0}^{s} h_2^{(n)}(s, \tau, x^{(n)}(\tau))d\xi(\tau)\Big)d\xi(s).$$

By using $(a+b+c)^2 \leqq 3(a^2+b^2+c^2)$, Schwarz inequality, Theorem 3.6 in [4, p. 70], conditions (2.1)-(2.4) and stochastic integral isometry, for any $t \leqq n$, we have,

$$E(\sup_{t_0 \leqq v \leqq t} |x^{(n)}(v)|^2)$$

$$\leqq 3E|x_0|^2 + \Big[3M_n^2(t+4)\Big\{1+M_n^2\frac{(t-t_0)^2}{2}+M_n^2(t-t_0)\Big\}\Big]X$$

$$\int_{t_0}^{t}(1+E|x^{(n)}(s)|^2)ds.$$

Now, the application of Lemma 1 yields

$$E(\sup_{t_0 \leqq v \leqq t} |x^{(n)}(v)|^2) \tag{2.8}$$

$$\leqq \Big[1+3M_n^2(n+4)\Big\{1+M_n^2\frac{(n-t_0)(n-t_0+2)}{2}\Big\}X$$

$$\Big\{\frac{e^{nC(M_n, n, t_0)}-e^{t_0C(M_n, n, t_0)}}{C(M_n, n, t_0)}\Big\}\Big](3E|x_0|^2+1)$$

$$=C_1(M_n, n, t_0)(3E|x_0|^2+1),$$

and the proof is complete.

We are now in a position to establish the following theorem which yields sufficient conditions for infinite explosion time.

THEOREM 1. *Let* $f_i(t, s, x)$, $h_i(t, s, x)$, $i=1, 2$, $F(t, x, y, u)$ *and* $H(t, x, y, u)$ *satisfy the assumption* (A) *and let*

$$|f_i(t, s, x)|^2+|h_i(t, s, x)|^2 \leqq \alpha_i(s)\beta_i(|x|^2), \quad i=1, 2, \ s \leqq t; \tag{2.9}$$

$$|F(t, x, y, u)|^2+|H(t, x, y, u)|^2 \tag{2.10}$$

$$\leqq(\alpha_3(t)\beta_3(|x|^2)+|y|^2+|u|^2)$$

for all $t \in R^+$ *and* $x, y, u \in R$ *where* $\alpha_i\colon R^+ \to R^+$, $i=1, 2, 3$ *are continuous and* $\beta_i\colon R^+ \to R^+$, $i=1, 2, 3$ *are monotonic increasing concave functions such that*

$$\int_0^{\infty}\frac{dv}{1+\sum_{i=1}^{3}\beta_i(v)}=\infty. \tag{2.11}$$

Then $P(e(t_0, x_0)=\infty)=1$ *for all* $t\in R^+$ *and* $x_0\in R$.

Proof. Let us consider the solution $x^{(n)}(t, t_0, x_0)$ of the equation (2.5) with initial condition $x^{(n)}(t_0)=x_0$, $x_0\in R$, for $n>\max\{|x_0|, t_0\}$. Suppose the time $\tau_n(t_0, x_0)$ denotes the first exist time for the solution $x^{(n)}(t, t_0, x_0)$ on the set $\{x, |x|<n\}$. Define $e_n(t_0, x_0)=\min\{n, \tau_n(t_0, x_0)\}$. For convenience we write $x^{(n)}(t, t_0, x_0)$, $\tau_n(t_0, x_0)$ and $e_n(t_0, x_0)$ as $x^{(n)}(t)$, τ_n and e_n respectively, by suppressing t_0 and x_0. Define

$$Z^{(n)}(t)=E(\sup_{t_0\leqq v\leqq t}|x^{(n)}(v)|^2)$$

for $t\in[t_0, n]$. By Lemma 2, $Z^{(n)}(t)$ is bounded.

By the definitions of $f_i^{(n)}$, $h_i^{(n)}$, $i=1, 2$, $F^{(n)}$ and $H^{(n)}$ and conditions (2.9), (2.10) we observe, for all $t\in R^+$ and $x\in R$, that

$$\text{(2.12)} \qquad |f_i^{(n)}(t, s, x)|^2+|h_i^{(n)}(t, s, x)|^2\leqq|f_i(t, s, x)|^2+|h_i(t, s, x)|^2$$
$$\leqq\alpha_i(s)\beta_i(|x|^2), \quad i=1, 2;$$

$$\text{(2.13)} \qquad |F^{(n)}(t, x, y, u)|^2+|H^{(n)}(t, x, y, u)|^2$$
$$\leqq|F(t, x, y, u)|^2+|H(t, x, y, u)|^2$$
$$\leqq(\alpha_3(t)\beta_3(|x|^2)+|y|^2+|u|^2).$$

From (2.7), by using $(a+b+c)^2\leqq3(a^2+b^2+c^2)$, Schwarz inequality, Theorem 3.6 in [4, p. 70], (2.12), (2.13) and stochastic integral isometry, we get,

$$E(\sup_{t_0\leqq v\leqq t}|x^{(n)}(v)|^2)$$
$$\leqq3|x_0|^2+3(t+4)\Big[\int_{t_0}^t(s-t_0)\int_{t_0}^s\alpha_1(\tau)E\beta_1(|x^{(n)}(\tau)|^2)d\tau ds$$
$$+\int_{t_0}^t\int_{t_0}^s\alpha_2(\tau)E\beta_2(|x^{(n)}(\tau)|^2)d\tau ds$$
$$+\int_{t_0}^t\alpha_3(s)E\beta_3(|x^{(n)}(s)|^2)ds\Big].$$

Since β_i, $i=1, 2, 3$ are monotone increasing, we observe that

$$\beta_i(|x^{(n)}(v)|^2)\leqq\beta_i(\sup_{t_0\leqq\theta\leqq v}|x^{(n)}(\theta)|^2)$$

holds for $v\leqq n$. Thus, we have,

$$Z^{(n)}(t)\leqq3|x_0|^2$$
$$+3(t+4)\alpha(t)\Big[\int_{t_0}^t(s-t_0)\int_{t_0}^sE\beta_1(\sup_{t_0\leqq\tau\leqq s}|x^{(n)}(\tau)|^2)d\tau\, ds$$

$$+\int_{t_0}^{t}\int_{t_0}^{s} E\beta_2(\sup_{t_0\leq\tau\leq s}|x^{(n)}(\tau)|^2)d\tau\, ds$$

$$+\int_{t_0}^{t} E\beta_3(\sup_{t_0\leq s\leq t}|x^{(n)}(s)|^2)ds\Big]$$

for all $t\in[t_0, n]$, $s\leqq t$, where

$$\alpha(t)=\max\{\max_{t_0\leq v\leq t}\alpha_1(v),\ \max_{t_0\leq v\leq t}\alpha_2(v),\ \max_{t_0\leq v\leq t}\alpha_3(v)\}.$$

Noting the fact that β_i, $i=1, 2, 3$ are concave, by using Jenson's inequality, we get,

$$Z^{(n)}(t)\leqq 3|x_0|^2+3(t'+4)\alpha(t')C(t')\int_{t_0}^{t}\Big[\sum_{i=1}^{3}\beta_i(Z^{(n)}(s))\Big]ds$$

for all $t\in[t_0, t'](t'<n)$ where $C(t)=\max\left\{\frac{(t-t_0)^2}{2}, (t-t_0), 1\right\}$. It is obvious that

$$\sum_{i=1}^{3}\beta_i(Z^{(n)}(t))\leqq\sum_{i=1}^{3}\beta_i(Z^{(n)}(t))+1.$$

Define

$$m(t)=3|x_0|^2+3(t'+4)\alpha(t')C(t')\int_{t_0}^{t}\Big[\sum_{i=1}^{3}\beta_i(Z^{(n)}(s))\Big]ds.$$

We have,

$$m(t_0)=3|x_0|^2,\qquad Z^{(n)}(t)\leqq m(t)$$

and

$$dm(t)\leqq 3(t'+4)\alpha(t')C(t')\Big[1+\sum_{i=1}^{3}\beta_i(m(t))\Big]dt.$$

Thus

$$\frac{dm(t)}{1+\sum_{i=1}^{3}\beta_i(m(t))}\leqq 3(t'+4)\alpha(t')C(t')dt.$$

On integrating from t_0 to t,

$$\int_{t_0}^{t}\frac{dm(s)}{1+\sum_{i=1}^{3}\beta_i(m(s))}\leqq 3(t'+4)\alpha(t')C(t')(t-t_0).$$

Substituting $v=m(s)$ and using the inequality $Z^{(n)}(t)\leqq m(t)$, we get

$$\int_{3|x_0|^2}^{Z^{(n)}(t)}\frac{dv}{1+\sum_{i=1}^{3}\beta_i(v)}\leqq C(t', t_0) \tag{2.14}$$

for all $t\in[t_0, t'](t'<n)$, where $C(t, t_0)=3(t+4)\alpha(t)C(t)(t-t_0)$. Suppose there exist some t_0 and x_0 such that $P(e(t_0, x_0)<T)\equiv\delta$, $\delta>0$ for some $T<\infty$. Let T' denote an arbitrary time such that $T'>T$ and be fixed. In the following we consider $x^{(n)}(t, t_0, x_0)$ for n, so large that $n>\max\{|x_0|, T'\}$. For any time t such that $T<t<T'$, we observe that

$$\{\tau_n<t\}=\{e_n<t\}\supseteqq\{e_n<T\}\supseteqq\{e(t_0, x_0)<T\}$$

and

$$\begin{aligned} Z^{(n)}(t) &\geqq E[\sup_{t_0\leqq v\leqq t}|x^{(n)}(v)|^2, \sup_{t_0\leqq v\leqq t}|x^{(n)}(v)|^2>n-1] \\ &\geqq(n-1)^2P(\sup_{t_0\leqq v\leqq t}|x^{(n)}(v)|>n-1) \\ &\geqq(n-1)P(\sup_{t_0\leqq v\leqq t}|x^{(n)}(v)|>n) \\ &\geqq(n-1)P(\tau_n<t)=(n-1)P(e_n<t) \\ &\geqq(n-1)\delta \end{aligned}$$

for all $t\in[T, T']$. Thus from (2.14) we get,

$$\int_{3|x_0|^2}^{(n-1)\delta}\frac{dv}{1+\sum_{i=1}^{3}\beta_i(v)}\leqq C(T', t_0). \tag{2.15}$$

It can be observed that as n tends to infinity the right side of (2.15) becomes finite while left side tends to infinity under (2.11). Therefore, it leads to a contradiction. Hence for any $t\geqq 0$, $x_0\in R$ and T we have $P(e(t_0, x_0)\geqq T)=1$. This completes the proof of the theorem.

3. Asymptotic behavior.

First we shall establish a lemma on a moment estimate for $x(t, t_0, x_0)$.

LEMMA 3. *Let $H(t, x, y, u)$ be such that*

$$H(t, x, y, u)=(c(t)a(x)+y+u) \tag{3.1}$$

where $c(t)$ is a nonzero continuous function defined for $t\in R^+$ and $a(x)$ is a real valued continuous function defined on R such that

$$|a(x)|\leqq K, \quad K>0 \text{ is a constant for all } x\in R, \tag{3.2}$$

$$B(t)=\left[\int_{t_0}^{t}|c(s)|^2ds\right]^{1/2} \tag{3.3}$$

and

$$\min_{t\to\infty}B(t)=\infty. \tag{3.4}$$

Suppose $f_i(t, s, x)$, $h_i(t, s, x)$, $i=1, 2$, $F(t, x, y, u)$ and $H(t, x, y, u)=(c(t)a(x)+y+u)$ satisfy the assumption (A). *Further suppose that $f_i(t, s, x)$, $h_i(t, s, x)$, $i=$*

1, 2 satisfy the condition (2.9) and $F(t, x, y, u)$ satisfies

$$|F(t, x, y, u)|^2 \leqq (\alpha_3(t)\beta_3(|x|^2)+|y|^2+|u|^2) \tag{3.5}$$

for all $t \in R^+$ and $x, y, u \in R$ where α_i, $i=1, 2, 3$ are nonnegative and continuous functions such that $A_i(t)=\int_{t_0}^t \alpha_i(s)ds<\infty$ and

$$\lim_{t\to\infty} \frac{D_i(2t)}{B(t)} \quad \text{exist}, \tag{3.6}$$

where $D_i(t)=(t-t_0+3)^3 A_i(t)$, and $\beta_i(v)$, $i=1, 2, 3$ are monotone increasing concave functions of u which are twice continuously differentiable in $v>0$, such that

$$\lim_{v\to\infty} v\beta_i'(v^2)=0. \tag{3.7}$$

Then, there exist some constants N_1 and N_2 such that

$$\sup_{t_0 \leqq s \leqq t} E(|x(s, t_0, x_0)|^2) \leqq N_1+N_2 B^2(t) \tag{3.8}$$

holds for all $t \in R^+$ and $x_0 \in R$.

Proof. For convenience we write $x(t)$ and e_n for $x(t)=x(t, t_0, x_0)$ and $e_n=e_n(t_0, x_0)$ respectively by suppressing t_0 and x_0. We shall denote the smallest of w and v by $w \wedge v$. Since $x(t)$ satisfies (1.1), by using $(a+b+c)^2 \leqq 3(a^2+b^2+c^2)$, Schwarz inequality, stochastic integral isometry, conditions (2.9), (3.1), (3.2) and (3.5), we observe for any $t \geqq t_0$, that

$$\begin{aligned} E|x(t\wedge e_n)|^2 &\leqq 3\Big[|x_0|^2+3K^2B^2(t)+(3+t-t_0)\frac{(t-t_0)^2}{2}\int_{t_0}^t \alpha_1(s)E\beta_1(|x(s\wedge e_n)|^2)ds \\ &\quad +(3+t-t_0)(t-t_0)\int_{t_0}^t \alpha_2(s)E\beta_2(|x(s\wedge e_n)|^2)ds \\ &\quad +(t-t_0)\int_{t_0}^t \alpha_3(s)E\beta_3(|x(s\wedge e_n)|^2)ds\Big]. \end{aligned} \tag{3.9}$$

Since β_i, $i=1, 2, 3$ are concave, by Jenson's inequality we observe,

$$E\beta_i(|x(s\wedge e_n)|^2) \leqq \beta_i(\sup_{t_0 \leqq s \leqq t} E|x(s\wedge e_n)|^2), \quad (t_0 \leqq s \leqq t). \tag{3.10}$$

Define

$$z_n(t)=\sup_{t_0 \leqq s \leqq t} E|x(s\wedge e_n)|^2. \tag{3.11}$$

Now, from (3.9)-(3.11) it follows that

$$z_n(t) \leqq 3\Big[|x_0|^2 + 3K^2B^2(t) + (3+t-t_0)^3\Big\{\beta_1(z_n(t))\int_{t_0}^t \alpha_1(s)ds \tag{3.12}$$
$$+\beta_2(z_n(t))\int_{t_0}^t \alpha_2(s)ds + \beta_3(z_n(t))\int_{t_0}^t \alpha_3(s)ds\Big\}\Big]$$
$$= 3\Big[|x_0|^2 + 3K^2B^2(t) + \sum_{i=1}^{3} D_i(t)\beta_i(z_n(t))\Big].$$

We define $p_{n,\delta}(t) = z_n(t)/3 + B^2(t) + \delta$, where $\delta > 0$ is arbitrary number. It follows from (3.4) that there exists some $t_1 > 0$ such that $3B^2(t) > 1$ for all $t > t_1$. From (3.6) and (3.7), for any $t > t_1$, we observe,

$$0 \leqq D_i(t)\beta_i'(3B^2(t)) \leqq \Big(\frac{D_i(2t)}{B(t)}\Big)B(t)\beta_i'(3B^2(t)) \longrightarrow 0$$

as $t \to \infty$, where $i = 1, 2, 3$. Therefore, we find some $t_2 > t_1$ such that

$$1 - 3D_i(t)\beta_i'(3B^2(t)) > 0$$

for all $t > t_2$ and $i = 1, 2, 3$. Further we assume that t be arbitrary and $t > \max\{t_0, t_2\}$. We note that $3p_{n,\delta}(t) > z_n(t)$ and $3p_{n,\delta}(t) > 3B^2(t)$. Thus from the hypothesis that β_i, $i = 1, 2, 3$ are monotone increasing, the inequality (3.12) yields

$$z_n(t) \leqq 3\Big[|x_0|^2 + 3K^2B^2(t) + \sum_{i=1}^{3} D_i(t)\beta_i(3p_{n,\delta}(t))\Big].$$

Hence, we have,

$$p_{n,\delta}(t) \leqq \delta + |x_0|^2 + (3K^2+1)B^2(t) + \sum_{i=1}^{3} D_i(t)\beta_i(3p_{n,\delta}(t)). \tag{3.13}$$

By the assumptions of concavity and differentiability on $\beta_i(v)$, $i = 1, 2, 3$, for any $v_2 > v_1 > 0$, we have $\beta_i(v_2) \leqq \beta_i(v_1) + (v_2 - v_1)\beta_i'(v_1)$. Define $r(t) = 3B^2(t)$. Then, we have,

$$\beta_i(3p_{n,\delta}(t)) \leqq \beta_i(t) + (r(t)-1)\beta_i'(r_\theta(t)) + (3p_{n,\delta}(t) - r(t))\beta_i'(r(t)),$$

where $r_\theta(t) = 1 + \theta(r(t)-1)$, $(0 < \theta < 1)$. By using this inequality in (3.13), we get,

$$p_{n,\delta}(t) \leqq \delta + |x_0|^2 + (3K^2+1)B^2(t)$$
$$+ \sum_{i=1}^{3} D_i(t)[\beta_i(1) + (r(t)-1)\beta_i'(r_\theta(t)) + (3p_{n,\delta}(t) - r(t))\beta_i'(r(t))].$$

Thus on letting δ tend to zero, we observe,

$$[z_n(t)/3 + B^2(t)]\Big[1 - 3\sum_{i=1}^{3} D_i(t)\beta_i'(r(t))\Big]$$
$$\leqq |x_0|^2 + (3K^2+1)B^2(t) + \sum_{i=1}^{3} D_i(t)[\beta_i(1) + (r(t)-1)\beta_i'(r_\theta(t)) - r(t)\beta_i'(r(t))]$$

and

$$\left[\frac{z_n(t)}{3}+B^2(t)\right]\Big/B^2(t)\leqq\frac{I(t)}{1-3\sum_{i=1}^{3}D_i(t)\beta_i'(r(t))} \tag{3.14}$$

where

$$I(t)=\frac{|x_0|^2}{B^2(t)}+(3K^2+1)+\sum_{i=1}^{3}\frac{D_i(t)}{B(t)}J_i(t),$$

and

$$J_i(t)=\frac{\beta_i(1)}{\beta(t)}+\left[\frac{3(r(t)-1)^2}{r(t)r_\theta(t)}\right]^{1/2}(r_\theta(t))^{1/2}\beta_i'(r_\theta(t))-(3r(t))^{1/2}\beta_i'(r(t)),$$

$i=1, 2, 3$. Note that $\beta_i(v)$ are concave and twice continuously differentiable in $v>0$. Then for $v>\varepsilon$, we have, $\beta_i(v)\leqq\beta_i(\varepsilon)+(v-\varepsilon)\beta_i'(\varepsilon)$. Thus from (3.2), conditions (2.9), (2.10) and (3.1) and using $(a+b+c)^2\leqq3(a^2+b^2+c^2)$, Schwarz inequality, stochastic integral isometry, we have,

$$\begin{aligned}
&\left|F\left(t,\ x(t),\ \int_{t_0}^{t}f_1(t,\ s,\ x(s))ds,\ \int_{t_0}^{t}f_2(t,\ s,\ x(s))d\xi(s)\right)\right|^2\\
&\quad+\left|H\left(t,\ x(t),\ \int_{t_0}^{t}h_1(t,\ s,\ x(s))ds,\ \int_{t_0}^{t}h_2(t,\ s,\ x(s))d\xi(s)\right)\right|^2\\
&\leqq(\alpha_3(t)\beta_3(|x(t)|^2))+\left|\int_{t_0}^{t}f_1(t,\ s,\ x(s))ds\right|^2+\left|\int_{t_0}^{t}f_2(t,\ s,\ x(s))d\xi(s)\right|^2\\
&\quad+\left|c(t)a(x)+\int_{t_0}^{t}h_1(t,\ s,\ x(s))ds+\int_{t_0}^{t}h_2(t,\ s,\ x(s))d\xi(s)\right|^2\\
&\leqq\int_{t_0}^{t}\gamma_1(s)(1+|x(s)|^2)ds+\int_{t_0}^{t}\gamma_2(s)(1+|x(s)|^2)ds+\gamma_3(t)(1+|x(t)|^2)
\end{aligned}$$

where

$$\gamma_1(s)=4(t-t_0)\beta(\varepsilon)\alpha_1(s),$$

$$\gamma_2(s)=4\beta(\varepsilon)\alpha_2(s),$$

$$\gamma_3(t)=3K^2c^2(t)+\beta(\varepsilon)\alpha_3(t),$$

and

$$\beta(\varepsilon)=\max\{\beta_i(\varepsilon),\ \beta_i'(\varepsilon)\},\qquad i=1,\ 2,\ 3.$$

Therefore, by Theorem 1, we get

$$P(e(t_0,\ x_0)=\infty)=1. \tag{3.15}$$

Let n tend to infinity in (3.14). Then from (3.15) and Fatou's lemma, we get,

$$\left[\frac{1}{3}\sup_{t_0\leqq s\leqq t}E|x(s)|^2+B^2(t)\right]\Big/B^2(t)\leqq\frac{I(t)}{1-3\sum_{i=1}^{3}D_i(t)\beta_i'(r(t))} \tag{3.16}$$

for all $t>\max\{t_0,\ t_2\}$. From (3.4), (3.6) and (3.7) we observe that

$$0 \leqq \left(\frac{D_i(t)}{B(t)}\right)|J_i(t)| \leqq \left(\frac{D_i(2t)}{B(t)}\right)|J_i(t)| \longrightarrow 0,$$

as $t \to \infty$ and

$$0 \leqq 3D_i(t)\beta_i'(r(t)) \leqq \left(\frac{D_i(2t)}{B(t)}\right) 3B(t)\beta_i'(r(t)) \longrightarrow 0,$$

as $t \to \infty$, $i=1, 2, 3$. Thus,

$$\frac{I(t)}{1-3\sum_{i=1}^{3} D_i(t)\beta_i'(r(t))} \longrightarrow 3K^2+1, \tag{3.17}$$

as $t \to \infty$. Hence, letting t tend to infinity in (3.16) and using (3.17), we have,

$$\limsup_{t\to\infty} \frac{1}{B^2(t)}\left[\frac{1}{3}\sup_{t_0 \leqq s \leqq t} E|x(s)|^2 + B^2(t)\right] \leqq 3K^2+1.$$

It follows that

$$\limsup_{t\to\infty} \frac{1}{B^2(t)}\left[\sup_{t_0 \leqq s \leqq t} E|x(s)|^2\right] \leqq 9K^2.$$

This implies that

$$\sup_{t_0 \leqq s \leqq t} E|x(s)|^2 \leqq N_1 + N_2 B^2(t)$$

for some constants $N_1 > 0$ and $N_2 > 0$. This completes the proof of the lemma.

We shall now prove our main result on the existence of the order of growth of solutions of equation (1.1).

THEOREM 2. *Let the conditions of Lemma* 3 *hold. Suppose that*

(i) $a(x)=1$,

and

(ii) *for any large* N,

$$b_i(N) = \sum_{k=0}^{\infty} \beta_i(B^2(N2^{k+1}))/B(N2^k),$$

$i=1, 2, 3$, *exist and*

$$\lim_{N\to\infty} b_i(N) = 0. \tag{3.18}$$

Then, we have

$$\frac{x(t, t_0, x_0)}{B(t)}$$

converges to the standard Gaussian Measure in law.

Proof. For simplicity we write $x(t, t_0, x_0)$ as $x(t)$ by suppressing t_0 and x_0. By (3.15), $x(t)$ satisfies

$$x(t)=x_0+\int_{t_0}^{t} c(s)d\xi(s)+\emptyset(t) \tag{3.19}$$

where

$$\emptyset(t)=\int_{t_0}^{t} F\Big(s,\ x(s),\ \int_{t_0}^{s} f_1(s,\ \tau,\ x(\tau))d\tau,\ \int_{t_0}^{s} f_2(s,\ \tau,\ x(\tau))d\xi(\tau)\Big)ds$$
$$+\int_{t_0}^{t}\int_{t_0}^{s} h_1(s,\ \tau,\ x(\tau))d\tau d\xi(s)$$
$$+\int_{t_0}^{t}\int_{t_0}^{s} h_2(s,\ \tau,\ x(\tau))d\xi(\tau)d\xi(s)$$

for all $t\geqq t_0$. The assertion of the theorem easily follows if we prove that

$$\frac{\emptyset(t)}{B(t)}\longrightarrow 0 \quad \text{in probability.} \tag{3.20}$$

Now set $e_n=e_n(t_0,\ x_0)$ and observe the process $|x(v\wedge e_n)|^2$ as well as the proof of Lemma 3. From (3.14), for large t, we note that

$$\sup_n [\{z_n(t)/3+B^2(t)\}/B^2(t)] \tag{3.21}$$
$$\leqq \frac{I(t)}{1-3\sum_{i=1}^{3} D_i(t)\beta_i'(r(t))}.$$

It is easily seen on taking superior limit as t tends to infinity in (3.21) and noting (3.17) that $\sup z_n(t)\leqq N_1'+N_2'B^2(t)$ where N_1' and N_2' are some constants greater than zero. There is no loss of generality if we consider $N_1'=N_1$ and $N_2'=N_2$ where N_1 and N_2 are constants involved in Lemma 3. We observe that

$$\sup_n [\sup_{t_0\leqq v\leqq t} E|x(v\wedge e_n)|^2]\leqq N_1+N_2B^2(t) \tag{3.22}$$

for all $t\geqq t_0$. From the conditions (2.9), (3.5) and the fact that $\beta_i(v)$, $i=1, 2, 3$ are concave and monotone increasing, by the application of Jenson's inequality and (3.22), we have,

$$E\Big(\int_{t_0}^{t\wedge e_n} F\Big(s,\ x(s),\ \int_{t_0}^{s} f_1(s,\ \tau,\ x(\tau))d\tau,\ \int_{t_0}^{s} f_2(s,\ \tau,\ x(\tau))d\xi(\tau)\Big)ds$$
$$+\int_{t_0}^{t\wedge e_n}\int_{t_0}^{s} h_1(s,\ \tau,\ x(\tau))d\tau d\xi(s)+\int_{t_0}^{t\wedge e_n}\int_{t_0}^{s} h_2(s,\ \tau,\ x(\tau))d\xi(\tau)d\xi(s)\Big)^2$$
$$\leqq 3E\Big(\int_{t_0}^{t\wedge e_n} F\Big(s,\ x(s),\ \int_{t_0}^{s} f_1(s,\ \tau,\ x(\tau))d\tau,\ \int_{t_0}^{s} f_2(s,\ \tau,\ x(\tau))d\xi(\tau)\Big)ds\Big)^2$$
$$+3E\Big(\int_{t_0}^{t\wedge e_n}\int_{t_0}^{s} h_1(s,\ \tau,\ x(\tau))d\tau d\xi(s)\Big)^2$$
$$+3E\Big(\int_{t_0}^{t\wedge e_n}\int_{t_0}^{s} h_2(s,\ \tau,\ x(\tau))d\xi(\tau)d\xi(s)\Big)^2$$

$$\leqq 3(t-t_0+1)\frac{(t-t_0)^2}{2}\beta_1(\sup_{t_0\leqq s\leqq t} E|x(s\wedge e_n)|^2)\int_{t_0}^t \alpha_1(s)ds$$

$$+3(t-t_0+1)(t-t_0)\beta_2(\sup_{t_0\leqq s\leqq t} E|x(s\wedge e_n)|^2)\int_{t_0}^t \alpha_2(s)ds$$

$$+3(t-t_0)\beta_3(\sup_{t_0\leqq s\leqq t} E|x(s\wedge e_n)|^2)\int_{t_0}^t \alpha_3(s)ds$$

$$\leqq 3\sum_{i=1}^3 C_i(t)\beta_i(N_1+N_2B^2(t))$$

where

$$C_1(t)=(t-t_0+1)\frac{(t-t_0)^2}{2}A_1(t),$$
$$C_2(t)=(t-t_0+1)(t-t_0)A_2(t),$$
$$C_3(t)=(t-t_0)A_3(t).$$

Now letting n tend to infinity, noting (3.15) and by the application of Fatou's lemma, we get,

$$E[(\varnothing(t))^2]\leqq 3\sum_{i=1}^3 C_i(t)\beta_i(N_1+N_2B^2(t)).$$

Since $\beta_i(v)$, $i=1,2,3$ are monotone increasing and twice continuously differentiable in $v>0$, we find some constants $Q_i>\max\{N_1+N_2+1,\ \beta_i(N_1+N_2+1)/\beta_i(1)\}$ such that

$$\beta_i(N_1+N_2v)\leqq\beta_i((N_1+N_2+1)v)\leqq Q_i\beta_i(v)$$

if $v\geqq 1$. Further, from (3.4) and (3.6) we find some $t'>0$ such that $B^2(t)\geqq 1$ and $\frac{C_i(2t)}{B(t)}\leqq C$, $i=1, 2, 3$ for all $t>t'$ where $C>0$ is a constant. Therefore, we have,

$$E[(\varnothing(t))^2]\leqq 3\sum_{i=1}^3 C_i(t)Q_i\beta_i(B^2(t)) \tag{3.23}$$

for all $t\geqq t''=\max\{t_0,\ t'\}$. For arbitrary $\varepsilon>0$ and $T_2>T_1>t''$, by using martingale inequality and (3.23), we get,

$$P\Big(\sup_{T_1\leqq t\leqq T_2}\Big|\frac{\varnothing(t)}{B(t)}\Big|>\varepsilon\Big)$$
$$\leqq\frac{1}{\varepsilon^2B^2(T_1)}E[(\varnothing(T_2))^2]$$
$$\leqq\frac{3}{\varepsilon^2B^2(T_1)}\sum_{i=1}^3 C_i(T_2)Q_i\beta_i(B^2(T_2)).$$

Thus, for an arbitrary N, $N>t''$ and using the fact that $\frac{C_i(2t)}{B(t)}\leqq C$, $i=1, 2, 3$ we observe,

$$P\Big(\sup_{N\leqq t}\Big|\frac{\emptyset(t)}{B(t)}\Big|>\varepsilon\Big)$$
$$\leqq\sum_{k=0}^{\infty}P\Big(\sup_{N2^k\leqq t\leqq N2^{k+1}}\Big|\frac{\emptyset(t)}{B(t)}\Big|>\varepsilon\Big)$$
$$\leqq\sum_{k=0}^{\infty}\Big[\frac{3}{\varepsilon^2B^2(N2^k)}\sum_{i=1}^{3}C_i(N2^{k+1})Q_i\beta_i(B^2(N2^{k+1}))\Big]$$
$$\leqq\frac{3C}{\varepsilon^2}\sum_{i=1}^{3}Q_ib_i(N)$$

for all $t>t''$. Define,

$$W_N=\sup_{N<t}\Big|\frac{\emptyset(t)}{B(t)}\Big|,$$

and note that it is monotonic decreasing as N increases. Thus, $\lim_{N\to\infty}W_N$ exists and (3.18) implies that

$$\lim_{N\to\infty}P(W_N>\varepsilon)=0.$$

It is easy to observe that (3.20) holds. This proves the theorem.

4. Examples.

In this section we give two examples which illustrate the assumptions used in our Theorems 1-2 and Lemma 3. Our examples are the modifications of the examples given by Narita in [11].

Example 1. Suppose that the functions f_i, h_i, $(i=1, 2)$, F, H involved in equation (1.1) satisfy the hypothesis (A) and the following conditions

$$\sup_{|x|\leqq 1}\{|f_i(t, s, x)|^2+|h_i(t, s, x)|^2\}\leqq C_1(s), \qquad t\geqq s\geqq 0,$$

and

$$|f_i(t, s, x)|^2+|h_i(t, s, x)|^2\leqq C_2(s)(1+|x|^{2\delta}), \qquad (\delta\leqq 1),$$

for $t\geqq s\geqq 0$ and $|x|\geqq 1$ where C_1 and C_2 are nonnegative, continuous functions defined on $[0, \infty)$, and

$$\sup_{|x|\leqq 1}\{|F(t, x, y, u)|^2+|H(t, x, y, u)|^2\}\leqq C_3(t), \qquad \text{for} \quad t\geqq 0,$$

and

$$|F(t, x, y, u)|^2+|H(t, x, y, u)|^2\leqq C_4(t)(1+|x|^{2\delta}+|y|^2|u|^2),$$

for $t\geqq 0$ and $|x|\geqq 1$ where C_3 and C_4 are nonnegative, continuous functions defined on $[0, \infty)$. By taking $\alpha_i(t)=C_1(t)+C_2(t)$, $i=1, 2$, $\alpha_3(t)=C_3(t)+C_4(t)\geqq 1$, $\beta_i(v)=1+v^\delta$ for $0\leqq\delta\leqq 1$, $\beta_i(v)=2$ for $\delta<0$, $i=1, 2, 3$ it is easy to verify that the conditions (2.9)-(2.11) of Theorem 1 are satisfied.

As observed in McKean [9, p. 66], the one dimensional stochastic differential equation

$$dX(t)=b(X(t))dt+dW(t)$$

with $b(x)=|x|^{\gamma}$ near large $|x|$, the explosion time is almost surely infinite or finite according as $\gamma\leqq 1$ or not.

Example 2. Let $c(t)=(t-t_0)^{7/2}$, $a(x)\equiv 1$ and $f_i(t, s, x)$, $h_i(t, s, x)$, $F(t, x, y, u)$ and $H(t, x, y, u)=((t-t_0)^{7/2}+y+u)$ satisfy the assumption (A) and suppose for $i=1, 2$,

$$\sup_{|x|\leqq 1}\{|f_i(t, s, x)|^2+|h_i(t, s, x)|^2\}\leqq K, \qquad \text{for all} \quad t\geqq s\geqq 0,$$

and

$$|f_i(t, s, x)|^2+|h_i(t, s, x)|^2\leqq K(1+|x|^{\gamma})$$

for all $t\geqq s\geqq 0$ and $|x|\geqq 1$ with some constant $K>0$ and $\gamma<1$, and

$$\sup_{|x|\leqq 1}\{|F(t, x, y, u)|^2+|H(t, x, y, u)|^2\}\leqq K, \qquad \text{for all} \quad t\geqq 0,$$

and

$$|F(t, x, y, u)|^2+H(t, x, y, u)|^2\leqq K(1+|x|^{\gamma}+|y|^2+|u|^2),$$

for all $t\geqq 0$ and $|x|\geqq 1$ with constant $K>0$ and $\gamma<1$. In Example 1 we have observed that the explosion time of the solution $x(t, t_0, x_0)$ of (1.1) with f_i, h_i, $i=1, 2$, F and H as above is infinite with probability one. Taking $\alpha_i(t)=K$, $K\geqq 1$, $i=1, 2, 3$,

$$\beta_i(v)=1+v^{\gamma/2}, \qquad (0\leqq\gamma<1),$$

(4.1)

$$\beta_i(v)=2, \qquad (\gamma<0),$$

$i=1, 2, 3$, it is easy to verify that the functions $f_i(t, s, x)$, $h_i(t, s, x)$, $(i=1, 2)$, $F(t, x, y, u)$ satisfy the conditions (2.9) and (3.5) and assumptions (3.4), (3.6)-(3.7) of the Lemma 3. Thus Lemma 3 implies

$$\sup_{t_0\leqq v\leqq t} E(|x(v, t_0, x_0)|^2)\leqq N_1+\frac{1}{8}N_2(t-t_0)^8$$

for all $t\in[0, \infty)$ and $x_0\in R$ where N_1 and N_2 are certain positive constants. Also we observe that

$$\lim_{t\to\infty}\frac{1}{B(t)}\left(\int_{t_0}^{t} E|s-t_0|^2 ds\right)^{1/2}=1.$$

We choose N large enough such that $N>\max\left\{2t, \frac{(t_0+1)}{2}\right\}$. Then it is easy to observe that

$$1\leqq B(N2^{k+1})=\frac{1}{2\sqrt{2}}(N2^{k+1}-t_0)^4\leqq\frac{1}{2\sqrt{2}}(N2^{k+1})^4, \qquad t_0\geqq 0,$$

and

$$B(N2^k)=\frac{1}{2\sqrt{2}}(N2^k-t_0)\geqq\frac{1}{2\sqrt{2}}(N2^{k-1})^4 \qquad \text{for} \quad k\geqq 0.$$

Further we notice that if $\beta_i(v)=1+v^{\gamma/2}(0\leqq\gamma<1)$ then $\beta_i(v)\leqq 2v^{\gamma/2}$ for $v\geqq 1$. Now from the definition of $b_i(N)$ in Theorem 2 and definition of $\beta_i(v)$ given in (4.1), for $i=1, 2, 3$ we observe that

$$\begin{aligned} b_i(N) &= \sum_{k=0}^{\infty}\frac{\beta_i(B^2(N2^{k+1}))}{B(N2^k)} \leqq \sum_{k=0}^{\infty}\frac{2[B^2(N2^{k+1})]^{\gamma/2}}{B(N2^k)} \\ &\leqq \sum_{k=0}^{\infty}\frac{2[1/2\sqrt{2}(N2^{k+1})^4]^{\gamma}}{1/2\sqrt{2}(N2^{k-1})^4} \\ &= 2^{3(\gamma+1)}\left(\frac{1}{\sqrt{2}}\right)^{\gamma-1} N^{-4(1-\gamma)}\sum_{k=0}^{\infty}2^{-4(1-\gamma)k} \end{aligned}$$

and if $\beta_i(v)=2$, then

$$b_i(N)=\sum_{k=0}^{\infty}\frac{2}{1/2\sqrt{2}(N2^k-t_0)^4}\leqq 4\sqrt{2}N^{-4}\sum_{k=0}^{\infty}\left(2^k-\frac{t_0}{N}\right)^{-4}.$$

Thus the conditions (3.18) hold. Now an application of Theorem 2 yields

$$\frac{x(t, t_0, x_0)}{1/2\sqrt{2}(t-t_0)^4}\longrightarrow 0 \qquad \text{in probability.}$$

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