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A TAUBERIAN THEOREM FOR CERTAIN CLASS OF MEROMORPHIC FUNCTIONS

Dedicated to Professor M. Ozawa on the occation of his 60th birthday

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§1. Introduction.

Let $f(z)$ be meromorphic in the plane. We define $m_2(r, f)$ by

$$m_2(r, f)^2 = \frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)^2 d\theta,$$

and denote by $N(r, c)$ the usual Nevanlinna counting function for the c-points of f in $|z| \leqq r$, then Miles and Shea had shown

$$(1) \qquad K_2(f) \equiv \limsup_{r\to\infty}\frac{N(r, 0)+N(r, \infty)}{m_2(r, f)} \geqq \frac{|\sin\pi\rho|}{\pi\rho}\left\{\frac{2}{1+\sin 2\pi\rho/2\pi\rho}\right\}^{1/2} \equiv C(\rho)$$

for $\rho \in [\mu_*(T(r, f)), \lambda_*(T(r, f))]$.
Further they had characterized those f for which equality holds in (1) as functions which are locally Lindelöffian (or the reciprocals of such).

Let M_p be the class of all meromorphic functions $f(z)$ of order ρ defined by $g(z)/g(-z)$ with the canonical product

$$g(z) = \prod_{n=1}^{\infty} E(z/a_n, q), \qquad q = [\rho].$$

Recently by making use of Fourier series method, Ozawa proved

THEOREM A. *Let $f(z)$ belongs to M_p, then*

$$(2) \qquad \limsup_{r\to\infty}\frac{N(r, 0)}{m_2(r, f)} \geqq \frac{\sqrt{2}}{\sqrt{\pi\rho}}\frac{|\cos\pi\rho/2|}{\{\pi\rho-\sin\pi\rho\}^{1/2}} \equiv B(\rho).$$

It is natural to hope that (2) holds for $\rho \in [\mu_*, \lambda_*]$ and that those f for which equality holds in (2) are $f(z) = g(z)/g(-z)$ with locally Lindelöffian g. But when ρ is an even integer, $B(\rho) > 0$ and the proof is not straightforward. We need some existence lemma of strong peaks for $f \in M_\rho$.

We assume that the reader is familiar with the fundamental concept of Nevanlinna theory and Fourier series method developed by Miles and Shea (See

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W. K. Hayman [3], Miles and Shea [5], [6] and Ozawa [7]). We use the terminology from [6] without comment.

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§ 2. Discussion of results.

Our first result is following.

THEOREM 1. *Let $f(z)$ be meromorphic in the plane and defined by $f(z)=g(z)/g(-z)$ with an entire function g. Then*

$$\limsup_{r\to\infty}\frac{N(r,0,f)}{m_2(r,f)}\geqq B(\rho) \tag{2.1}$$

for $\rho\in[\mu_*(T(r,f)), \lambda_*(T(r,f))]$, $\rho>0$.

Next we have

THEOREM 2. *Under the same assumption as in theorem 1 and if*

$$\limsup_{r\to\infty}\frac{N(r,0,f)}{m_2(r,f)}=B(\rho) \tag{2.2}$$

for some $\rho\in[\mu_*(T(r,f)), \lambda_*(T(r,f))]$, $\rho\neq$ an odd integer.

Then there exist positive sequences $r_n\to\infty$ and $\eta_n\to 0$ such that

$$N(r,0)\sim N(r_n,0)(r/r_n)^{\rho}, \tag{2.3}$$

$$N(r,0)\sim B(\rho)m_2(r,f), \tag{2.4}$$

uniformly for $r\in[\eta_n r_n, \eta_n^{-1}r_n]$ as $n\to\infty$. Further there exist $\delta_n\to 0$ and $\theta_n\in[0,2\pi)$ such that if

$$S_n=\{z:\ \delta_n\leqq\arg z-\theta_n\leqq 2\pi-\delta_n\},$$

then

$$N(r,0;S_n)=o(N(r,0,f)),\qquad \eta_n r_n\leqq r\leqq\eta_n^{-1}r_n \tag{2.5}$$

as $n\to\infty$, where $N(r,0;S_n)$ denote the counting function for the number of zeros of f in the sector S_n.

If (2.2) holds with $\rho=$ an odd integer, i.e. $N(r,0)=o(m_2(r,f))$ as $r\to\infty$, then $\rho=\mu_*=\lambda_*$ and

$$m_2(r,f)=r^{\rho}L(r),\qquad \lim_{r\to\infty}\frac{L(\sigma r)}{L(r)}=1\quad(0<\sigma<\infty) \tag{2.6}$$

holds.

Theorem 1 and 2 have extensions.

THEOREM 3. *Let $f(z)$ be meromorphic in the plane defined by $f(z)=g(z)g(e^{i\pi a}z)$ with an entire function g and $0<a\leqq 1$. Then*

$$\limsup_{r\to\infty}\frac{N(r,0,f)}{m_2(r,f)}\geqq B(a,\rho)\equiv\left\{2\sum_{m=-\infty}^{+\infty}(1-\cos ma)\frac{4}{m^2-\rho^2}\right\}^{-1/2} \tag{2.7}$$

for $\rho\in[\mu_*(T(r,f)),\lambda_*(T(r,f))]$, $\rho\neq 0$.

THEOREM 4. *Under the same assumption as in theorem 3 and if equality holds in (2.7) for some $\rho\in[\mu_*(T(r,f)),\lambda_*(T(r,f))]$, $\mu\neq 0$ and $B(a,\rho)=0$. Then there exist sequences $r_n\to\infty$, $\eta_n\to 0$, $\delta_n\to 0$ and $\theta_n\in[0,2\pi]$ satisfying (2.2)-(2.5). If ρ satisfies $B(a,\rho)=0$ and $\rho>0$, then $\rho=\mu_*=\lambda_*$ and (2.6) holds.*

Especially, if $a=1$, then we have theorem 2. Proofs of theorem 3 and 4 are quite similar as to theorem 1 and 2. It will be done by improving the lemma 3 and be left to the reader.

Theorem 1 and 3 are not new, essentially they were proved by Ozawa ([7] theorem 4 and its extension in § 11).

§ 3. Preliminaries.

To prove (2.1) we need some lemmas.

LEMMA 1. *Let $f(z)$ be meromorphic in the plane defined by $f(z)=g(z)/g(-z)$ with an entire function g. Put a_n be zeros of g and $W(z)$ by*

$$\log|f(z)|=\sum_{s<|a_n|\leqq R}\log\frac{|E(z/a_n,q)|}{|E(-z/a_n,q)|}+W(z) \tag{3.1}$$

where $0<2s\leqq|z|=r\leqq R/2$. Then if $q\geqq 1$.

$$|W(z)|\leqq V_q(s,r,R)\equiv A[(r/s)^{q_0-1}\{m_2(s,g)+N(2s,0)\}+(r/R)^{q_0+1}\{m_2(R,g)+N(2R,0)\}], \tag{3.2}$$

where A is an absolute constant and $q_0=2[(q+1)/2]$; if $q=0$,
$|W(z)|\leqq V_0(s,r,R)\equiv A\{N(2s,0)\log(r/s)+(r/R)(m_2(R,g)+N(2R,0))\}$.

Proof. According to the proof of theorem 3.b in [2],

$$W(z)=\mathrm{Re}\left\{\sum_{m=1}^{q}d_m(s)z^m+\sum_{m=q+1}^{\infty}d_m(R)z^m+\log\prod_{|a_n|\leqq s}\frac{|1-z/a_n|}{|1+z/a_n|}\right\}$$

where

$$d_m(t)=\frac{1}{\pi}\int_0^{2\pi}\log|f(te^{i\theta})|t^{-m}e^{-im\theta})d\theta$$
$$+\frac{1}{mt^m}\sum_{|a_n|\leqq t}\{(\bar{a}_n/t)^m-(-\bar{a}_n/t)^m\},\ t\in[s,R].$$

Hence we have $d_{2m}(t)=0$ and

$$\frac{1}{4}|d_{2p+1}(t)|\leqq\frac{|c_{2p+1}(t,\ g)|}{t^{2p+1}}+\frac{n(t,\ 0)}{2(2p+1)t^{2p+1}}.$$

Next we have

$$\left|\sum_{|a_n|\leqq s}\log|(1-z/a_n)/(1+z/a_n)|\right|\leqq 2N(s,\ 0)+2\frac{n(s,\ 0)}{2q+1}(r/s)^q.$$

Consequently

$$|W(z)|\leqq\sum_{p=0}^{[(q+1)/2]-1}(r/s)^{2p+1}\left\{4m_2(s,\ g)+2\frac{n(s,\ 0)}{2p+1}\right\}$$

$$+\sum_{[(q+1)/2]}^{\infty}(r/R)^{2p+1}\left\{4m_2(R,\ g)+\frac{2n(R,\ 0)}{2p+1}\right\}$$

$$+2N(s,\ 0)+\frac{2n(s,\ 0)}{q}(r/s)^q,$$

and hence we have the desired result when $q\geqq 1$.

If $q=0$, we have

$$W(z)=\mathrm{Re}\left\{\sum_{m=1}^{\infty}d_m(R)z^m+\log\prod_{|a_n|\leqq s}(1-z/a_n)/(1+z/a_n)\right\}$$

and

$$\log\prod_{|a_n|\leqq s}|(1-z/a_n)(1+z/a_n)|\leqq 2N(s,\ 0)+n(s,\ 0)(1+\log r/s),$$

and this completes the proof of lemma 1.

LEMMA 2. *Under the same assumption as in lemma* 1, *we have*

$$m_2(r,\ f)\leqq K_q r^{q_0+1}\int_s^R\frac{N(t,\ 0,\ g)}{t^{q_0}(t+r)^2}dt+BV_q(s,\ r,\ R), \tag{3.4}$$

where K_q and B are constants depending only on $q\geqq 0$.

Proof. By lemma 1,

$$\log|f(z)|=\log\prod_{s<|a_n|\leqq R}|E(z/a_n,\ q)/E(-z/a_n,\ q)|+W(z),$$

hence we have by Minkowski's inequality,

$$m_2(r,\ f)\leqq\sum_{s<|a_n|\leqq R}m_2(r/|a_n|,\ E(z,\ q)/E(-z,\ q))+V_q(s,\ r,\ R). \tag{3.5}$$

For $q\geqq 1$, we have by calculating the m-th Fourier coefficients

$$(3.6)\qquad m_2(r, G)^2=\begin{cases} 2\sum_{k=q_0}^{\infty}\dfrac{r^{4k+2}}{(2k+1)^2}, & r<1 \\ 2\sum_{k=0}^{q_0-1}\left\{\dfrac{r^{4k+2}}{(2k+1)^2}-\dfrac{2}{(2k+1)^2}\right\}+2\sum_{k=0}^{\infty}\dfrac{r^{-(4k+2)}}{(2k+1)^2}, & r\geqq 1\end{cases}$$

and if $q=0$

$$(3.7)\qquad m_2(r, G)^2=\begin{cases} \sum_{k=0}^{\infty}\dfrac{r^{4k+2}}{(2k+1)^2}, & r<1 \\ \sum_{k=0}^{\infty}\dfrac{r^{-(4k+2)}}{(2k+1)^2}, & r\geqq 1\end{cases}$$

where $G(z)=G(z, q)\equiv E(z, q)/E(-z, q)$ and $q_0=2[(q+1)/2]$. Hence we obtain from (3.6) and (3.7)

$$(3.8)\qquad m_2(r, G)\leqq\begin{cases} 2r^{q_0+1}, & r<1 \\ 2r^{q_0-1}, & r\geqq 1.\end{cases}$$

Thus we have from (3.5) and (3.8),

$$(3.9)\qquad m_2(r, f)\leqq 2\int_s^r (r/t)^{q_0-1}d(n(t, 1/g)-n(s, 1/g))$$
$$+2\int_r^R (r/t)^{q_0+1}d(n(t, 1/g)-n(r, 1/g))+V_q(s, r, R).$$

Integration by parts applied twice to (3.9) yields (3.4).

LEMMA 3. *Let g be a entire function and put f by $f(z)=g(z)/g(-z)$. Suppose $\mu_*(m_2(r, f))<\infty$ and $K_2(f)<\infty$.*

If $\mu_*(m_2(r, f))<\rho<\lambda_*(m_2(r, f))$ and ρ is not an odd integer, then there exist sequences s_n, r_n and R_n tending to ∞ and $\xi_n\to 0$ such that

$$(3.10)\qquad s_n=o(r_n),\qquad r_n=o(R_n)\qquad \text{as}\quad n\to\infty,$$

$$(3.11)\qquad N(t, 0)\leqq N(r_n, 0)(t/r_n)^{\rho}\qquad s_n\leqq t\leqq R_n$$

$$(3.12)\qquad \begin{aligned} m_2(t, f)&\leqq \xi_n N(r_n, 0)(s_n/r_n) && s_n\leqq t\leqq 2s_n \\ m_2(t, f)&\leqq \xi_n N(r_n, 0)(R_n/r_n) && R_n\leqq t\leqq 2R_n.\end{aligned}$$

If $\mu_*(m_2(r, f))=\lambda_*(m_2(r, f))$ and $\mu_*(m_2(r, f))$ is not an odd integer, then

$$\liminf_{r\to\infty}\frac{N(r, 0)}{m_2(r, f)}>0.$$

Proof. We first observe that there exist sequences s_n, r_n, R_n and A_n tending to ∞ and $\delta_n\to 0$, such that

$$(3.13)\qquad s_n=o(r_n),\qquad t_n=o(R_n)\qquad \text{as}\quad n\longrightarrow\infty$$

$$(3.14)\quad \begin{aligned} &m_2(t, f) \leqq m_2(t_n, f)(t/t_n), && s_n \leqq t \leqq 2R_n, \\ &m_2(t, f) \leqq \delta_n m_2(t_n, f)(t/t_n), && s_n \leqq t \leqq A_n s_n \quad \text{or} \quad R_n/A_n \leqq t \leqq 2R_n. \end{aligned}$$

To see this, choose $\varepsilon > 0$ so that $\mu_* < \rho - \varepsilon$, $\rho + \varepsilon < \lambda_*$, then there exist x_n, y_n and A_n tending to ∞ and $\gamma_n \to 0$ such that

$$(3.15)\quad \begin{aligned} &m_2(t, f) \leqq (1+\gamma_n) m_2(x_n, f)(t_n/x_n)^{\rho+\varepsilon}, && A_n^{-2} x_n \leqq t \leqq A_n^2 x_n \\ &m_2(t, f) \leqq (1+\gamma_n) m_2(y_n, f)(t/y_n)^{\rho-\varepsilon} && A_n^{-2} y_n \leqq t \leqq 2A_n^2 y_n. \end{aligned}$$

And we may assume $A_n^2 x_n < A_n^{-2} y_n$. Choose $t_n \in [A_n^{-1} x_n, A_n y_n]$ so that

$$m_2(t_n, f) t_n^{-\rho} \geqq m_2(t, f) t^{-\rho} \qquad A_n^{-1} x_n \leqq t \leqq A_n y_n.$$

Then

$$\begin{aligned} m_2(t, f) &< (1+\gamma_n)(t/x_n)^{\varepsilon}(t/x_n)^{\rho}(x_n/t_n)^{\rho} m_2(t_n, f) \\ &\leqq \delta_n (t/t_n)^{\rho} m_2(t_n, f), \qquad A_n^{-2} x_n \leqq t \leqq A_n^{-1} x_n, \\ &\qquad\qquad\qquad\qquad\qquad\quad A_n y_n \leqq t \leqq 2A_n^2 y_n. \end{aligned}$$

Thus (3.13) and (3.14) hold with $s_n = A_n^{-2} x_n$ and $R_n = A_n^2 y_n$.
Choose $r_n \in [s_n, 2R_n]$ so that

$$N(r_n, 0) r_n^{-\rho} \geqq N(t, 0) t^{-\rho}, \qquad s_n \leqq t \leqq 2R_n.$$

By lemma 2 and $K_2(f) < \infty$

$$m_2(t_n, f) \leqq K_q t_n^{q_0+1} \int_{s_n}^{R_n} \frac{N(t, 0)}{t^{q_0}(t_n+t)^2} dt + o(m_2(t_n, f)).$$

Hence

$$(1+o(1)) m_2(t_n, f) \leqq K_q N(r_n, 0)(t_n/r_n)^{\rho} \int_0^{\infty} \frac{u^{\rho}}{u^{q_0}(u+1)^2} du.$$

Since $|\rho - q_0| < 1$, the integral is convergent. We have

$$m_2(t_n, f) \leqq (1+o(1)) \hat{K}_\rho N(r_n, 0)(t_n/r_n)^{\rho} \qquad \text{as} \quad n \to \infty.$$

Thus for n large enough,

$$(3.16)\quad m_2(t, f) \leqq \delta_n m_2(t_n, f)(t/t_n)^{\rho} \leqq 2\hat{K}_\rho \delta_n N(r_n, 0)(t/r_n)^{\rho},$$
$$s_n \leqq t \leqq A_n s_n, \quad A_n^{-1} R_n \leqq t \leqq 2R_n.$$

Next we note from (3.14), (3.16) and $K_2(f) < \infty$ that

$$r_n \in [A_n s_n, A_n^{-1} R_n] \qquad (n \geqq n_0),$$

and (3.10) follows.

To prove last assertion of lemma 3 suppose first that for any $\sigma>1$, there exist a sequence $\tau_k\to\infty$ such that $\tau_k=N(\sigma t_k)/m_2(t_k, f)\to 0$. Let $q_0=2[(\rho+1)/2]$, and use

$$m_2(r, f)\leqq K_{q_0}\int_0^\infty \frac{r^{q_0+1}N(t, 0)}{t^{q_0}(t+r)^2}dt,$$

where K_{q_0} is a constant depending only on q_0.

Since $\mu_*(m_2)=\lambda_*(m_2)=\rho$, given $\varepsilon>0$ their exist $A=A(\varepsilon)$ and $x_0=x_0(\varepsilon)$ such that for any $x\geqq x_0$, there is a peak $y\in[x, Ax]$:

$$\begin{aligned} &m_2(t, f)\leqq m_2(y, f)(t/y)^{\rho-\varepsilon} \qquad (x_0\leqq t\leqq y), \\ &m_2(t, f)\leqq m_2(y, f)(t/y)^{\rho+\varepsilon} \qquad (y\leqq t<\infty). \end{aligned} \tag{3.17}$$

(See [1], p. 410 and [6], p. 178). Choose $\varepsilon>0$ so that $\rho+\varepsilon<q_0+1$ and $q_0-1<\rho-\varepsilon$. Then for all large k there exist peaks $y_k\in[t_k/A, t_k]$; if $s_k\in(x_0, y_k)$,

$$\begin{aligned} m_2(y_k, f)\leqq &\left\{BK_2(f)\int_{x_0}^{s_k}m_2(t, f)+\int_{s_k}^{\sigma t_k}N(\sigma t_k, 0)\right. \\ &\left.+BK_2(f)\int_{\sigma t_k}^\infty m_2(t, f)\right\}\frac{y_k^{q_0+1}}{t^{q_0}(t+y_k)^2}dt+O(y_k^{q_0-1}) \end{aligned}$$

$$\begin{aligned} 1\leqq &BK_2(f)\int_{y_k/s_k}^{y_k/x_0}\frac{u^{q_0-\rho+\varepsilon}}{(1+u)^2}du+\tau_k A^{\rho+\varepsilon}\int_{y_k/\sigma t_k}^{y_k/s_k}\frac{u^{q_0}}{(1+u)^2}du \\ &+BK_2(f)\int_0^{y_k/\sigma t_k}\frac{u^{q_0-\rho-\varepsilon}}{(1+u)^2}du+o(1). \end{aligned}$$

We determine the s_k so that $s_k\to\infty$, $y_k/s_k\to\infty$ and

$$\tau_k\int_0^{y_k/s_k}\frac{u^{q_0}}{(1+u)^2}du\to 0$$

Then since $y_k\leqq t_k\leqq Ay_k$,

$$1\leqq BK_2(f)\int_0^{1/\sigma}\frac{u^{q_0-\rho-\varepsilon}}{(1+u)^2}du+o(1) \qquad \text{as} \quad k\to\infty,$$

a contradiction if σ has been chosen large enough. Thus $m_2(r, f)\leqq C_1N(\sigma r, 0)$. Since $K_2(f)<\infty$, we have $\mu_*(N)=\mu_*(m_2)=\lambda_*(N)=\lambda_*(m_2)=\rho$. In particular $N(\sigma r, 0)\leqq C_2N(r, 0)$, and we have

$$\liminf_{r\to\infty}\frac{N(r, 0)}{m_2(r, f)}>0.$$

§4. Proof of Theorems 1 and 2.

We may assume $K_2(f)<\infty$, since otherwise (2.1) is trivial. Hence we have $\mu_*(m_2)=\mu_*(T)$ and $\lambda_*(m_2)=\lambda_*(T)$.

Let $\rho \in [\mu_*, \lambda_*](\rho>0)$ be not odd and choose $a=a(\rho)\in(0, e^{-1})$ by

$$(4.1)\qquad 1-\log a>\rho^{-1}, \qquad (ae)^{\rho}(1-\log a)<1.$$

Let $q_0=2[(\rho+1)/2]$ and put $f_n(z)$ by

$$f_n(z)=\prod_{s_n<|a_n|\leqq aR_n} E(z/a_n, q^0)/E(-z/a_n, q_0)$$

where s_n, r_n and R_n satisfy (3.10), (3.12) and $\gamma_n\to 0$

$$(4.2)\qquad N(t, 0)\leqq(1+\gamma_n)N(r_n, 0)(t/r_n)^{\rho}, \qquad s_n\leqq t\leqq R_n.$$

Define associated functions $G_n(z)$ and $F_n(z)$ by

$$G_n(z)=\prod_{s_n<|a_n|\leqq aR_n} E(z/|a_n|, q_0),$$

$$F_n(z)=G_n(z)/G_n(-z).$$

Put $N_n(t, 0)=N(t, 1/F_n)$ so that by (3.12) and (4.2)

$$(4.3)\qquad N_n(r_n, 0)\sim N(r_n, 0) \qquad \text{as} \quad n\to\infty.$$

$$(4.4)\qquad N_n(t, 0)\leqq(1+\gamma_n)N(r_n, 0)(t/r_n)^{\rho}, \qquad 0<t<\infty.$$

We apply lemma 1 on $|z|=r_n$, and obtain

$$\log|f(z)|=\log|f_n(z)|+o(N(r_n, 0)), \qquad \text{as} \quad n\to\infty.$$

Since $m_2(r, f)\leqq m_2(r, F_n)$, we have

$$(4.5)\qquad m_2(r_n, f)\leqq(1+o(1))m_2(r_n, F_n), \qquad \text{as} \quad n\to\infty.$$

Let

$$L_n(z)=\prod_{k=1}^{\infty} E(z/d_k, q_0)/E(-z/d_k, q_0)$$

be the meromorphic function with positive zeros d_k and negative poles $-d_k$ satisfying

$$n(t, 0)=[\rho(t/r_n)^{\rho}N(r_n, 0)] \qquad 0<t<\infty.$$

Then for each $n\geqq 1$

$$(4.6)\qquad \begin{aligned} &N(t, 1/L_n)\leqq(t/r_n)^{\rho}N(r_n, 0) \qquad 0<t<\infty, \\ &N(t, 1/L_n)\sim(t/r_n)^{\rho}N(r_n, 0) \qquad \text{as} \quad t\to\infty. \end{aligned}$$

and

$$(4.7)\qquad \begin{aligned} &c_{2p}(r_n, L_n)=0 \\ &|c_{2p+1}(r_n, L_n)|\sim N(r_n, 0)\frac{\rho^2}{|(2p+1)^2-\rho^2|} \qquad \text{as} \quad n\to\infty, \end{aligned}$$

uniformly in p. Hence

$$|c_m(r_n, F_n)| \leqq (1+o(1))|c_m(r_n, L_n)| \quad \text{as} \quad n\to\infty, \tag{4.8}$$

uniformly in m. We deduce

$$m_2(r_n, F_n) \leqq (1+o(1))m_2(r_n, L_n) = (1+o(1))N(r_n, 0)B(\rho)^{-1}$$

and thus (2.1) follows (See [4] p. 185).

Proof of theorem 2. Let $\rho>0$ satisfy (2.2) and be not odd. Then by the proof of theorem 1 there exist meromorphic f_n and associated G_n, F_n, and L_n satisfying (3.10), (3.12) and (4.2). Let $M>1$ be large and suppose that there exist $x_n\in[r_n, Mr_n]$ and $\sigma\in(0, 1)$ such that

$$N(x_n, 0) < \sigma^2 N(r_n, 0)(x_n/r_n)^{\rho} \qquad \text{for infinitely many} \quad n.$$

Then

$$\limsup_{n\to\infty} \frac{|c_{q_0+1}(r_n, L_n)|}{|c_{q_0+1}(r_n, F_n)|} > 1,$$

a contradiction. We conclude

$$N(x, 0) = (1+o(1))N(r_n, 0)(x/r_n)^{\rho}, \qquad r_n \leqq x \leqq Mr_n, \tag{4.9}$$

uniformly as $n\to\infty$. Thus by lemma 1 and lemma 3

$$\log|f(z)| = \log|f_n(z)| + o(m_2(r, f)), \qquad r_n \leqq |z| = r \leqq Mr_n,$$

uniformly as $n\to\infty$ and we have

$$m_2(r, f) \leqq (1+o(1))m_2(r, F_n) \leqq (1+o(1))B(\rho)^{-1}N(r, 0) \tag{4.10}$$

uniformly on $r_n \leqq |z| = r < Mr_n$ as $n\to\infty$; by (2.2) equality holds throughout in (4.10).

Now by (4.8) there exist ε_n tending to 0 with

$$|c_m(r_n, f_n)| > (1-\varepsilon_n)|c_m(r_n, F_n)| \tag{4.11}$$

for $m=q_0+1$, q_0+3. By lemma (2.2) of [6], there exist $\delta_n\to 0$ and $\phi_n, \psi_n\in [0, 2\pi)$ such that if

$$\hat{I}_n = \bigcap_{j=0}^{q_0} S(\phi_n + 2j\pi/(q_0+1), \delta_n) \qquad \tilde{I}_n = \bigcap_{j=0}^{q_0+2} S(\psi_n + 2j\pi/(q_0+3), \delta_n) \tag{4.12}$$

$$\hat{G}_n(z) = \prod_{\substack{s_n < |a_\nu| \leqq aR_n \\ a_\nu \in \hat{I}_n}} E(z/a_n, q_0) \qquad \tilde{G}_n(z) = \prod_{\substack{s_n < |a_\nu| \leqq aR_n \\ a_\nu \in \tilde{I}_n}} E(z/a_n, q_0) \tag{4.13}$$

and put $\hat{F}_n(z) = \hat{G}_n(z)/\hat{G}_n(-z)$ and $\tilde{F}_n(z) = \tilde{G}_n(z)/\tilde{G}_n(-z)$, then

$$\begin{aligned} &|c_{q_0+1}(r_n, F_n)| < \sqrt{\varepsilon_n}\,|c_{q_0+1}(r_n, F_n)|, \\ &|c_{q_0+3}(r_n, F_n)| < \sqrt{\varepsilon_n}\,|c_{q_0+1}(r_n, F_n)|. \end{aligned} \tag{4.14}$$

One consequence of (4.14) is

$$N(t, 0, F_n)+N(t, 0, F_n) \leqq M^{-\rho/8} N(t, 0, L_n) \qquad M^{3/4} r_n \leqq t \leqq M r_n . \tag{4.15}$$

And (4.15) shows (2.5). (See [6] p. 183).

If (2.2) holds for $\rho = a$ positive odd integer, the proof of (2.6) is quite similar to (14) of [6] and will be left to the reader.

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