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A CERTAIN SPACE-TIME METRIC AND SMOOTH GENERAL CONNECTIONS

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Introduction.

For a manifold M with a general connection Γ we say a connected subset A is a black hole, if it has a neighborhood U such that if any one going on along a geodesic enters U, then he will be finally swallowed in A. The present author gave a way in [8] by which we can construct a general connection Γ for any Riemannian manifold (M, g) and any point p of M such that Γ has p as a black hole and has the same system of geodesics as the one of (M, g) outside of a neighborhood.

In the theory of general relativity, the Eddington-Finkelstein metric g is given by

$$d\tau^2=-\left(1-\frac{2m}{r}\right)dt^2+2dtdr+r^2(d\theta^2+\sin^2\theta\, d\varphi^2), \tag{1}$$

where (r, θ, φ) are the polar coordinates of the space R^3 with the coordinates (x_1, x_2, x_3) as

$$r=\sqrt{\Sigma x_i^2}, \quad x_1=r\sin\theta\cos\varphi, \quad x_2=r\sin\theta\sin\varphi, \quad x_3=r\cos\theta.$$

As is well known, the curve $r=0$ in the space-time is a black hole as is mentioned above, even though the metric (1) loses the meaning along this curve, (1) is locally equivalent to the Schwarzschild metric

$$d\tau^2=-\frac{r-2m}{r}dt^2+\frac{r}{r-2m}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2), \tag{2}$$

through the change of time t in (2) to $t-r-\log|r-2m|^{2m}$. (2) loses its meaning where $r=0$ and $r=2m$ but (1) is everywhere regular except $r=0$.

Now, we denote the affine connection made by the Christoffel symbols from the space-time metric (1) by Γ_g. Taking a tensor field P of type (1, 1), consider the general connection $\Gamma=P\Gamma_g$. Then, any geodesic of Γ_g is also a geodesic with respect to Γ. Conversely any geodesic of Γ is also a geodesic with respect to Γ_g, where P is an isomorphism on the tangent space of $R\times(R^3-\{0\})$. We consider a problem: Taking P suitably, is it possible $\Gamma=P\Gamma_g$ to extend smoothly

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over $R\times R^3=R^4$ with the canonical coordinates (x_0, x_1, x_2, x_3)? Let $\Gamma=(P^i_j, \Gamma^i_{jk})$, where P^i_j and Γ^i_{jk} are the components of Γ with respect to the coordinates

$$t=u_1, \quad r=u_2, \quad \theta=u_3, \quad \varphi=u_4.$$

We have the Christoffel symbols $\left\{ {i \atop jk} \right\}$ made by (1) as follows:

$$\left(\left\{ {1 \atop jk} \right\}\right)=\begin{pmatrix} m/u_2u_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -u_2 & 0 \\ 0 & 0 & 0 & -u_2\sin^2 u_3 \end{pmatrix},$$

$$\left(\left\{ {2 \atop jk} \right\}\right)=\begin{pmatrix} mB/u_2u_2 & -m/u_2u_2 & 0 & 0 \\ -m/u_2u_2 & 0 & 0 & 0 \\ 0 & 0 & 2m-u_2 & 0 \\ 0 & 0 & 0 & (2m-u_2)\sin^2 u_3 \end{pmatrix},$$

$$\left(\left\{ {3 \atop jk} \right\}\right)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/u_2 & 0 \\ 0 & 1/u_2 & 0 & 0 \\ 0 & 0 & 0 & -\sin u_3 \cos u_3 \end{pmatrix},$$

and

$$\left(\left\{ {4 \atop jk} \right\}\right)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/u_2 \\ 0 & 0 & 0 & \cot u_3 \\ 0 & 1/u_2 & \cot u_3 & 0 \end{pmatrix},$$

where $B=1-2m/r$. Since we have by definition $\Gamma^i_{jk}=\Sigma P^i_h \left\{ {h \atop jk} \right\}$ and from the condition that Γ is extended smoothly to R^4, P^i_j must be of the forms as

$$P^i_j=F^i_j u_2u_2+2mF^i_2u_2, \quad P^i_2=F^i_2u_2u_2, \quad P^i_3=F^i_3u_2, \quad P^i_4=F^i_4u_2\sin u_3, \tag{3}$$

where, F^i_j are continuous near $r=0$. Hence we have

$$(\Gamma^i_{jk})=\begin{pmatrix} m(F^i_1+F^i_2) & -mF^i_2 & 0 & 0 \\ -mF^i_2 & 0 & F^i_3 & F^i_4\sin u_3 \\ 0 & F^i_3 & -(F^i_1+F^i_2)(u_2)^3 & F^i_4u_2\cos u_3 \\ 0 & F^i_4\sin u_3 & F^i_4 u_2\cos u_3 & * \end{pmatrix} \tag{4}$$

where $*$ is $-(F^i_1+F^i_2)(u_2)^3\sin^2 u_3-F^i_3u_2\sin u_3\cos u_3$. This expression tells us that if we compute the components of Γ in the canonical coordinates (x_0, x_1, x_2, x_3) of $R\times R^3$, it is possible to make it continuous but impossible to make it smooth. Since we have the expression of g in the coordinates (x_0, x_1, x_2, x_3) as

$$d\tau^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{2}{r}\sum_{i=1}^{3} x_i dt dx_i+\sum_{i=1}^{3} dx_i dx_i-\left(\sum_{i=1}^{3}\frac{x_i}{r}dx_i\right)^2$$

and the coefficients of the quadratic form $rd\tau^2$ are continuous but some of them are not differentiable at the points where $r=0$. This fact may be the reason which implies the above situation on the general connection.

§ 1. A certain space-time metric.

In this section, we shall give a space-time metric on $R\times(R^3-\{0\})$ with the curve $r=0$ as a black hole and make smooth general connections on R^4 having the same system of geodesics with the one of this pseudo-Riemannian metric in $R\times(R^3-\{0\})$.

First we consider a space-time metric g given by

$$d\sigma^2=-\left(1-\frac{4m^2}{r^2}\right)dt^2+\frac{2}{r}dtdr+r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \tag{1.1}$$

in the same coordinates (t, r, θ, φ) in Introduction and setting $d\sigma^2=\sum_{i,j} g_{ij}du_i du_j$, where $t=u_1$, $r=u_2$, $\theta=u_3$ and $\varphi=u_4$. Then we have the Christoffel symbols $\left\{ {i \atop jk} \right\}$ made by (1.1) as follows.

$$\left(\left\{ {1 \atop jk} \right\}\right)=\begin{pmatrix} 4m^2/u_2u_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -u_2u_2 & 0 \\ 0 & 0 & 0 & -u_2u_2\sin^2 u_3 \end{pmatrix},$$

$$\left(\left\{ {2 \atop jk} \right\}\right)=\begin{pmatrix} 4m^2B/u_2 & -4m^2/u_2u_2 & 0 & 0 \\ -4m^2/u_2u_2 & -1/u_2 & 0 & 0 \\ 0 & 0 & -B(u_2)^3 & 0 \\ 0 & 0 & 0 & -B(u_2)^3\sin^2 u_3 \end{pmatrix},$$

$$\left(\left\{ {3 \atop jk} \right\}\right)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/u_2 & 0 \\ 0 & 1/u_2 & 0 & 0 \\ 0 & 0 & 0 & -\cos u_3 \sin u_3 \end{pmatrix},$$

$$\left(\left\{ {4 \atop jk} \right\}\right)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/u_2 \\ 0 & 0 & 0 & \cot u_3 \\ 0 & 1/u_2 & \cot u_3 & 0 \end{pmatrix},$$

where $B=1-4m^2/r^2$. Hence the equation of a geodesic with respect to this space-time metric are

$$(1.2)\qquad \begin{cases} \dfrac{d^2t}{dp^2}+\dfrac{4m^2}{r^2}\Big(\dfrac{dt}{dp}\Big)^2-r^2\Big(\dfrac{d\theta}{dp}\Big)^2-r^2\sin^2\theta\Big(\dfrac{d\varphi}{dp}\Big)^2=0, \\ \dfrac{d^2r}{dp^2}+\dfrac{4m^2B}{r}\Big(\dfrac{dt}{dp}\Big)^2-\dfrac{8m^2}{r^2}\dfrac{dt}{dp}\dfrac{dr}{dp}-\dfrac{1}{r}\Big(\dfrac{dr}{dp}\Big)^2 \\ \qquad -Br^3\Big(\dfrac{d\theta}{dp}\Big)^2-Br^3\sin^2\theta\Big(\dfrac{d\varphi}{dp}\Big)^2=0, \\ \dfrac{d^2\theta}{dp^2}+\dfrac{2}{r}\dfrac{dr}{dp}\dfrac{d\theta}{dp}-\cos\theta\sin\theta\Big(\dfrac{d\varphi}{dp}\Big)^2=0, \\ \dfrac{d^2\varphi}{dp^2}+\dfrac{2}{r}\dfrac{dr}{dp}\dfrac{d\varphi}{dp}+2\cot\theta\dfrac{d\theta}{dp}\dfrac{d\varphi}{dp}=0, \end{cases}$$

where p is the canonical parameter of the geodesic as

$$(1.3)\qquad \frac{d\sigma^2}{dp^2}=-\Big(1-\frac{4m^2}{r^2}\Big)\Big(\frac{dt}{dp}\Big)^2+\frac{2}{r}\frac{dt}{dp}\frac{dr}{dp}+r^2\Big\{\Big(\frac{d\theta}{dp}\Big)^2+\sin^2\theta\Big(\frac{d\varphi}{dp}\Big)^2\Big\}$$

$$=c=\begin{cases}-1\\ \ \ 0\\ \ \ 1\end{cases}$$

according to the sign of $\sum\limits_{i,j} g_{ij}(dui/dp)(duj/dp)$.

Next, consider a geodesic which pass through a given point $q_0=(t_0, r_0, \theta_0, \varphi_0)$ and $(dq/dp)_0=(\xi_0, \eta_0, \lambda_0, \mu_0)$. Then we may put

$$\theta_0=\frac{\pi}{2} \quad\text{and}\quad \lambda_0=0$$

without loss of generality, because the metric (1.1) is spherical symmetric with respect to (x_1, x_2, x_3). Noticing that the third of (1.2) is satisfied with $\theta\equiv\pi/2$, we put $\theta=\pi/2$ in (1.2) and (1.3), we obtain the following equations:

$$(1.2')\qquad \begin{cases} \dfrac{d^2t}{dp^2}+\dfrac{4m^2}{r^2}\Big(\dfrac{dt}{dp}\Big)^2-r^2\Big(\dfrac{d\varphi}{dp}\Big)^2=0, \\ \dfrac{d^2r}{dp^2}+\dfrac{4m^2B}{r}\Big(\dfrac{dt}{dp}\Big)^2-\dfrac{8m^2}{r^2}\dfrac{dt}{dp}\dfrac{dr}{dp}-\dfrac{1}{r}\Big(\dfrac{dr}{dp}\Big)^2-Br^3\Big(\dfrac{d\varphi}{dp}\Big)^2=0, \\ \dfrac{d^2\varphi}{dp^2}+\dfrac{2}{r}\dfrac{dr}{dp}\dfrac{d\varphi}{dp}=0 \end{cases}$$

and

$$(1.3')\qquad -B\Big(\frac{dt}{dp}\Big)^2+\frac{2}{r}\frac{dt}{dp}\frac{dr}{dp}+r^2\Big(\frac{d\varphi}{dp}\Big)^2=c,$$

From the third of (1.2′) we see that $r^2(d\varphi/dp)$ is constant along the geodesic and so we put this constant as

$$r^2\frac{d\varphi}{dp}=r_0^2\mu_0=J. \tag{1.4}$$

Using this fact, the first and second of (1.2′) become

$$\begin{cases}\dfrac{d^2t}{dp^2}=-\dfrac{4m^2}{r^2}\Big(\dfrac{dt}{dp}\Big)^2+\dfrac{J^2}{r^2}, \\ \dfrac{d^2r}{dp^2}+\dfrac{4m^2B}{r}\Big(\dfrac{dt}{dp}\Big)^2-\dfrac{8m^2}{r^2}\dfrac{dt}{dp}\dfrac{dr}{dp}-\dfrac{1}{r}\Big(\dfrac{dr}{dp}\Big)^2-\dfrac{B}{r}J^2=0,\end{cases}$$

from which we obtain

$$\frac{d^2r}{dp^2}-\frac{8m^2}{r^2}\frac{dt}{dp}\frac{dr}{dp}-\frac{1}{r}\Big(\frac{dr}{dp}\Big)^2-Br\frac{d^2t}{dp^2}=0$$

by cancelling J and hence

$$\frac{d}{dp}\Big(\frac{dr}{dp}-Br\frac{dt}{dp}\Big)=\frac{1}{r}\frac{dr}{dp}\Big(\frac{dr}{dp}-Br\frac{dt}{dp}\Big).$$

Therefore we see that $(1/r)(dr/dp-Br(dt/dp))$ is also constant along the geodesic and we put this constant as

$$\frac{1}{r}\Big(\frac{dr}{dp}-Br\frac{dt}{dp}\Big)=\frac{\eta_0}{r_0}-\Big(1-\frac{4m^2}{r_0^2}\Big)\xi_0=A. \tag{1.5}$$

Finally using (1.4) and (1.5) for (1.3′) we have

$$-\Big(B\frac{dt}{dp}-\frac{1}{r}\frac{dr}{dp}\Big)^2+\frac{1}{r^2}\Big(\frac{dr}{dp}\Big)^2+\frac{B}{r^2}J^2=cB$$

and so

$$-A^2+\frac{1}{r^2}\Big(\frac{dr}{dp}\Big)^2=-B\Big(\frac{J^2}{r^2}-c\Big) \quad \text{or} \quad \Big(\frac{d\log r}{dp}\Big)^2=A^2-B\Big(\frac{J^2}{r^2}-c\Big). \tag{1.6}$$

From (1.6) we see the following fact. When $c=-1$ or 0, if $r\leqq 2m$ $(B\leqq 0)$, which implies

$$\Big|\frac{d\log r}{dp}\Big|\geqq|A|.$$

Let t_1 be the moment such that the geodesic passes through the hypersurface $r=2m$ at the point $(t_1, 2m, \pi/2, \varphi_1)$ then we have from (1.5)

$$A=\frac{1}{2m}\eta_1, \qquad \eta_1=\frac{dr}{dp}\Big|_{t=t_1}.$$

Therefore the geodesic enters the hypersurface $r=2m$ with $\eta_1<0$, then the decreasing ratio of $\log r$ is greater than $|A|$.

THEOREM 1. *The space-time metric* (1.1) *has the curve* $r=0$ *in* $R\times R^3$ *as a black hole for the system of visible geodesics, i. e.* $c=-1$ *or* 0 *in* (1.3).

§ 2. Smooth general connections with the same system of geodesics of (1.1).

In the canonical coordinates (x_0, x_1, x_2, x_3) of $R\times R^3$ with $x_0=t$, (1.1) can be represented as

$$d\sigma^2=-\left(1-\frac{4m^2}{r^2}\right)dt^2+\frac{2}{r^2}\sum_{i=1}^{3} x_i dt dx_i+\sum_{i=1}^{3} dx_i dx_i-\frac{1}{r^2}\left(\sum_{i=1}^{3} x_i dx_i\right)^2 \tag{2.1}$$

and setting the right hand side of (2.1) as $\sum_{\alpha,\beta=0}^{3} g_{\alpha\beta}dx_\alpha dx_\beta$, we have

$$g_{00}=-\left(1-\frac{4m^2}{r^2}\right)=-B,\quad g_{0j}=g_{j0}=\frac{x_i}{r^2},\quad g_{ij}=g_{ji}=\delta_{ij}-\frac{x_i x_j}{r^2}, \tag{2.2}$$

from which $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$ are given as

$$g^{00}=0,\quad g^{0i}=g^{i0}=x_i,\quad g^{ij}=\delta_{ij}+\left(B-\frac{1}{r^2}\right)x_i x_j. \tag{2.3}$$

Making use of (2.2) and (2.3), the Christoffel symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$ of (1.1) in the canonical coordinates (x_α) are given by the formulas as follows

$$\begin{cases} \left(\left\{ {0 \atop \beta\gamma} \right\}\right)=\dfrac{1}{r^2}\begin{pmatrix} 4m^2 & 0 \\ 0 & -(r^2\delta_{ij}-x_i x_j) \end{pmatrix}, \\ \left(\left\{ {h \atop \beta\gamma} \right\}\right)=\dfrac{x_h}{r^4}\begin{pmatrix} 4m^2(r^2-4m^2) & -4m^2 x_j \\ -4m^2 x_i & -x_i x_j-(r^2-4m^2-1)(r^2\delta_{ij}-x_i x_j) \end{pmatrix}. \end{cases} \tag{2.4}$$

Now, take a tensor field P of type (1, 1) with local components P^α_β and let Γ be the general connection $P\Gamma_g$, where Γ_g is the affine connection with the components $\left\{ {\alpha \atop \beta\gamma} \right\}$. Since for $\Gamma=(P^\alpha_\beta, \Gamma^\alpha_{\beta\gamma})$ we have

$$\Gamma^\alpha_{\beta\gamma}=\sum_{\tau=0}^{3} P^\alpha_\tau\left\{ {\tau \atop \beta\gamma} \right\} \tag{2.5}$$

and so in order to be determined Γ so that it is smooth near $r=0$ and has the same system of geodesics as the one of (2.1) in $R\times(R^3-\{0\})$, it is necessary and sufficient to put

$$P^\alpha_0=F^\alpha_0 r^2,\qquad P^\alpha_i=F^\alpha_i r^4, \tag{2.6}$$

where F^α_β are smooth near $r=0$, and

$$|F^\alpha_\beta|\neq 0 \quad \text{where} \quad r\neq 0. \tag{2.7}$$

Then, Γ is regular where $r\neq 0$. Thus we obtain

$$(\Gamma^\alpha_{\beta\gamma})=F^\alpha_0\begin{pmatrix} 4m^2 & 0 \\ 0 & -(r^2\delta_{ij}-x_i x_j) \end{pmatrix} \tag{2.8}$$

$$+\sum_h F_h^\alpha x_h \begin{pmatrix} 4m^2(r^2-4m^2) & -4m^2 x_j \\ -4m^2 x_i & -x_i x_j-(r^2-4m^2-1)(r^2\delta_{ij}-x_i x_j) \end{pmatrix}$$

There are many freedoms of the choice of P for the purpose mentioned above for (2.1). We request that

$$\delta_{\beta,0}^\alpha=0, \tag{2.9}$$

where "," denotes the covariant differentiation with respect to Γ. If we put for F the condition

$$\frac{\partial F_\beta^\alpha}{\partial t}=0, \tag{2.10}$$

then we have for the covariant components of Γ:

$$\Lambda_{\beta\gamma}^\alpha:=\Gamma_{\beta\gamma}^\alpha-\frac{\partial P_\beta^\alpha}{\partial x_\gamma}$$

the equalities

$$\Lambda_{\beta 0}^\alpha=\Gamma_{\beta 0}^\alpha,$$

and hence

$$\delta_{\beta,0}^\alpha=\sum_\rho \Gamma_{\rho 0}^\alpha P_\beta^\rho-\sum_\rho P_\rho^\alpha \Lambda_{\beta 0}^\rho=\sum_\rho \Gamma_{\rho 0}^\alpha P_\beta^\rho-\sum_\rho P_\rho^\alpha \Gamma_{\beta 0}^\rho=0.$$

From (2.6) and (2.8), we have

$$(P_\beta^\alpha)=r^2(F_0^\alpha,\ r^2F_j^\alpha), \qquad (\Gamma_{\beta 0}^\alpha)=4m^2(F_0^\alpha+(r^2+4m^2)V^\alpha,\ -V^\alpha x_j),$$

where $V^\alpha:=\sum_h F_h^\alpha x_h$. Hence, setting $W_\alpha=\sum F_\alpha^h x_h$, and using (2.7), the condition $\delta_{0,0}^\alpha=0$ is equivalent to

$$\begin{cases} (r^2-4m^2)V^0=0, \\ r^2\{F_0^j+(r^2-4m^2)V^j\}=\{(r^2-4m^2)F_0^0-W_0\}\,x_j \end{cases} \tag{2.11}$$

and the condition $\delta_{j,0}^\alpha=0$ is equivalent to

$$\begin{cases} -V^0 x_j=r^2F_j^0, \\ -V^h x_j=\{(r^2-4m^2)F_j^0-W_j\}\,x_h. \end{cases} \tag{2.12}$$

From the first of (2.11) and (2.12) we obtain

$$\begin{cases} F_j^0=0, \\ \sum_h F_h^i x_h=\lambda x_i, \quad \sum_h x_h F_j^h=\lambda x_j, \end{cases} \tag{2.13}$$

where λ is an auxiliary function. Then using (2.13) for the second of (2.11) we can put

$$F_0^i=\mu x_i, \tag{2.14}$$

where μ is an auxiliary function, and hence we obtain

(2.15) $$2r^2\mu=(r^2-4m^2)(F_0^0-\lambda r^2).$$

Thus, we see that

LEMMA 2.1. *Supposing that P does not depend on t, then $\delta_{\beta,0}^{\alpha}=0$ is equivalent to* (2.13), (2.14) *and* (2.15).

Now, considering (2.13), we take a special one such that

(2.16) $$F_0^0=r^2F \quad \text{and} \quad F_j^i=\lambda\delta_j^i,$$

then from (2.15) we obtain

$$\lambda=-\frac{2\mu}{r^2-4m^2}+F$$

and so if we put

(2.17) $$\mu=(r^2-4m^2)G,$$

then we obiain from the above equality

(2.18) $$\lambda=F-2G.$$

Thus, we obtain a special $P=(P_\beta^\alpha)$ implying $\delta_{\beta,0}^{\alpha}=0$ given by

(2.19) $$(P_\beta^\alpha)=r^2\begin{pmatrix} r^2F & 0 \\ (r^2-4m^2)Gx_i & r^2(F-2G)\delta_j^i \end{pmatrix},$$

where F and G are smooth, $F\neq 0$ and $F-2G\neq 0$.

THEOREM 2. *The general connection $\Gamma=P\Gamma_g$ with P given by* (2.19) *is smooth on $R\times R^3$, has the same system of geodesics as the one of the space-time metric* (2.1) *where $r\neq 0$, and satisfies the conditions:*

$$\delta_{\beta,0}^{\alpha}=0 \quad \textit{and} \quad \tilde{g}_{\alpha\beta,0}=0,$$

where $\tilde{g}_{\alpha\beta}=r^2g_{\alpha\beta}$.

Proof. Except the last condition $\tilde{g}_{\alpha\beta,0}=0$, the rest ones are evident from the above argument. In fact, we have

$$(\tilde{g}_{\alpha\beta})=\begin{pmatrix} -(r^2-4m^2) & x_j \\ x_i & r^2\delta_{ij}-x_ix_j \end{pmatrix}$$

and

$$\tilde{g}_{\alpha\beta,0}:=\sum_{\rho,\sigma}P_\alpha^\rho P_\beta^\sigma\frac{\partial\tilde{g}_{\rho\sigma}}{\partial x_0}-\sum_{\rho,\sigma}\tilde{g}_{\rho\sigma}\Lambda_{\alpha 0}^{\rho}P_\beta^\sigma-\sum_{\rho,\sigma}\tilde{g}_{\rho\sigma}P_\alpha^\rho\Lambda_{\beta 0}^{\sigma},$$

into which substituting (2.19) and

$$(\Lambda^\alpha_{\beta 0})=(\Gamma^\alpha_{\beta 0})=4m^2\begin{pmatrix} r^2F & 0 \\ (r^2-4m^2)(F-G)x_i & -(F-2G)x_ix_j\end{pmatrix}$$

we can easily obtain $\tilde{g}_{\alpha\beta,0}=0$. Q.E.D.

Finally, we give the components $\Gamma^\alpha_{\beta\gamma}$ of $\Gamma=P\Gamma_g$ in Theorem 2, which are

$$(\Gamma^\alpha_{\beta\gamma})=F^\alpha_0\begin{pmatrix} 4m^2 & 0 \\ 0 & -(r^2\delta_{ij}-x_ix_j)\end{pmatrix} \tag{2.20}$$

$$+\sum_h F_h x_h\begin{pmatrix} 4m^2(r^2-4m^2) & -4m^2x_j \\ -4m^2x_i & -x_ix_j-(r^2-4m^2-1)(r^2\delta_{ij}-x_ix_j)\end{pmatrix}$$

where

$$F^0_0=r^2F,\quad F^i_0=(r^2-4m^2)Gx_i,\quad F^0_j=0,\quad F^i_j=(F-2G)\delta^i_j.$$

§ 3. The curvature form for a special general connection in Theorem 2.

In this section, we shall give the curvature form for the special general connection in Theorem 2 given by

$$F\equiv 1 \quad\text{and}\quad G\equiv 0. \tag{3.1}$$

Then we have

$$P^\alpha_\beta=r^4\delta^\alpha_\beta \tag{3.2}$$

and (2.20) becomes

$$(\Gamma^0_{\beta\gamma})=r^2\begin{pmatrix} 4m^2 & 0 \\ 0 & -r^2\delta_{jh}+x_jx_h\end{pmatrix},$$
$$(\Gamma^i_{\beta\gamma})=x_i\begin{pmatrix} 4m^2(r^2-4m^2) & -4m^2x_h \\ -4m^2x_j & -x_jx_h+(r^2-4m^2-1)(x_jx_h-r^2\delta_{jh})\end{pmatrix}. \tag{3.3}$$

The connection $\Gamma=P\Gamma_g$ in Theorem 2 is smooth on $R\times R^3$ and hence we can obtain the curvature along $r=0$ by taking its limit from the outside of the curve. Where $r\neq 0$, Γ is regular, i.e. det $(P^\alpha_\beta)\neq 0$, therefore the curvature form Ω^α_β can be computed from the one of Γ_g by the formula ([2], § 7)

$$\Omega^\alpha_\beta=\sum P^\alpha_\gamma P^\gamma_\rho{}'\Omega^\rho_\sigma P^\sigma_\beta+\sum P^\sigma_\rho{}'DP^\rho_\sigma\wedge{}'DP^\sigma_\beta, \tag{3.4}$$

where $'D$ denote the covariant differentiation with respect to Γ_g. Hence (3.4) becomes in this case as

$$\Omega^\alpha_\beta=r^{12}\,{}'\Omega^\alpha_\beta. \tag{3.5}$$

Since we obtain the connection forms of Γ_g from (2.4) as follows:

$$'\omega^0_0=\frac{4m^2}{r^2}dt,\quad '\omega^0_j=-dx_j+x_jd\log r,\quad '\omega^i_0=\frac{4m^2}{r^2}x_i(Bdt-d\log r), \tag{3.6}$$

$$'\omega^i_j = -\Big(B - \frac{1}{r^2}\Big)x_i dx_j + x_i x_j \Big\{-\frac{4m^2}{r^4}dt + \Big(B - \frac{2}{r^2}\Big)d\log r\Big\},$$

the curvature forms of Γ_g

$$'\Omega^\alpha_\beta := d\,'\omega^\alpha_\beta + \sum_\rho {'\omega^\alpha_\rho} \wedge {'\omega^\rho_\beta}$$

are given by a little long computation as follows:

$$'\Omega^0_0 = \frac{8m^2}{r^2}dt \wedge d\log r,$$

$$'\Omega^0_j = -\frac{4m^2}{r^2}dt \wedge dx_j + \frac{4m^2}{r^2}x_j dt \wedge d\log r + dx_j \wedge d\log r,$$

$$'\Omega^i_0 = -\frac{4m^2 B}{r^2}dt \wedge dx_i + \frac{12m^2}{r^2}Bx_i dt \wedge d\log r - \frac{4m^2}{r^2}dx_i \wedge d\log r,$$

$$'\Omega^i_j = -\Big(B - \frac{1}{r^2}\Big)dx_i \wedge dx_j + \frac{4m^2+1}{r^2}x_i dx_j \wedge d\log r$$
$$-\frac{12m^2}{r^2}x_i x_j dt \wedge d\log r + \frac{4m^2}{r^4}x_j dt \wedge dx_i + \Big(B - \frac{2}{r^2}\Big)x_j dx_i \wedge d\log r.$$

We see that the curvature forms vanish on the curve $r=0$.

REFERENCES

[1] T. OTSUKI, Tangent bundles of order 2 and general connections, Math. J. Okayama Univ., Vol. **8** (1958), 143-179.

[2] T. OTSUKI, On general connections I, Math. J. Okayama Univ., Vol. **9** (1960), 99-164.

[3] T. OTSUKI, On general connections II, Math. J. Okayama Univ., Vol. **10** (1961), 113-124.

[4] T. OTSUKI, On metric general connections, Proc. Japan Acad., Vol. **37** (1961), 183-188.

[5] T. OTSUKI, A note on metric general connections, Proc. Japan Acad., Vol. **38** (1962), 409-413.

[6] T. OTSUKI, On normal general connections, Kōdai Math. Sem. Reports, Vol. **13** (1961), 152-166.

[7] T. OTSUKI, General connection $A\Gamma A$ and the parallelism of Levi-Civita, Kōdai Math. Sem. Reports, Vol. **14** (1962), 40-52.

[8] T. OTSUKI, A construction of space with general connections which have points swallowing geodesics, Math. J. Okayama Univ., Vol. **24** (1982), 157-165.

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