S. SATO
KODAI MATH. J.
8 (1985), 5—13

AN INEQUALITY FOR THE SPECTRAL RADIUS OF MARKOV PROCESSES

BY SADAO SATO

1. Introduction.

Let A be a second-order uniformly elliptic operator in a bounded domain D. Consider the eigenvalue problem

$$Au+\lambda u=0 \tag{1.1}$$

with mixed boundary conditions:

$$\begin{aligned} u&=0 \quad \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n}+\alpha(x)u&=0 \quad \text{on } \Gamma_2, \end{aligned} \tag{1.2}$$

where n stands for the outer normal and $\partial D=\Gamma_1\cup\Gamma_2$. Let λ_0 be the first eigenvalue. When A is symmetric, J. Barta proved that

$$\inf\{-Au/u\}\leqq\lambda_0\leqq\sup\{-Au/u\}, \tag{1.3}$$

where u is any positive C^2-function satisfying the same boundary conditions (1.2) (see [1]).

When A is nonsymmetric, M.H. Protter and H.F. Weinberger [7] proved the left hand of (1.3) for any function u satisfying

$$\begin{aligned} u&>0 \quad \text{on } D\cup\partial D \\ \frac{\partial u}{\partial n}+\alpha(x)u&\geqq 0 \quad \text{on } \Gamma_2. \end{aligned} \tag{1.4}$$

Let $\alpha(x)$ be positive. Then there exists a diffusion process with the generator A whose domain is the collection of C^2-functions satisfying (1.2).

For a Markov process, we can define the spectral radius λ_0 by

$$\lambda_0=\lim_{t\to\infty}-\frac{1}{t}\log\|T_t\|, \tag{1.5}$$

where $\{T_t\}$ is the associated semigroup and $\|T_t\|=\sup_x T_t1(x)$.

Our main purpose is to prove the inequality (1.3) for the spectral radius of a Markov process satisfying some conditions. We will show that the spectral

Received January 31, 1984

radius is equal to the first eigenvalue if the first eigenfunction exists. Thus as a corollary we can see that the inequality (1.3) holds for a nonsymmetric diffusion process. For the proof, the existence of a stationary measure will play a fundamental role.

2. Notations.

Let (P_x, X_t) be a right continuous strong Markov process on a state space S which is a locally compact separable Hausdorff space. Then the resolvent operator G_α of (X_t) is defined by

$$G_\alpha u(x)=E_x\left[\int_0^\sigma e^{-\alpha s}u(X_s)ds\right], \tag{2.1}$$

where u is a bounded measurable function, and σ is the life time of (X_t). Let $\bar{S}$ be the one point compactification of S, and denote

$$\bar{S}=S\cup\{\partial\}. \tag{2.2}$$

In the probabilistic sense, ∂ is called the death point and related to the life time σ by

$$X_t\in S \quad \text{for all} \quad t<\sigma \quad \text{and} \quad X_t=\partial \quad \text{for all} \quad t\geqq\sigma.$$

We define the spaces of real-valued functions with the supremum norm as follows:

$$\begin{aligned} C(S)&=\{u\,;\ u \text{ is bounded continuous on } S\}, \\ C_+(S)&=\{u\in C(S)\,;\ u\geqq 0 \text{ and } u(x)>0 \text{ for some } x\in S\}, \\ B(S)&=\{u\,;\ u \text{ is bounded Borel measurable on } S\}, \\ B_+(S)&=\{u\in B(S)\,;\ u\geqq 0 \text{ and } u(x)>0 \text{ for some } x\in S\}. \end{aligned} \tag{2.3}$$

We also define the spaces of measures on the topological Borel field as follows:

$$\begin{aligned} M(\bar{S})&=\{m\,;\ m \text{ is a bounded Borel measure on } \bar{S}\}, \\ \Pi(\bar{S})&=\{P;\ P \text{ is a probability measure on } \bar{S}\}. \end{aligned} \tag{2.4}$$

In the most of the paper we assume the following conditions.

(A. 1) (X_t) is a Feller process, that is $G_\alpha : C(S)\to C(S)$.

(A. 2) $\lim_{x\to\partial} G_\alpha 1(x)=0$ (if S is non-compact).

If S is compact, we demand $P_x(\sigma<\infty)>0$ for some $x\in S$.

(A. 3) For every non-void open set G in S and $x\in S$, $P_x(\sigma_G<\infty)>0$, where σ_G is the first hitting time for G.

We set $G_\alpha u(\partial)=0$ for every $u\in B(S)$. Under the conditions (A.1) and (A.2), we can regard G_α as the operator on $C(\bar{S})$. We denote by G_α^* the dual operator of G_α on $M(\bar{S})$. Note that the condition (A.2) implies that

(2.5) $$G_\alpha^* m(\partial)=0 \quad \text{for every} \quad m\in M(\bar{S}).$$

and the condition (A.3) implies that

(2.6) $$G_\alpha u>0 \quad \text{for every} \quad u\in C_+(\bar{S})$$
$$(\text{support}(G_\alpha^* m)=S \quad \text{for every} \quad m\in M(S)).$$

LEMMA 2.1. *For every* $u\in B(S)$, *we have*

$$G_\alpha^n u(x)=E_x\left[\int_0^\sigma e^{-\alpha s}s^{n-1}u(X_s)ds\right]\Big/(n-1)\,!\,.$$

Proof. Though this formula is well-known, we give a proof for the convenience. Since $\|G_\alpha u\|\leqq\|u\|/\alpha$, we can define

$$v=\sum_{n=1}^{\infty}\lambda^n G_\alpha^n u \qquad \text{for} \quad |\lambda|<\alpha\,.$$

By the resolvent equation, we can easily see

$$v=\lambda G_{\alpha-\lambda}u\,.$$

Therefore we have

$$\sum_{n=1}^{\infty}\lambda^n G_\alpha^n u=\lambda E_x\left[\int_0^\sigma e^{-\alpha s+\lambda s}u(X_s)ds\right]$$
$$=\sum_{n=1}^{\infty}\lambda^n E_x\left[\int_0^\sigma e^{-\alpha s}s^{n-1}u(X_s)ds\right]\Big/(n-1)\,!\,.$$

3. Spectral radius and Barta's inequality.

At the first we consider the semigroup T_t and the resolvent G_α as the operators on $B(S)$.

Since $\|T_t\|=\sup_{x\in S}\{P_x(t<\sigma)\}$ is submultiplicative in t, there exists the limit

(3.1) $$\lambda_0=\lim_{t\to\infty}-\frac{1}{t}\log\|T_t\|,$$

which will be called the spectral radius of the Markov process (X_t).

THEOREM 3.1.

(3.2) $$\lambda_0=\lim_{n\to\infty}\|G_\alpha^n\|^{-1/n}-\alpha$$
$$=\sup\{\lambda\,;\ \sup_{x\in S}E_x[e^{\lambda\sigma}]<\infty\}\,.$$

Proof. We denote the right hands of (3.2) by λ_G and λ_F respectively. Note that $\lim_{n\to\infty}\|G_\alpha^n\|^{-1/n}$ is the spectral radius of G_α. Therefore $T_\lambda=\sum_{n=1}^\infty(\lambda+\alpha)^{n-1}G_\alpha^n$ is a continuous operator on $B(S)$ for any $\lambda<\lambda_G$. From Lemma 2.1, the norm is given by

$$\|T_\lambda\|=\sup_{x\in S}T_\lambda 1(x)=\sup_{x\in S}\{E_x[e^{\lambda\sigma}]-1\}/\lambda .$$

Thus we have $\lambda_G\leqq\lambda_F$.

If $\lambda<\lambda_F$, we have

$$e^{\lambda t}\|T_t\|=e^{\lambda t}\sup_x P_x(t<\sigma)\leqq\sup_x E_x[e^{\lambda\sigma}]<\infty .$$

This implies $\lambda\leqq\lambda_0$ and so $\lambda_F\leqq\lambda_0$.

If $\lambda<\lambda_0$, we have $\|T_t\|\leqq\exp(-\lambda t)$ for large t. Since

$$\|G_\alpha^n\|\leqq\int_0^\infty e^{-\alpha t}t^{n-1}\|T_t\|dt/(n-1)!\,,$$

we can easily obtain $\lambda\leqq\lambda_G$. Thus the theorem is proved.

COROLLARY 3.2. *The following conditions are equivalent:*

(i) $\lambda_0>0$,
(ii) $\|T_t\|<1$ *for some* $t>0$,
(iii) $\|G_\alpha\|<1/\alpha$ *for some* $\alpha>0$,
(iv) $\sup_x E_x[\sigma]<\infty$.

Remark 1. The expression λ_F is due to A. Friedman. He proved that λ_F is the principal eigenvalue, when (X_t) is a smooth diffusion process and S is a bounded domain in R^n with C^2-boundary (see [3]). Note that the equality (3.2) does not hold for a semigroup on $C(S)$ in general.

THEOREM 3.3. *For any* $u\in B_+(S)$, *we have*

$$\lambda_0\leqq\sup\{u/G_\alpha u\}-\alpha . \tag{3.3}$$

Suppose that u is uniformly positive on S. Then we have

$$\inf\{u/G_\alpha u\}-\alpha\leqq\lambda_0 . \tag{3.4}$$

Proof. Set $\lambda=\sup\{u/G_\alpha u\}$. Then we have $u\leqq\lambda^n G_\alpha^n u$. Thus for some $x\in S$, we have

$$0<u(x)^{1/n}\leqq\lambda\|G_\alpha^n\|^{1/n}\|u\|^{1/n},$$

which proves (3.3). Set $\lambda=\inf\{u/G_\alpha u\}$. If $\lambda=0$, then (3.4) is trivial. If $\lambda>0$, then we have

$$0<(\inf u)\cdot\lambda^n G_\alpha^n 1\leqq\lambda^n G_\alpha^n u\leqq u .$$

Therefore we obtain

$$0<(\inf u)^{1/n}\cdot\lambda\|G_\alpha^n\|^{1/n}\leqq\|u\|^{1/n},$$

which proves (3.4).

Remark 2. By Theorem 3.3, we have shown that the right hand side of the Barta's inequality (1.3) holds for every Markov process. In particular (3.4) implies

$$1/\sup_x E_x[\sigma] \leqq \lambda_0 .$$

However, for the proof of (3.4) for every positive function u, we need the conditions (A. 1)-(A. 3) for the Markov process.

LEMMA 3.4. *Let the conditions* (A. 1) *and* (A. 3) *be satisfied. In order that* λ_0 *be positive, it is necessary and sufficient that*

$$\lim_{x\to\partial}\sup G_\alpha 1(x) < 1/\alpha \tag{3.5}$$

(or $P_x(\sigma<\infty)>0$ *for some* $x\in S$ *if* S *is compact)*

Proof. From Corollary 3.2 the necessity is obvious. For the sufficiency, we must prove $\sup G_\alpha 1<1/\alpha$. Suppose that $\|G_\alpha\|=1/\alpha$. Since $G_\alpha 1$ is continuous, there exists a point $y\in S$ such that $G_\alpha 1(y)=1/\alpha$ by (3.5). Let $k=(\lim\sup_{x\to\partial} G_\alpha 1(x)+\alpha^{-1})/2$ and $G=\{x\,;\ G_\alpha 1(x)<k\}$. By the strong Markov property, we have

$$\alpha^{-1}=G_\alpha 1(y)\leqq \alpha^{-1}P_y(\sigma_G=\infty)+k\ P_y(\sigma_G<\infty)\,,$$

which contradicts to the assumption (A. 3). If S is compact, the above condition implies that $G=\{x\,;\ G_\alpha 1(x)<\alpha^{-1}-\varepsilon\}$ is a nonvoid open set for some $\varepsilon>0$. If $\|G_\alpha\|=\alpha^{-1}$, then we have for some y

$$\alpha^{-1}=G_\alpha 1(y)\leqq \alpha^{-1}P_y(\sigma_G=\infty)+(\alpha^{-1}-\varepsilon)P_y(\sigma_G<\infty)\,,$$

which completes the proof.

LEMMA 3.5. *If* λ_0 *is positive, then we have*

$$\sup_x E_x[\exp(\lambda_0\sigma)]=+\infty\,. \tag{3.6}$$

Under the conditions (A. 1) *and* (A. 3), *we have*

$$\lambda_0<+\infty\,. \tag{3.7}$$

Proof. Define

$$T_\lambda=\int_0^\infty dt\,\exp(\lambda t)T_t\,.$$

Then we have

$$\|T_\lambda\|=(\sup_x E_x(e^{\lambda\sigma})-1)/\lambda\,.$$

Suppose that $\sup E_x[\exp(\lambda_0\sigma)]$ be finite. Then T_{λ_0} is a bounded operator. Since

$T_{\lambda_0+\varepsilon}=\sum_{n=1}^{\infty}\varepsilon^{n-1}T_{\lambda_0}^n$, $T_{\lambda_0+\varepsilon}$ is bounded for $0<\varepsilon<1/\|T_{\lambda_0}\|$. However this means $\sup E_x[\exp((\lambda_0+\varepsilon)\sigma)]<+\infty$, which is a contradiction. Let (A. 1) and (A. 3) be satisfied. Let u be a continuous function with compact support. From (2.6) and Theorem 3.3, we obtain (3.7).

By Lemma 3.4 and 3.5 we know that λ_0 is a finite positive number under the conditions (A. 1)-(A. 3). Then the Green operator $G=G_0$ is continuous operator on $B(S)$ (or $C(\bar{S})$). In the following, we use G instead of G_α.

THEOREM 3.6. *Assume that the conditions* (A. 1)-(A. 3) *be satisfied. Then there exists a probability measure P on S such that*

$$P=\lambda_0 G^*P\,, \tag{3.8}$$

where G^* is the dual operator of G.

Proof. For $m\in M(\bar{S})$, we define

$$K_\lambda m=\sum_{n=0}^{\infty}\lambda^n G^{*n}m\,.$$

If $\lambda<\lambda_0$, we have

$$K_\lambda m=m+\lambda G^*K_\lambda m \tag{3.9}$$

and

$$K_\lambda m(\bar{S})=\int E_x[e^{\lambda\sigma}]dm(x)\,. \tag{3.10}$$

From Lemma 3.5, we can take the sequences $\{x_n\}$ and $\{\lambda_n\}$ such that $\lambda_n\uparrow\lambda_0$ and

$$a_n=E_{x_n}[\exp(\lambda_n\sigma)]\to+\infty \qquad \text{as} \quad n\to\infty\,. \tag{3.11}$$

Let m_n be the Dirac measure $\delta(x_n)$, and put

$$P_n=K_{\lambda_n}m_n/a_n\,.$$

From (3.9) and (3.10), we have $P_n\in\Pi(\bar{S})$ and

$$P_n=\lambda_n G^*P_n+m_n/a_n\,. \tag{3.12}$$

Since $\Pi(\bar{S})$ is compact in the weak*-topology, we can take a subsequence of $\{P_n\}$ which converges to some element P of $\Pi(\bar{S})$. From (3.11) and (3.12) P must satisfy (3.8). By (2.5) P is a probability measure on S. The theorem is proved.

Remark 3. For the existence of the above P, the condition (3.5) is not sufficient. To see this, consider the semigroup $e^{-kt}T_t$, where (T_t) is a conservative semigroup. Then $\lambda_0=k$ and from (3.8) P must be a finite invariant measure. However it does not exist in general.

In the remainder of this paper, we always assume the conditions (A. 1)-(A. 3), and P denotes the above probability measure.

THEOREM 3.7. *We have*

$$\inf\{u/Gu\} \leqq \lambda_0 \leqq \sup\{u/Gu\} \qquad \text{for every} \quad u \in C_+(\bar{S}). \tag{3.13}$$

Proof. Set $\lambda = \inf\{u/Gu\}$. Since $u \geqq \lambda Gu$, we have

$$\lambda_0 \int Gu\, dP = \int u\, dP \geqq \lambda \int Gu\, dP.$$

By (2.6), we obtain $\lambda_0 \geqq \lambda$. Similarly we can get the right hand inequality.

Remark 4. Since $A = -G^{-1}$, (3.13) is identical to (1.3). For the left hand inequality, we have

$$\sup_{u \in C_+(\bar{S})} \inf (u/Gu) = \lambda_0. \tag{3.14}$$

To see this, let $u = E_x[e^{\lambda \sigma}]$ for $\lambda < \lambda_0$. Then we have $u = \lambda Gu + 1$, and so $u \geqq \lambda Gu$, which proves (3.14).

Now we study the connection between λ_0 and the first eigenvalue.

DEFINITION. A bounded continuous complex valued function u is called an eigenfunction if it is nontrivial and satisfies

$$u = \lambda Gu, \tag{3.15}$$

where λ is some complex number which we call an eigenvalue.

THEOREM 3.8.
(i) *If there exists a nonnegative eigenfunction, then the eigenvalue is* λ_0.
(ii) *Suppose that* λ_0 *is an eigenvalue. Then problem* (3.15) *has a unique normalized nonnegative eigenfunction. The eigenvalue* λ_0 *has the smallest real part of all eigenvalues and is simple.*

Proof. (i) is clear from (3.13). Let λ be a complex number and $T_\lambda = \int_0^\infty dt \exp(\lambda t) T_t$. By the definition of λ_0, T_λ is bounded if $\mathrm{Re}(\lambda) < \lambda_0$. Therefore, if λ is an eigenvalue then we have $\mathrm{Re}(\lambda) \geqq \lambda_0$. Let $u = \lambda_0 Gu$. Since λ_0 is real, we can assume that u is a real function. Let $u^+ = \max(u, 0)$. We can assume that u^+ is nontrivial. Then we have

$$\lambda_0 Gu^+ \geqq \lambda_0 Gu = u.$$

Thus we get $\lambda_0 Gu^+ \geqq u^+$. On the other hand, by virtue of Theorem 3.6, we have

$$\int \lambda_0 G u^+ dP = \int u^+ dP,$$

which implies $\lambda_0 G u^+ = u^+$ by (2.6). From (2.6), u^+ is positive on S and so $u^+ = u$. If v is another eigenfunction, we set

$$w = u\int v\, dP - v\int u\, dP.$$

Then w is also an eigenfunction and $w \geqq 0$ by the above argument and we have

$$\int w\, dP = 0,$$

which implies $w=0$. The uniqueness is proved.

Recall that if λ_0 is not simple, there exists a natural number $n \geqq 2$ such that

$$(\lambda_0 G - I)^n u = 0 \quad \text{and} \quad (\lambda_0 G - I)^{n-1} u \neq 0,$$

where I is the identity operator. Set $v = (\lambda_0 G - I)^{n-1} u$ and $w = (\lambda_0 G - I)^{n-2} u$. Then v is an eigenfunction. On the other hand, we have

$$\int v\, dP = \int \lambda_0 G w\, dP - \int w\, dP = 0,$$

which is a contradiction. The theorem is proved.

Remark 5. The existence of the positive eigenfunction can be found in M. A. Krasnosel'skii [6] for the smooth diffusion process in a bounded domain with smooth boundary. The uniqueness and the simplicity of the first eigenfunction are also proved in it by a different manner.

REFERENCES

[1] BANDLE, M., Isoperimetric inequalities and applications, Boston-London-Melbourne, Pitman 1980.

[2] DONSKER, M. D. AND S. R. S. VARADHAN, On the principal eigenvalue of second-order elliptic differential operators, Comm. Pure Appl. Math. XXIX, 595-621 (1976).

[3] FRIEDMAN, A., Stochastic differential equations and applications, vol. 2, New York-London, Academic Press 1976.

[4] GIHMAN, I. I. AND A. V. SKOROHOD, The theory of stochastic processes II, Berlin-Heidelberg-New York, Springer 1975.

[5] KAC, M., On some connections between probability theory and differential and integral equations, Proc. 2nd Berkeley Symp. Math. Statist., Probability, 189-215 (1951).

[6] KRASNOSEL'SKII, M. A., Positive solutions of operator equations, Groningen, Noordhoff 1964.

[7] PROTTER, M. H. AND H. F. WEINBERGER, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, N. J., 1967.

Department of Mathematics
Faculty of Engineering
Tokyo Electrical Engineering College
Kanda-Nishikicho, Chiyoda-ku, Tokyo 101
Japan