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ON STRONG NORMALITIES

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In the paper [1], the author asks for an example in a complete K-metric space where K is a strongly normal cone of a reflexive infinite dimensional Banach space. Our main purpose is to present such an example.

Let V be a normed space. A set $K \subset V$ is said to be a cone if and only if

(1) K is closed;

(2) If $u, v \in K$, then $au+bv \in K$ for all $a, b \geqq 0$;

(3) $K \cap (-K)=\{\theta\}$ where θ is the zero of the space V, and

(4) $K^0=\emptyset$ where K^0 is the interior of K.

We say $u \geqq v$ if and only if $u-v \in K$. The cone K is said to be strongly normal if there is $c>0$ such that if $z=\sum_{i=1}^{n} b_i x_i$, $x_i \in K$, $\|x_i\|=1$, $\sum_{i=1}^{n} b_i=1$, $b_i \geqq 0$ implies $\|z\|>c$. The mapping $\phi: K \to K$ is said to be lower semicontinuous if $\{u_n\}$ and $\{\phi(u_n)\}$ are both weakly convergent, then $\lim \phi(u_n) \geqq \phi(\lim u_n)$. In a finite dimensional space, the weak topology and the strong topology are same, but, in an infinite dimensional space, they are different. Therefore if we can get an example in a complete K-metric space where K is a strongly normal cone of a reflexive infinite dimensional Banach space, the above definition of the lower semicontinuity will be more significant; we also generalize the value of K-metric $d(x, y)$ to an infinite dimensional space and improve [1, 2].

From now on, we assume that $(V, \langle \cdot, \cdot \rangle)$ is an inner product space over R (all real numbers). $\langle \cdot, \cdot \rangle$ is an inner product on V, and $\|x\|=\langle x, x\rangle^{1/2}$, $x \in V$.

LEMMA 1 (Parallelogram Identity [4]). *Let V be an inner product space over R. Then*

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \qquad (x, y \in V).$$

LEMMA 2 (Polarization Identity [4]). *Let V be an inner product space over R. Then*

$$\langle x, y\rangle=\left\|\frac{x+y}{2}\right\|^2-\left\|\frac{x-y}{2}\right\|^2 \qquad (x, y \in V).$$

Remark. Let $0<c<1$. From Lemma 1, if $\|x-y\| \leqq c$, $\|x\|=1$, and $\|y\|=1$,

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we see $\left\|\frac{x+y}{2}\right\| \geqq 1-(c^2/4)$.

THEOREM 1. *Let V be an inner product space over R, K be a nonempty cone of V, and K have the following property (P). (P) If $x, y \in K$, and $\|x\|=\|y\|=1$, then $\|x-y\| \leqq c$, where $0<c<1$. Then K is strongly normal.*

Proof. Let $x_i \in K$, $1 \leqq i \leqq n$, $\|x_i\|=1$, $b_i \geqq 0$, and $\sum_{i=1}^{n} b_i=1$. Consider

$$\left\langle \sum_{i=1}^{n} b_i x_i,\ \sum_{i=1}^{n} b_i x_i \right\rangle = \sum_{i=1}^{n} b_i^2 + \sum_{i \neq j} b_i b_j \langle x_i, x_j \rangle .$$

From Lemma 2 and Remark, we have

$$\langle x_i, x_j \rangle \geqq 1-(c^2/2) > 0 .$$

Therefore, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} b_i x_i \right\|^2 &\geqq \sum_{i=1}^{n} b_i^2 + \sum_{i \neq j} b_i b_j \left(1-\frac{1}{2}c^2\right) \\
&\geqq \left(1-\frac{1}{2}c^2\right) \sum_{i=1}^{n} b_i^2 + \sum_{i \neq j} b_i b_j \left(1-\frac{1}{2}c^2\right) \\
&\geqq \left(1-\frac{1}{2}c^2\right)\left(\sum_{i=1}^{n} b_i^2 + \sum_{i \neq j} b_i b_j \right) \\
&\geqq \left(1-\frac{1}{2}c^2\right)(b_1+b_2+\cdots+b_n)^2 \\
&\geqq 1-\frac{1}{2}c^2 .
\end{aligned}$$

We get $\left\| \sum_{i=1}^{n} b_i x_i \right\| \geqq \left(1-\frac{1}{2}c^2\right)^{1/2} = \delta(c) > 0$. Hence K is strongly normal. This completes the proof.

Let $B=\left\{y\,;\ y \in V,\ \|y-x_0\| \leqq \frac{1}{8}\right\}$, where $x_0 \in V$, and $\|x_0\|=1$.

THEOREM 2. *Let $K=\{rx\,;\ x \in B$, and $r \geqq 0\}$. Then K is a strongly normal cone.*

Proof. We divide the proof into five steps.

(1) K is closed: Let $\{y_n\}$ be a sequence in K, which converges to $y \neq \theta$. There exist two sequences $\{a_n\} \subset R$, $\{z_n\} \subset B$ such that $y_n = a_n z_n$. Since $\{a_n\}$ is bounded, there exists a subsequence $\{a_{n(i)}\}$ of $\{a_n\}$ such that $\{a_{n(i)}\}$ converges to $a \neq 0$, and $\{z_{n(i)} = (1/a_{n(i)}) y_{n(i)}\}$ converges to $(1/a)y$. Since B is closed, we get $(1/a)y \in B$. Hence $y \in K$, and K is closed.

(2) K^0 is nonempty: It is clear because $x_0 \in K^0$.

(3) If $u, v \in K$, then $au+bv \in K$ for all $a, b \geqq 0$: Let $a>0$, $b>0$ and let $u=rx$, $v=sy$ where $r \geqq 0$, $s \geqq 0$, and $x, y \in B$. Then if $ar+bs=0$, $au+bv=0 \in K$. If

$ar+bs\neq 0$, $au+bv=arx+bsy=(ar+bs)\left[\frac{arx}{ar+bs}+\frac{bsy}{ar+bs}\right]\in K$.

(4) $K\cap(-K)=\{\theta\}$: If $t\in K\cap(-K)$ and $t\neq\theta$, then there exist points x, $y\in B$, and positive numbers r, p such that $t=-rx=py$. Hence we get $x=-(p/r)y$. Now $\frac{1}{4}\geqq\|x-y\|=\|y+(p/r)y\|=\|(1+(p/r))y\|\geqq\|y\|\geqq 1-\frac{1}{8}$. We get $\frac{3}{8}\geqq 1$, which is a contradiction. Hence $K\cap(-K)=\{\theta\}$.

(5) If x, $y\in K$, and $\|x\|=1$, $\|y\|=1$, then $\|x-y\|\leqq\frac{4}{7}$: Let $x\in K$, and $\|x\|=1$. There exist a positive number a, a point $z\in B$ such that $x=az$. Since $\|z-x_0\|\leqq\frac{1}{8}$, we get $\frac{7}{8}\leqq\|z\|\leqq\frac{9}{8}$. So $\frac{8}{9}\leqq a\leqq\frac{8}{7}$. Consider the distance

$$\begin{aligned}\|az-x_0\|&=\|az-ax_0+(a-1)x_0\|\\&\leqq a\|z-x_0\|+|(a-1)|\\&\leqq\frac{2}{7}.\end{aligned}$$

Therefore, if x, $y\in K$, and $\|x\|=1$, $\|y\|=1$, we have $\|x-y\|\leqq\frac{4}{7}$.

Combining (1) through (5) and Theorem 1, we see K is a strongly normal cone. This completes the proof.

The set K in Theorem 2 is an example of a strongly normal cone of a reflexive infinite dimensional Banach space if we let V be a Hilbert space.

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