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ON THE ORDER OF AUTOMORPHISM GROUP OF A COMPACT BORDERED RIEMANN SURFACE OF GENUS FOUR

Dedicated to Professor Mitsuru Ozawa on his 60th birthday

BY TAKAO KATO

§0. **Introduction.** For non-negative integers g and k ($2g+k-1\geqq 2$), let $N(g, k)$ be the maximum of the orders of the automorphism groups of compact bordered Riemann surfaces of genus g having k boundary components. Oikawa [9] proved that every automorphism group of a compact bordered Riemann surface is isomorphic to a subgroup of the automorphism group of a compact Riemann surface of the same genus and that $N(g, k)$ is equal to the maximum of the order of the automorphisms groups of k-times punctured compact Riemann surfaces of genus g. Hurwitz [3] proved that $N(g, 0)\leqq 84(g-1)$. For infinitely many values of g, $N(g, 0)$ were determined by [1, 6, 7, 8]. But, for infinitely many g, $N(g, 0)$ are not known. For every $g\geqq 0$, $N(g, 1)$, $N(g, 2)$ and $N(g, 3)$ were determined by the author [4], for every $k\geqq 0$, $N(0, k)$, $N(1, k)$, $N(2, k)$ and $N(3, k)$ were determined by [2, 9, 11, 12] and for many other special pairs of g and k, $N(g, k)$ were determined by Ouchi [10]. In this paper we shall determine $N(4, k)$ for every $k\geqq 0$. Wiman [14] showed the equations of all the compact Riemann surfaces of genus 4 which have non-trivial automorphism groups and proved that $N(4, 0)=120$. To determine $N(4, k)$, we shall study subgroups of groups which Wiman showed.

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§1. **Lemmas**: Let S be a compact Riemann surface of genus 4 and let G be a group of automorphisms of S. S/G has the conformal structure induced by the conformal structure of S such that the natural projection π of S onto S/G is holomorphic. Then, there are at most finite number of points $P_1, \cdots, P_t$ on S/G over which π is ramified with multiplicities $\nu_1, \cdots, \nu_t$ ($\nu_j\geqq 2$), respectively. Then Riemann-Hurwitz's relation shows

$$6/N=2\tilde{g}-2+\sum_{j=1}^{t}(1-1/\nu_j),$$

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where N is the order of G and $\tilde{g}$ is the genus of S/G. Note that if $\tilde{g}=0$, then $t \geqq 3$ and that $\pi^{-1}(P)$ consists of N/ν_j points if $P=P_j$ $(j=1, \cdots, t)$ and N points otherwise. We call such a group G a $(\tilde{g}; \nu_1, \cdots, \nu_t)$ group. For simplicity's sake we shall denote $(0; \nu_1, \cdots, \nu_t)$ by $(\nu_1, \cdots, \nu_t)$.

Using these notations we have a sequence of Lemmas.

LEMMA 1. *For any point P on S/G, all the points of $\pi^{-1}(P)$ have the same Weierstrass gap sequences.*

Proof. For any two points Q_1, Q_2 of $\pi^{-1}(P)$, there is an element of G, i.e., an automorphism of S, which maps Q_1 to Q_2.

LEMMA 2. *Assume $\tilde{g}=0$. Let*

$$k=mN+\sum_{j=1}^{t} \varepsilon_j(N/\nu_j),$$

where m is a non-negative integer and $\varepsilon_j=0$ or 1 $(j=1, \cdots, t)$. Then, $N(4, k) \geqq N$.

Proof. Choose m points $P_{t+1}, \cdots, P_{t+m}$ on $S/G-\{P_1, \cdots, P_t\}$ arbitrarily. Delete the set of points $\pi^{-1}(P_j)$, $(j=t+1, \cdots, t+m)$ and the set of points $\pi^{-1}(P_j)$, if $\varepsilon_j=1$ $(j=1, \cdots, t)$ from S. Then we have a k-times punctured Riemann surface of genus 4 such that G is a group of automorphisms of it. Thus $N(4, k) \geqq N$.

LEMMA 3. (Hurwitz [3]). *Assume* i) $\tilde{g} \geqq 1$ *and G is not a* $(1; 2)$ *group, or* ii) $t \geqq 6$. *Then, $N<12$, especially $N \leqq 5$ provided if N is prime.*

LEMMA 4. *N cannot be divided by any prime number greater than* 5.

Proof. Assume N is divisible by a prime number $N' \geqq 7$. Using Sylow's theorem we may assume $N=N'$. By Lemma 3, we have $\tilde{g}=0$. By Lemma 1, $\nu_1=\cdots=\nu_t=N$ and $t \geqq 3$. Using Riemann-Hurwitz's relation, we have

$$6/N=-2+t(1-1/N).$$

Hence, $N=(t+6)/(t-2)$ which is a contradiction.

For any point Q of $\pi^{-1}(P_j)$ $(j=1, \cdots, t)$, there is an element ϕ of G which fixes Q and generate a cyclic group $\langle\phi\rangle$ of order ν_j. Thus we have:

LEMMA 5. *There is a cyclic subgroup of G of order ν_j $(j=1, \cdots, t)$.*

But the order of a cyclic group is restricted from above such as:

LEMMA 6. (Wiman [13], Kato [4]). *If G is a cyclic group, then $N \leqq 18$.*

The next lemma is a well known property of hyperelliptic surface.

LEMMA 7. *Assume S is the hyperelliptic surface defined by*

$$y^2=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{10}).$$

Then, every automorphism of S induces a linear transformation of the x-sphere which maps the set $\{\alpha_1, \cdots, \alpha_{10}\}$ onto itself. Hence, $N\leqq 40$.

In the following two Lemmas we shall show properties of cyclic trigonal surface of genus 4.

LEMMA 8. (Kato [5]). *Assume S is defined by*

$$y^3=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_6).$$

Then, every automorphism of S induces a linear transformation of the x-sphere which maps the set $\{\alpha_1, \alpha_2, \cdots, \alpha_6\}$ onto itself. Hence, $N\leqq 72$.

LEMMA 9. (Kato [5]). *Assume S is defined by*

$$y^3=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)^2(x-\alpha_5)^2(x-\alpha_6)^2.$$

Then, the number of Weierstrass points whose gap sequences are $\{1, 2, 4, 5\}$ *is* 6, 9, *or* 12. *If there are* 12 *such Weierstrass points, then N is a multiple of* 36, *i.e.*, $N=36$ *or* 72.

§2. Models.

In this section we shall list up Riemann surfaces which are used to determine $N(4, k)$ and show some properties of those surfaces.

Let S_j $(j=1, \cdots, 19)$ be the Riemann surfaces of genus 4 defined by the equations as follows, throughout these equations α, β and γ are mutually distinct complex numbers which are neither 0 nor 1:

$$S_1 : y^5=x^3(x-1)^2(x+1),$$

$$S_2 : y^3=x(x^4-1),$$

$$S_3 : y^3=(x^3-1)/(x^3+1),$$

$$S_4 : y^2=x^{10}-1,$$

$$S_5 : y^2=x(x^8-1),$$

$$S_6 : y^2=x^9-1,$$

$$S_7 : y^3=x^6-1,$$

$$S_8 : y^5=x^3-1,$$

$$S_9 : y^6=x(x^2-1),$$

$$S_{10} : y^4=x^3(x-1)(x-\alpha)(x-\beta),$$

$S_{11}: y^4=x^3(x-1)(x-\alpha)^2(x-\beta)^2(x-\gamma)^2$,

$S_{12}: y^2=x(x^4-1)(x^4-\alpha)$,

$S_{13}: y^{10}=x(x-1)^3$,

$S_{14}: y^6=x(x-1)(x-\alpha)^2$,

$S_{15}: y^6=x(x-1)(x-\alpha)$,

$S_{16}: y^6=x(x-1)^2(x-\alpha)^4$,

$S_{17}: y^5=x(x-1)(x-\alpha)$,

$S_{18}: y^5=x^4(x-1)(x-\alpha)$,

$S_{19}: y^5=x^3(x-1)^2(x-\alpha)$.

We shall show the automorphism group $\mathrm{Aut}\, S_j$ of S_j $(j=1, \cdots, 9)$ in the following Properties 1-9.

Property 1. Choose as basis of holomorphic differentials on S_1 such as $\theta_1=dx/y$, $\theta_2=xdx/y^2$, $\theta_3=x(x-1)dx/y^3$ and $\theta_4=x^2(x-1)dx/y^4$. Then we have a canonical embedding of S_1 into $\boldsymbol{P}^3(\theta)$ with projective coordinates $(\theta_1, \theta_2, \theta_3, \theta_4)$. Embed $\boldsymbol{P}^3(\theta)$ into $\boldsymbol{P}^4(X)$ such as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ \eta^4 & \eta^3 & \eta^2 & -\eta \\ \eta^3 & \eta & \eta^4 & -\eta^2 \\ \eta^2 & \eta^4 & \eta & -\eta^3 \\ \eta & \eta^2 & \eta^3 & -\eta^4 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix},$$

where $\eta=e^{2\pi i/5}$. Then $\boldsymbol{P}^3(\theta)$ is mapped onto the hyperplane $X_1+X_2+X_3+X_4+X_5=0$ in $\boldsymbol{P}^4(X)$ and the image of S_1 is mapped onto the intersection of the hyperplane and two hypersurfaces,

$$X_1^2+X_2^2+X_3^2+X_4^2+X_5^2=0,$$

$$X_1^3+X_2^3+X_3^3+X_4^3+X_5^3=0.$$

This is known as Bring's curve [14] and its automorphism group is of order 120 which is a (2, 4, 5) group. Let ψ_j $(j=1, 2)$ be the automorphisms of $\boldsymbol{P}^4(X)$ defined by

$$\psi_1: (X_1, X_2, X_3, X_4, X_5) \longrightarrow (X_2, X_1, X_3, X_4, X_5),$$

$$\psi_2: (X_1, X_2, X_3, X_4, X_5) \longrightarrow (X_1, X_3, X_4, X_5, X_2).$$

Let ϕ_{1j} $(j=1, 2)$ be the automorphisms of S_1 corresponding to ψ_j. Then, $\mathrm{Aut}\, S_1=\langle\phi_{11}, \phi_{12}\rangle$ and $\phi_{13}=\phi_{11}\circ\phi_{12}$ is of order 5.

Property 2. Let ϕ_{21}, ϕ_{22} and ϕ_{23} be the automorphisms of S_2 defined by

$$\phi_{21}:(x,\ y)\longrightarrow(e^{\pi i/2}/x,\ e^{\pi i/2}y/x^2),$$

$$\phi_{22}:(x,\ y)\longrightarrow\left(\frac{i-x}{i+x},\ 2e^{\pi i/6}y/(x+i)^2\right),$$

$$\phi_{23}=\phi_{22}\circ\phi_{21}.$$

Then the orders of ϕ_{21}, ϕ_{22} and ϕ_{23} are 2, 3, 12, respectively and $\langle\phi_{21},\ \phi_{22}\rangle$ is a (2, 3, 12) group of order 72. By Lemma 8 we have Aut $S_2=\langle\phi_{21},\ \phi_{22}\rangle$.

Property 3. Let ϕ_{31}, ϕ_{32} and ϕ_{33} be the automorphisms of S_3 defined by

$$\phi_{31}:(x,\ y)\longrightarrow(1/x,\ -y),$$

$$\phi_{32}:(x,\ y)\longrightarrow(y,\ -e^{2\pi i/3}/x),$$

$$\phi_{33}=\phi_{32}\circ\phi_{31}.$$

Then, the order of ϕ_{31}, ϕ_{32} and ϕ_{33} are 2, 4 and 6, respectively, and $\langle\phi_{31},\ \phi_{32}\rangle$ is a (2, 4, 6) group of order 72. Since $N(4,\ 0)=120$, Aut $S_3=\langle\phi_{31},\ \phi_{32}\rangle$.

Property 4. Let ϕ_{41}, ϕ_{42} and ϕ_{43} be the automorphisms of S_4 defined by

$$\phi_{41}:(x,\ y)\longrightarrow(e^{\pi i/5}/x,\ e^{\pi i/2}y/x^5),$$

$$\phi_{42}:(x,\ y)\longrightarrow(1/x,\ e^{\pi i/2}y/x^5),$$

$$\phi_{43}=\phi_{41}\circ\phi_{42}.$$

Then, the orders of ϕ_{41}, ϕ_{42} and ϕ_{43} are 2, 4 and 10, respectively. Since S_4 is hyperelliptic, by Lemma 6 we have Aut $S_4=\langle\phi_{41},\ \phi_{42}\rangle$ which is a (2, 4, 10) group of order 40.

Property 5. Let ϕ_{51}, ϕ_{52} and ϕ_{53} be the automorphisms of S_5 defined by

$$\phi_{51}:(x,\ y)\longrightarrow(e^{\pi i/4}/x,\ e^{5\pi i/8}y/x^5),$$

$$\phi_{52}:(x,\ y)\longrightarrow(1/x,\ e^{\pi i/2}y/x^5),$$

$$\phi_{53}=\phi_{51}\circ\phi_{52}.$$

Then the orders of ϕ_{51}, ϕ_{52}, ϕ_{53} are 2, 4 and 16, respectively and by Lemma 6 Aut $S_5=\langle\phi_{51},\ \phi_{52}\rangle$ which is a (2, 4, 16) group of order 32.

Property 6. Let ϕ_{61} be the automorphism of S_6 defined by

$$\phi_{61}:(x,\ y)\longrightarrow(e^{2\pi i/9}x,\ -y).$$

Then, by Lemma 6 we have Aut $S_6=\langle\phi_{61}\rangle$ which is a (2, 9, 18) group of order 18.

Property 7. Let ϕ_{71}, ϕ_{72} and ϕ_{73} be the automorphisms of S_7 defined by

$$\phi_{71}:(x,\ y)\longrightarrow(e^{\pi i/3}/x,\ e^{\pi i/3}y/x^2),$$

$$\phi_{72}:(x,\ y)\longrightarrow(e^{\pi i/3}x,\ y),$$

$$\phi_{73}=\phi_{71}\circ\phi_{72}.$$

Then, the orders of ϕ_{71}, ϕ_{72} and ϕ_{73} are 2, 6, 6, respectively, and $\langle\phi_{71},\ \phi_{72}\rangle$ is a (2, 6, 6) group of order 36. Assume the order of $\mathrm{Aut}\,S_7$ is 72. By Lemma 7, the cubic group is a subgroup of $\mathrm{Aut}\,S_7$. But in this case $\langle\phi_{71},\ \phi_{72}\rangle$ induces a dihedral group of the x-sphere. It is a contradiction. Hence, $\mathrm{Aut}\,S_7=\langle\phi_{71},\ \phi_{72}\rangle$.

Property 8. Let ϕ_{81} be the automorphism of S_8 defined by

$$\phi_{81}:(x,\ y)\longrightarrow(e^{2\pi i/3}x,\ e^{2\pi i/5}y).$$

By Lemma 7 we have $\mathrm{Aut}\,S_8=\langle\phi_{81}\rangle$ which is a (3, 5, 15) group of order 15.

Property 9. Let ϕ_{91} be the automorphism of S_9 defined by

$$\phi_{91}:(x,\ y)\longrightarrow(-x,\ e^{\pi i/6}y).$$

We have $\mathrm{Aut}\,S_9=\langle\phi_{91}\rangle$ which is a (4, 6, 12) group of order 12. In fact, points over $x=0,\ 1,\ -1$ are Weierstrass points whose gap sequences are $\{1, 2, 3, 7\}$. Hence, meromorphic functions of order 3 on S_9 are linear fractions in y. As a covering of the y-sphere, S_9 has 12 branch points over $y^{12}=4/27$. Hence, automorphisms are possibly induced from $y\to1/y$, $y\to e^{\pi i/6}y$ and these compositions. However, the gap sequences of the three points over $y=0$ are $\{1, 2, 3, 7\}$ and those over $y=\infty$ are $\{1, 2, 3, 5\}$. Hence, an automorphism induced from $y\to1/y$ does not exist.

§ 3. **Estimate of** $N(4,\ k)$.

To determine $N(4,\ k)$, we have to consider the possibility of $(\tilde{g},\ \nu_1,\ \cdots,\ \nu_t)$ group. However, giving an estimate of $N(4,\ k)$ from below, we shall not need to consider groups of small order.

PROPOSITION (10). *$N(4,\ k)\geqq10$ for all k.*

Proof. The group $\langle\phi_{72}^3,\ \phi_{73}\rangle$ is a (2, 2, 3, 6) group of order 12. Since every even number can be represented as $12m+6\varepsilon_1+6\varepsilon_2+4\varepsilon_3+2\varepsilon_4$ by a suitable non-negative integer m and $\varepsilon_j=0$ or 1 $(j=1,\ \cdots,\ 4)$, by Lemma 2 we have $N(4,\ k)\geqq12$ if $k\equiv0 \pmod 2$. $\langle\phi_{42}\circ\phi_{41}\rangle$ is a (5, 10, 10) group of order 10. Hence, by Lemma 2, if $k\equiv0,\ 1,\ 2,\ 3,\ 4 \pmod{10}$, then $N(4,\ k)\geqq10$. $\langle\phi_{42}^2,\ \phi_{43}^2\rangle$ is a (2, 2, 5, 5) group of order 10. Hence, again by Lemma 2, if $k\equiv0,\ 2,\ 4 \pmod 5$, then $N(4,\ k)\geqq10$. Therefore, we have $N(4,\ k)\geqq10$ for all k.

Thus, it is not necessary to consider groups of order less than or equal to 10. We shall list up possible groups of order more than 10.

Making the following table we are assuming Lemmas 3, 4 and 6.

Possible order	*Possible group*
144	(2, 3, 8)
120	(2, 4, 5)
108	(2, 3, 9)
90	(2, 3, 10)
72	(2, 3, 12), (2, 4, 6), (3, 3, 4)
60	(2, 3, 15), (2, 5, 5)
54	(2, 3, 18)
48	(2, 4, 8)
45	(3, 3, 5)
40	(2, 4, 10)
36	(2, 4, 12), (2, 6, 6), (3, 3, 6), (3, 4, 4), (2, 2, 2, 3)
32	(2, 4, 16)
30	(2, 5, 10)
27	(3, 3, 9)
24	(2, 6, 12), (2, 8, 8), (3, 3, 12), (3, 4, 6), (4, 4, 4), (2, 2, 2, 4)
20	(2, 10, 10), (4, 4, 5), (2, 2, 2, 5)
18	(2, 9, 18), (3, 6, 6), (2, 2, 2, 6), (2, 2, 3, 3)
16	(2, 16, 16), (4, 4, 8), (2, 2, 2, 8)
15	(3, 5, 15), (5, 5, 5)
12	(3, 12, 12), (4, 6, 12), (6, 6, 6), (2, 2, 3, 6), (2, 2, 4, 4), (2, 3, 3, 3), (2, 2, 2, 2, 2), (1 ; 2)

It is known that $N(4, 0) \leqq 120$ [4]. Hence, $N(4, k) \leqq 120$ for all k. It is also known that an automorphism group of order 108 or 90 does not exist [14]. We shall give an alternative proof of this facts.

PROPOSITION (120). *A* $(2, 3, r)$ *group does not exist for* $r=8$, 9 *or* 10. *Hence, if* $k \equiv 0, 24 \pmod{30}$, *then* $N(4, k)=120$.

Proof. Assume $r=8$. Since the total weights of Weierstrass points on S is 60, by Lemma 1, we have

$$144\alpha_1+72\alpha_2+48\alpha_3+18\alpha_4=60$$

for some nonnegative integers α_1, α_2, α_3 and α_4. But it is impossible.

Assume $r=9$. By Lemmas 3 and 5, S is defined by

$$y^9=x^\lambda(x-1)^\mu, \qquad 3 \nmid \lambda,\ \mu,\ \lambda+\mu\,.$$

Hence, it is conformally equivalent to S_6 which is hyperelliptic. Hence, the order of Aut S is 18.

Assume $r=10$. As is the case of $r=8$, there are 30 Weierstrass points of weight 2. On the other hand, by Lemmas 3 and 5, S is conformally equivalent to S_4 or S_{13}. But S_4 is hyperelliptic and S_{13} has a Weierstrass point of weight 4 at $(x, y)=(0, 0)$. Contradiction.

Using Lemma 1 and Property 1, we can prove that if $k\equiv 0, 24 \pmod{30}$, then $N(4, k)=120$.

PROPOSITION (72). *If* $k\equiv 0 \pmod 6$ *and* $N(4, k)\neq 120$, *then* $N(4, k)=72$.

Proof. Using Lemma 2 and Properties 1, 2 and 3, we can prove this Proposition. As a fact, there is no (3, 3, 4) group. It is proved by Wiman [14]. But we shall give an alternative proof. Assume there is a (3, 3, 4) group and $\nu_1=\nu_2=3$, $\nu_3=4$. Then, by Lemma 1 and the fact that the total weights of Weierstrass points is 60, all the points of $\pi^{-1}(P_3)$ are Weierstrass points of weight 2 and all the points of one of the sets $\pi^{-1}(P_1)$ and $\pi^{-1}(P_2)$, say $\pi^{-1}(P_1)$, are Weierstrass points of weight 1. The possibilities of gap sequences of Weierstrass points of weight 2 are $\{1, 2, 4, 5\}$ and $\{1, 2, 3, 6\}$. Since there are 18 points in $\pi^{-1}(P_3)$, the gap sequences of these points are $\{1, 2, 3, 6\}$, (cf. Kato [5, Theorem 1]). Assume α is an automorphism of S which fixes a point Q of $\pi^{-1}(P_3)$ and f is a meromorphic function on S which has a pole of order 4 at Q and is holomorphic elsewhere. Then, $(f+f\circ\alpha+f\circ\alpha^2+f\circ\alpha^3)^{1/4}$ is a single valued meromorphic function on $S/\langle\alpha\rangle$, whose order is 1. Hence, $S/\langle\alpha\rangle$ is the sphere. Therefore, S is conformally equivalent to S_{10} or S_{11}. But S_{10} has a Weierstrass point of weight 3 at $(x, y)=(0, 0)$ and S_{11} has a Weierstrass point of weight 6 at $(x, y)=(0, 0)$. Both of them contradict our assumption.

PROPOSITION (60). *A* (2, 3, 15) *group does not exist. For each* k, $N(4, k)\neq 60$.

Proof. Assume a subgroup of Aut S is a (2, 3, 15) group. By Lemma 5, Aut S has an element of order 15. Hence, S is conformally equivalent to S_8. But the order of Aut S_8 is 15. This is a contradiction. Assume k is an integer such that $N(4, k)=60$. Then, by Lemma 2

$$k=60m+\varepsilon_1(60/2)+\varepsilon_2(60/5)+\varepsilon_3(60/5),$$

for a nonnegative integer m and $\varepsilon_j=0$ or 1 $(j=1, 2, 3)$. Thus, $k\equiv 0 \pmod 6$. But for such a k, $N(4, k)\geqq 72$ by Proposition (72).

PROPOSITION (54). *A* (2, 3, 18) *group does not exist.*

Proof. Assume a subgroup of Aut S is a (2, 3, 18) group. Then, there is an automorphism of S of order 18. Hence, S is conformally equivalent to S_6. But the order of Aut S_6 is 18. This is a contradiction.

PROPOSITION (48). *A* (2, 4, 8) *group does not exist.*

Proof. Assume a subgroup of Aut S is a (2, 4, 8) group. Then, there is an automorphism of S of order 8 and S is conformally equivalent to S_{12}. Since S_{12}

is hyperelliptic, by Lemma 7 we have a contradiction.

PROPOSITION (45). *A* (3, 3, 5) *group does not exist.*

Proof. Assume a subgroup of Aut S is a (3, 3, 5) group. Then, there is an automorphism of S of order 5 and S is conformally equivalent to S_{17}, S_{18} or S_{19}. Assume S_{17} admits a (3, 3, 5) group G and P is the point of S_{17}/G which corresponds to a fixed point of an automorphism of order 5. Then, $\pi^{-1}(P)$ consists of 9 points and the points corresponding to $x=0$, 1, α and ∞ are in $\pi^{-1}(P)$. However, the gap sequences of these points are $\{1, 2, 3, 7\}$ for $x=0$, 1, α and $\{1, 2, 4, 7\}$ for $x=\infty$. This is a contradiction. Since S_{18} is hyperelliptic, the order of Aut S_{18} is 40. Assume S_{19} admits a (3, 3, 5) group. There is an automorphism of S_{19} whose order is 2. Hence, the order of Aut S_{19} is a multiple of 90. This contradicts Proposition (120).

PROPOSITION (40). *If* $k\equiv 0, 4 \pmod{10}$ *and* $N(4, k)\neq 120, 72$, *then* $N(4, k)=40$.

Proof. Observe Property 4 and Lemma 2.

PROPOSITION (36). *If* $k\equiv 9$ *or* $21 \pmod{36}$, *then* $N(4, k)=36$. *A* (2, 4, 12) *group does not exist.*

Proof. $\langle\phi_{33}^2, \phi_{32}\rangle$ is a (3, 4, 4) group. Therefore, if $k\equiv 0, 9, 12, 21$, or $30 \pmod{36}$, then $N(4, k)\geqq 36$. But if $k\equiv 0, 12, 18, 30 \pmod{36}$, then $N(4, k)\geqq 72$. By virtue of Proposition (72) it is not necessary to consider the possibility of cases (2, 6, 6), (3, 3, 6) and (2, 2, 2, 3) groups.

Assume S admits a (2, 4, 12) group. Then, there is an automorphism of S of order 12. Hence, S is defined by either

$$y^{12}=x(x-1)^4$$

or

$$y^{12}=x(x-1)^2.$$

The former is conformally equivalent to S_2 and the latter is to S_9. On S_2 there are exactly 6 Weierstrass points whose gap sequences are $\{1, 2, 4, 7\}$. Let $Q_1, \cdots, Q_6$ be these points. Then,

$$^{\#}\pi^{-1}(\{\pi(Q_1), \cdots, \pi(Q_6)\})=6,$$

by Lemma 1. However, it is impossible because 6 cannot be represent as $36m+(36/2)\varepsilon_1+(36/4)\varepsilon_2+(36/12)\varepsilon_3$ for a nonnegative integer m and $\varepsilon_j=0$ or 1 ($j=1, 2, 3$). Hence, S_2 does not admit a (2, 4, 12) group. On S_9 there are 3 Weierstrass points whose gap sequences are $\{1, 2, 4, 7\}$. Hence they should be fixed points of automorphisms of order 12. For nonnegative integers m_1, m_2 and m_3, $36m_1+(36/2)m_2+(36/4)m_3$ is divisible by 9. On the other hand the total weight of Weierstrass points except for the above 3 points is 48 which cannot be divided by 9. Hence, by Lemma 1 S_9 does not admit a (2, 4, 12) group.

PROPOSITION (32). *If* $k\equiv 0, 2 \pmod 8$ *and* $N(4, k)\neq 120, 72, 40$, *then* $N(4, k) =32$.

Proof. Observe Property 5 and Lemma 2.

PROPOSITION (30). *A* (2, 5, 10) *group does not exist.*

Proof. Assume S admits a (2, 5, 10) group. Then, there is an automorphism of S of order 10. Hence, S is conformally equivalent to S_4 or S_{13}. On S_4 there are exactly 10 hyperelliptic Weierstrass points. By Lemma 1, S_4 does not admit a (2, 5, 10) group. On S_{13} the gap sequence corresponding to $x=0$ is $\{1, 2, 4, 7\}$ and that to $x=1$ is $\{1, 2, 3, 6\}$. Both of these points are fixed points of an automorphism of order 10. Hence, by Lemma 1 S_{13} does not admit a (2, 5, 10) group.

PROPOSITION (27). *A* (3, 3, 9) *group does not exist.*

Proof. Assume S admits a (3, 3, 9) group. Then S has an automorphism of order 9. Hence, S is conformally equivalent to S_6. But the order of Aut S_6 is 18. This is a contradiction.

PROPOSITION (24). *If* $k\equiv 2, 4 \pmod{12}$ *and* $N(4, k)\neq 40, 32$, *then* $N(4, k)=24$. *None of the following groups exists*: (2, 8, 8), (3, 3, 12) *and* (3, 4, 6) *groups. It is not necessary to consider* (4, 4, 4) *and* (2, 2, 2, 4) *groups.*

Proof. $\langle\phi_{21}, \phi_{23}^2\rangle$ is a (2, 6, 12) group. Hence, by Lemma 2, if $k\equiv 0, 2, 4$, or 6, then $N(4, k)\geqq 24$. By Lemma 2 and Proposition (72), it is not necessary to consider (4, 4, 4) and (2, 2, 2, 4) groups. Assume S admits a (2, 8, 8) group. Then S has an automorphism of order 8 and S is conformally equivalent to S_{12}. Since S_{12} is hyperelliptic, by Lemma 1 we have a contradiction. Assume S admits a (3, 3, 12) group. Since S has an automorphism of order 12, S is conformally equivalent to S_2 or S_9. Observing Weierstrass points whose gap sequences are $\{1, 2, 4, 7\}$, by a similar argument as Proposition (36), we have a contradiction. Assume S admits a (3, 4, 6) group. Then, S is conformally equivalent to S_{14}, S_{15} or S_{16}. S_{14} is conformally equivalent to the surface defined by

$$y^3=x(x^2-1)(x^2-\alpha/(\alpha-1)).$$

Hence, S_{14} has 6 Weierstrass points whose gap sequences are $\{1, 2, 4, 7\}$. At least 2 of those points are fixed points of an automorphism of order 6. Therefore, by Lemma 1 S is not conformally equivalent to S_{14}. Assume S is conformally equivalent to S_{15}. Let ϕ be an automorphism of S_{15} defined by

$$\phi:(x, y)\longrightarrow(x, e^{\pi i/3}y).$$

Then, ϕ has exactly 3 fixed points. On the other hand, there exist 4 points on

S_{15} which are fixed by automorphisms of order 6. This is a contradiction. Assume S is conformally equivalent to S_{16}. S_{16} is conformally equivalent to the surface defined by

$$y^3 = x(x^2-1)(x^2-\alpha)^2 .$$

Hence, by Lemma 9 there exist at least 6 Weierstrass points whose gap sequences are $\{1, 2, 4, 5\}$. Especially the fixed points of automorphisms of order 6 are among those points. There are exactly 4 such points. Hence, by Lemma 1 and Lemma 9 the 8 fixed points of automorphisms of order 3 also have the gap sequence $\{1, 2, 4, 5\}$. Therefore, again by Lemma 9, the order of $\mathrm{Aut}\, S_{16}$ is 36 or 72. Since the order of (3, 4, 6) group is 24, the order of $\mathrm{Aut}\, S_{16}$ is 72. Hence, S is conformally equivalent to S_3. Observing a (2, 4, 6) group, i.e., $\mathrm{Aut}\, S_3$, there are 12 Weierstrass points of weight 2 which are the fixed points of order 6 and there are either 18 Weierstrass points of weight 2 which are the fixed points of order 4 or 36 Weierstrass points of weight 1 which are the fixed points of order 2 (as a fact, the latter case does occur). But both the cases contradict Lemma 1.

PROPOSITION (20). *If* $k \equiv 2, 5, 9, 12 \pmod{20}$ *and* $N(4, k) \neq 72, 36, 32, 24$, *then* $N(4, k)=20$.

Proof. $\langle \phi_{42}^2, \phi_{43} \rangle$ is a (2, 10, 10) group and $\langle \phi_{42}, \phi_{43}^2 \rangle$ is a (4, 4, 5) group. Hence, by Lemma 2, we have this proposition.

PROPOSITION (18). *If* $k \equiv 0 \pmod 3$ *or* $k \equiv 1, 2 \pmod 9$ *and* $N(4, k)=120, 72, 40, 36, 32, 24, 20$, *then* $N(4, k)=18$. *It is not necessary to consider* (2, 2, 2, 6) *groups.*

Proof. $\langle \phi_{61} \rangle$ is a (2, 9, 18) group, $\langle \phi_{73}^2, \phi_{72} \rangle$ is a (3, 3, 6) group and $\langle \phi_{73}^3, \phi_{72}^2 \circ \phi_{73}^3, (\phi_{72} \circ \phi_{71})^2 \rangle$ is a (2, 2, 3, 3) group. Apply Lemma 2.

PROPOSITION (16). *If* $k \equiv 1, 4, 6 \pmod{16}$ *and* $N(4, k) \neq 120, 72, 40, 36, 24, 20, 18$, *then* $N(4, k)=16$. *It is not necessary to consider* (2, 2, 2, 8) *groups.*

Proof. $\langle \phi_{53} \rangle$ is a (2, 16, 16) group and $\langle \phi_{52}, \phi_{53}^2 \rangle$ is a (4, 4, 8) group. Apply Lemma 2.

PROPOSITION (15). *If* $k \equiv 1, 4, 5, 8 \pmod{15}$ *and* $N(4, k) \neq 40, 32, 24, 20, 18, 16$, *then* $N(4, k)=15$. *It is not necessary to consider* (5, 5, 5) *groups.*

Proof. $\langle \phi_{81} \rangle$ is a (3, 5, 15) group. Apply Lemma 2.

PROPOSITION (12). *If* $k \equiv 1, 3, 5 \pmod{12}$ *or* $k \equiv 0 \pmod 2$ *and* $N(4, k) \neq 120, 72, 40, 32, 24, 20, 18, 16, 15$, *then* $N(4, k)=12$. *It is not necessary to consider* (6, 6, 6), (2, 2, 4, 4), (2, 3, 3, 3) *and* (2, 2, 2, 2, 2) *groups.*

Proof. $\langle\phi_{23}\rangle$ is a (3, 12, 12) group and $\langle\phi_{91}\rangle$ is a (4, 6, 12) group. Apply Lemma 2.

Summing up these Propositions we have:

THEOREM:

$$
\begin{aligned}
N(4, k) = 120,\ & \textit{if}\ \ k\equiv 0,\ 24 \pmod{30}, \\
72,\ & \textit{if}\ \ k\equiv 0 \pmod 6\ \textit{and}\ N(4, k)\neq 120, \\
40,\ & \textit{if}\ \ k\equiv 0,\ 4 \pmod{10}\ \textit{and}\ N(4, k)\neq 120,\ 72, \\
36,\ & \textit{if}\ \ k\equiv 9,\ 21 \pmod{36}, \\
32,\ & \textit{if}\ \ k\equiv 0,\ 2 \pmod 8\ \textit{and}\ N(4, k)\neq 120,\ 72,\ 40, \\
24,\ & \textit{if}\ \ k\equiv 2,\ 4 \pmod{12}\ \textit{and}\ N(4, k)\neq 40,\ 32, \\
20,\ & \textit{if}\ \ k\equiv 2,\ 5,\ 9,\ 12 \pmod{20}\ \textit{and}\ N(4, k)\neq 72,\ 36,\ 32,\ 24, \\
18,\ & \textit{if}\ \ k\equiv 0 \pmod 3\ \text{or}\ k\equiv 1,\ 2 \pmod 9\ \textit{and}\ N(4, k)\neq 120, \\
& \qquad 72,\ 40,\ 36,\ 32,\ 24,\ 20, \\
16,\ & \textit{if}\ \ k\equiv 1,\ 4,\ 6 \pmod{16}\ \textit{and}\ N(4, k)\neq 120,\ 72,\ 40,\ 36,\ 24, \\
& \qquad 20,\ 18, \\
15,\ & \textit{if}\ \ k\equiv 1,\ 4,\ 5,\ 8 \pmod{15}\ \textit{and}\ N(4, k)\neq 40,\ 32,\ 24,\ 20, \\
& \qquad 18,\ 16, \\
12,\ & \textit{if}\ \ k\equiv 1,\ 3,\ 5 \pmod{12}\ \text{or}\ k\equiv 0 \pmod 2\ \textit{and}\ N(4, k) \\
& \qquad \neq 120,\ 72,\ 40,\ 32,\ 24,\ 20,\ 18,\ 16,\ 15 \\
10,\ & \textit{otherwise}.
\end{aligned}
$$

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